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PHY422/820: Classical Mechanics

FS 2021
Worksheet #2 (Sep 8 – 10)

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1 Preparation

- Lemos, Section 1.2–1.3
- Goldstein, Sections 1.3–1.4

2 Notes

2.1 Configuration Space and Configuration Manifold

The purpose of the present discussion is to define some of the terminology we will use to describe the "setting" of Lagrangian (and later, Hamiltonian) dynamics, chief among them the **configuration space** and **configuration manifold**.

2.1.1 A Single Particle

Let us start with the simple case of a point particle that has three degrees of freedom, associated with the motion in the directions associated with our chosen basis vectors for three-dimensional space, e.g., $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$. Then the trajectory of the particle that is obtained by solving the equations of motion can be expressed as

$$\vec{r}(t) = x(t)\vec{e}_x + y(t)\vec{e}_y + z(t)\vec{e}_z\,, \tag{1}$$

which is obviously an element of the space $\mathbb{R}^3$ at all times t — thus, we say the $\mathbb{R}^3$ is the configuration space of the particle.

Let us now assume that the particle is not able to move freely in space due to some constraint — for instance, it might be connected to a rod of fixed length l whose end is fixed at all times. Thus, the particle is now constrained to move on a sphere with fixed radius l, which means the relation

$$x^2 + y^2 + z^2 = l^2 = \text{const.}\,, \tag{2}$$

must hold at all times for an allowed trajectory. This reduces the degrees of freedom of the particle from three to two: essentially, we can describe the dynamics in terms of two angules that uniquely identify the position of the particle on the sphere. We may be tempted to say that the configuration space of the particle is $\mathbb{R}^2$ now, but this would be wrong: The curvature of the sphere prevents us from simply treating its surface as a vector space, and we must resort to the more fundamental

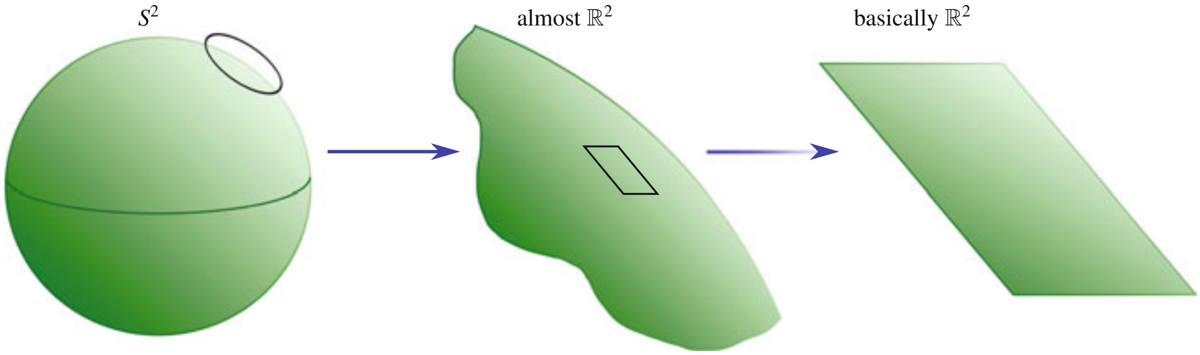

Figure 1: Zooming in on a two-dimensional spherical surface.

concept of a manifold, which for our purposes we take to be a continuous, smooth set of points (more below). Thus, we we say that the sphere defines a two-dimensional **configuration manifold** in the configuration space $\mathbb{R}^3$ of our particle.

In general, if we have a particle whose n degrees of freedom are labeled by **generalized** coordinates $q_1, \ldots, q_n$ and that is subject to c constraints of the form[1]

$$f_i(q_1, \ldots, q_n) = 0\,, \quad i = 1, \ldots, c \tag{3}$$

we will take the configuration space to be $\mathbb{R}^n$, and the configuration manifold to be an $(n-c)-$dimensional surface embedded in this space that is defined as[2]

$$\mathcal{M} = \left\{ (q_1, \ldots, q_n) \in \mathbb{R}^n : \quad \bigwedge_{i=1}^{c} f_i(q_1, \ldots, q_n) = 0 \right\} . \tag{4}$$

2.1.2 Manifolds in a Nutshell

In the previous subsection, we have already given an operating definition of a manifold that will be useful for this course: We can think of it as a continuous set of points $q = (q_1, \ldots, q_n) \in \mathbb{R}^n$ that defines a smooth surface of dimension $d \leq n$, but does **not** have the structure of a vector space in general. Because the set is supposed to be continuous and smooth, we will be able to take derivatives at each point of the manifold, which allows us to construct **local tangent vectors** that span a d-dimensional **tangent space** — this is essentially what we have been doing to construct the basis vectors in cylindrical and spherical coordinates! Geometrically, the underlying idea is that any smooth $d-$dimensional surface will start to look like $\mathbb{R}^d$ if you zoom in on an infinitesimally small region around the point q (see Fig. 1 for the example of a two-dimensional sphere).

Since we can define a tangent space at each point (see Fig. 2), a manifold is accompanied by an infinite set of tangent spaces that we refer to as a **tangent bundle**. In general, one has to be careful when one wants to compare quantities that are defined in one tangent space to those defined in the tangent space at another point of the manifold. This should be evident from Fig. 2, which shows the different orientations of the tangent spaces for the circle, sphere, and torus. In

[1] Note that we can easily write Eq. (2) in the required form by moving l^2 to the left-hand side.

[2] Read: the set of points in $\mathbb{R}^3$ that simultaneously satisfies the constraints $f_1 = 0$ through $f_{N_c} = 0$. $\bigwedge$ is analogous to the sum $\sum_n$ or product $\prod_n$, but for the logical "and" operation $\wedge$.

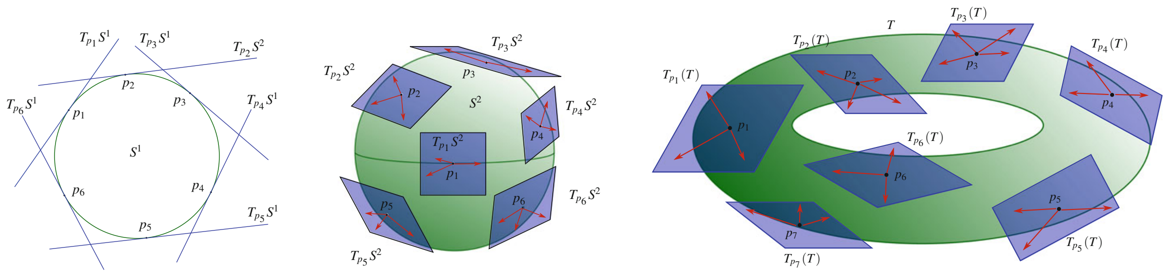

Figure 2: Tangent spaces of a circle, sphere, and torus.

Classical and Quantum Field Theory, for example, one needs to study how fields evolve as they interact with particles that move through the space time manifold, which leads to the introduction of **connections** on the tangent bundle, that are commonly referred to as **covariant derivatives** in these domains — we will encounter one of the simplest examples when we study the motion of a particle in an electromagnetic field in a few weeks.

Examples

- The set of points $q = (x, y, z) \in \mathbb{R}^3$ is a manifold but **not** a vector space, a priori. The reason is that q describes a *position* in $\mathbb{R}^3$, but not a direction! We can turn the manifold into a vector space if we introduce a unique mapping that identifies points with (column) vectors,

$$\Phi : \quad q \to \vec{r} = (x, y, z)^T = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \tag{5}$$

  The vector now has a direction because we tacitly assume that its tail is the origin of a chosen coordinate system, and the tip the point q.

  The local tangent vectors of our manifold are $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ *everywhere*, which means that the tangent spaces are $\mathbb{R}^3$ vector spaces themselves, and they can be identified with each other as well as the manifold itself if we "promote it" to a vector space by the map Φ.

- A circle and a sphere with arbitrary fixed radius, conventionally chosen to be $r = 1$ since one can easily account for scaling factors, are manifolds denoted S^1 and S^2, respectively. Note that they both satisfy equations of the form

$$\sum_{i=1}^{n} x_i^2 = r^2 \tag{6}$$

  for $n = 2$ (circle) and $n = 3$ (sphere), respectively. Thus, one can view the circle as a "sphere" in a two-dimensional space, and from there generalize the notion of a sphere to spheres S^n in higher-dimensional spaces.

- We can build more complicated manifolds by taking products of the "primitive" manifolds $\mathbb{R}^n$ and S^n. Products of $\mathbb{R}$ simply give rise to manifolds of higher dimension, e.g., $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. A cylinder in three dimensions with fixed radius is a manifold of type $\mathbb{R} \times S^1$, and a torus can be expressed as a product of two circles, $S^1 \times S^1$. (cf. Fig. 2). Tori in higher dimension can be defined through the n-fold products $S^1 \times \ldots \times S^1$

Concluding Remarks

- Manifolds that *contain* their boundaries are called **compact**. This is a generalization of the notion of a closed interval ($[a,b]$) in the real numbers, which can be understood as a sub-manifold of the real numbers $\mathbb{R}$. In contrast, an open interval — commonly denoted as $]a,b[$ or (a,b) — is not compact, and neither is $\mathbb{R}$, because it does not contain $\pm\infty$.

- The spheres S^n are compact: For S^1, this stems from the fact that the angles 0 and 2π are identified with each other. The same holds for the azimuthal angle ϕ on the sphere S^2, while the polar angle θ is defined on the finite interval $[0,\pi]$. Since S^1 is compact, tori in arbitrary dimensions are compact manifolds as well.

- One can define proper notions of integration on manifolds, through the use of differential forms — we might discuss this a bit more elsewhere. We implicitly make use of the concepts associated with the integration on manifolds when we define surface and volume elements in curvilinear coordinates (think $dV = r\sin\theta dr d\theta d\phi$).

- In the present discussion, we defined manifolds as set of points over some chosen coordinates, e.g., in $\mathbb{R}^n$. Essentially, we treated a point q as an n-dimensional function of the coordinates,

$$q : \mathbb{R}^n \to \mathbb{R}^n, \quad q(q_1, \ldots, q_n) = (q_1, \ldots, q_n)\,. \tag{7}$$

  The notion of a manifold does not rely on q simply being a point. We already saw that this is true when we noted that the *vector space* $\mathbb{R}^n$ $\vec{r}(q_1, \ldots, q_n)$ is a manifold as well.

  In fact, we can construct manifolds from sets of general mathematical objects as long as they depend smoothly on some variables: matrices, elements of mathematical groups, etc. In the discussion of the rigid body dynamics later in the semester, we will see that the orientations of the rigid body are described by the elements of the rotation group SO(3), which can be parameterized by a set of three angles (e.g., the Euler angles you may have heard of).

2.1.3 A-Particle Systems

Let us now apply the notions introduced in the previous sections to the description of the dynamics of an A-particle system.

The configuration space for A particles with n degrees of freedom each is obtained by simply taking the product of the configuration spaces for the individual particles, and then we impose constraints of the form

$$f_i(q_1, \ldots, q_{An}) = 0\,, \quad i = 1, \ldots, c\,, \tag{8}$$

analogous to what we did for a single particle. **It is important to note that the number of constraints is in general not A times the number of the constraints on an individual particle of the system:**

- Constraints on the motion of individual particles **cannot** account for the coupling of degrees of freedom of several particles, e.g., if particles are connected pairwise through fixed rods. Such constraint must explicitly depend on the variables of several particles.

- Constraints that couple the motion of particles might also make constraints on individual particles redundant, i.e., they might be automatically satisfied and no longer constrain the dynamics meaningfully (see Section 2.2).

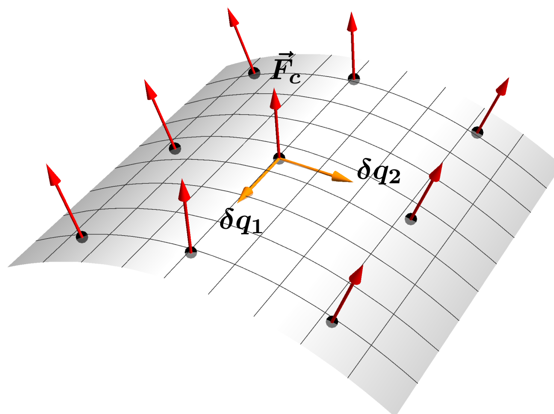

Figure 3: Configuration manifold for a single particle and constraint forces.

2.2 Constraints and Constraint Forces

In many mechanics applications, we are dealing with constrained motion: We would naively assign Cartesian coordinates to all masses of interest because that is easy to picture, and subsequently solve the equations of motion resulting from Newton's Second Law, modeling any restrictions to the motion with the help of constraint forces like the normal force, tension, etc. The problem with this approach is that the constraint forces can usually only be determined after the fact, when we have solved for all trajectories.

What we would really like to do is exploit the constraints immediately, so that we can deal with the "true" degrees of freedom directly. Let us discuss this using the motion of a point mass under gravity that is sliding without friction on a smooth curved surface (see Fig. 3). The configuration space of the particle is $\mathbb{R}^3$, and a natural choice of coordinates would be (x, y, z). However, the motion is constrained to a curved manifold that is defined by a constraint

$$f(x, y, z) = 0\,. \tag{9}$$

Such a constraint that is called **holonomic**. It can be used to eliminate one of the coordinates, so we can parameterize the dynamics in terms of the **generalized coordinates** q_1 and q_2:

$$\vec{r}(x, y, z) \quad \to \quad \vec{r}(q_1, q_2)\,. \tag{10}$$

Virtual Displacements

We can now introduce (unnormalized) tangent vectors to the manifold by computing the derivatives of $\vec{r}$ with respect to the generalized coordinates at a given point of $q_0 = (q_{10}, q_{20})$ of the manifold (see Fig. 3):

$$\delta\vec{q}_i = \frac{\partial\vec{r}}{\partial q_i}\delta q_i\,, \quad i = 1, 2\,. \tag{11}$$

Traditionally, these tangent vectors or the scalar quantities δq_i are referred to as **virtual displacements** in the literature. They are often described as *"instantaneous infinitesimal changes of the coordinates that are allowed under constraints"*. In this context,

- *instantaneous* means that any time argument appearing in $\vec{r}$ is held fixed, and
- *allowed under the constraints* means that the change of coordinates occurs in the configuration manifold (or tangential to it, for vectors).

The geometric interpretation given here will hopefully be more intuitively accessible.

An arbitrary virtual displacement of the particle is given by

$$\delta\vec{r} = \delta\vec{q}_1 + \delta\vec{q}_2 = \sum_i \frac{\partial\vec{r}}{\partial q_i}\delta q_i \,, \tag{12}$$

which is distinct from the "real" displacement given by the total differential

$$d\vec{r} = \sum_i \frac{\partial\vec{r}}{\partial q_i}dq_i + \frac{\partial\vec{r}}{\partial t}dt \,, \tag{13}$$

which is subject to both the constraints and the non-constraint forces acting on the particle, i.e., gravity in our example.

Constraint Forces

In Figure 3, we have also indicated the constraint force $\vec{F}_C$. The sole function of $\vec{F}_C$ is to keep the particle constrained to the surface at all points and times. Since the particle is free to move tangentially to the surface, $\vec{F}_C$ has no impact on the tangential motion, but it must prevent any motion that is orthogonal to the surface, i.e., any acceleration that would cause the particle to sink through the surface or lift off from it. These considerations imply that we can make the following ansatz for the constraint force:

$$\vec{F}_C = \lambda\vec{\nabla}f \,, \tag{14}$$

since $\vec{\nabla}f$ will be orthogonal to any surface for which

$$f(x, y, z) = \text{const.} \tag{15}$$

(This is analogous to conservative forces being orthogonal to the equipotential surfaces of the underlying potential.) **It is important to point out that we will usually not be able to determine the overall sign of the factor λ a priori, but must do so on physical grounds while we are solving the problem.**

In our present example, the physical interpretation of the constraint force is clear: It is the normal force that the curved surface must exert on the particle to counteract the normal component of gravity.

Types of Constraints

The preceding discussion, including the definition (11), holds not only for holonomic constraints of the form

$$f(x, y, z) = 0 \,, \tag{16}$$

but also for cases where the constraint surface becomes explicitly time dependent:

$$f(x, y, z, t) = 0 \,. \tag{17}$$

This simply meands that the configuration manifold and its tangent spaces evolve in time, as determined by Eq. (17), and we perform the construction of the preceding sections at a fixed

$t = t_0$. Both types of constraints are holonomic, and they can be used to eliminate variables. In the case (16), we refer to the dynamics as **scleronomous** (from Greek *skleros*, "stiff, tough") and the constraint is called **holonomic-scleronomic** or **scleronomic** for short. In the case (17), the dynamics is called **rheonomous** (from Greek *rheo*, "to flow, run"), and the constraint is called **(holonomic-)rheonomic**.

In principle, we can also have constraints that not only depend on the generalized coordinates q_i, but also on their corresponding generalized velocities $\dot{q}_i$ — usually, this happens when we can only relate the rate of change of coordinates to each other. An important example we will encounter is rolling without slipping. In such cases, we speak of a **nonholonomic** constraint, because we won't be able to use the constraints to eliminate variables. Under certain conditions, we may be able to integrate the nonholonomic constraint, which then just turns out to be a holonomic constraint in disguise.

Finally, we can have constraints that involve inequalities, e.g., when the dynamics of particles is not restricted to a surface, but a restricted volume. An example would be a mass sliding off a hemisphere of radius R under the influence of gravity: The constraint in spherical coordinates will be given by

$$x^2 + y^2 + z^2 \geq R^2 \,, \tag{18}$$

i.e., the mass cannot penetrate the hemisphere, but is allowed to have periods of contact. The various types of constraints are summarized in Box 2.1.

Box 2.1: Summary: Types of Constraints

Type	Form	Examples
	holonomic	
scleronomic	$f(q_1, ..., q_n) = 0$	rigid body, pendulum ...
rheonomic	$f(q_1, ..., q_n, t) = 0$	bead on rotating wire, pendulum with moving suspension
	nonholonomic	
	$f(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) = 0$	disk rolling without slipping
	$f(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = 0$	processes with friction
	other	
	$f(q_1, \ldots, q_n) \geq 0$	mass sliding off a hemisphere

2.3 D'Alembert's Principle

In the present section, we will derive **D'Alembert's principle** using the geometric view of constraints we introduced in the previous section.

For our discusion, we use the example of a bead that can slide without friction on a spiral wire with cross section radius a (see Fig. 4). The coordinates of the bead are

$$x = a\cos\phi\,, \tag{19}$$

$$y = a\sin\phi\,, \tag{20}$$

$$z = b\phi\,, \tag{21}$$

where we choose ϕ as our generalized coordinate. We can parameterize multiple loops of the spiral by allowing $\phi \in]-\infty, \infty[$ — otherwise, we would have to eliminate the angle in favor of the coordinate z, which leads to messier equations for x and y.)
The definition of z is one of the constraints on the bead; the other is given by

$$x^2 + y^2 = a^2\,. \tag{22}$$

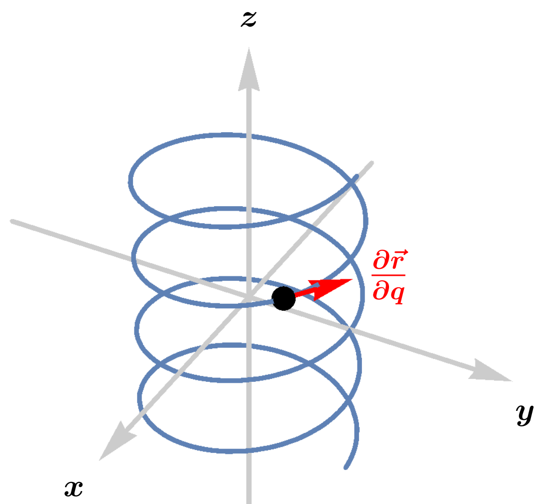

Figure 4: Bead gliding on a spiral wire without friction.

Together, the two constraints define the configuration manifold, i.e., the spiral wire.

Now consider Newton's Second Law,

$$m\ddot{\vec{r}} = \vec{F}_A + \vec{F}_C\,, \tag{23}$$

where $\vec{F}_A$ is the total "applied" force acting on the bead, and $\vec{F}_C$ is the total constraint force. Here, the applied force is gravity, so

$$\vec{F}_A = -mg\vec{e}_z\,, \tag{24}$$

and the explicit form of $\vec{F}_C$ can only be determined *after* we have solved the equations of motion for the bead. We know from Sec. 2.2 that the constraint forces are proportional to the gradient of the constraints defining the configuration manifold. and orthogonal to the manifold itself in each point. This means that the scalar products of $\vec{F}_C$ with the vectors from the manifold's tangent space at the same point must vanish.

As discussed above, we obtain a tangent vector to the manifold by taking the derivative of the coordinates with respect to ϕ,

$$\frac{\partial\vec{r}}{\partial\phi} = \left(-a\sin\phi,\ a\cos\phi,\ b\right)^T = a\vec{e}_\phi + b\vec{e}_z\,, \tag{25}$$

where we have introduced the usual unit vectors of a cylindrical coordinate system, and we have

$$\vec{F}_C\cdot\frac{\partial\vec{r}}{\partial\phi} = 0 \tag{26}$$

(see Exercise 2.1).

Exercise 2.1: Constraint Forces for a Bead on a Spiral Wire

Show that the constraints defining the spiral wire can be expressed in cylindrical coordinates as

$$f_1(\rho, \phi, z) = \rho - a = 0\,, \tag{27}$$
$$f_2(\rho, \phi, z) = z - b\phi = 0\,. \tag{28}$$

Construct the constraint forces (up to the unknown factor λ), and show that they are orthogonal to the configuration manifold.

We can exploit the orthogonality condition (26) by projecting Eq. (23) onto the tangent vector:

$$m\ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial \phi} = \vec{F}_A \cdot \frac{\partial \vec{r}}{\partial \phi} + \underbrace{\vec{F}_C \cdot \frac{\partial \vec{r}}{\partial \phi}}_{=0}\,. \tag{29}$$

In this way, we do not have to worry about the constraint forces. In fact, we can now drop the distinction between applied and constraint forces in this equation and simply write

$$m\ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial \phi} = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \phi}\,, \tag{30}$$

because the constraint forces are projected out from $\vec{F}$ automatically!

Let us now consider the left-hand side of Eq. (30). The trajectory of the bead can be written as $\vec{r}(t) = \vec{r}(\phi(t))$, i.e., the time dependence is entirely contained in the time dependence of the generalized coordinate ϕ, but all steps would work analogously if $\vec{r}$ were explicitly time dependent. Using the chain rule, we have

$$\dot{\vec{r}} = \frac{\partial \vec{r}}{\partial \phi}\,\dot{\phi}\,, \tag{31}$$
$$\ddot{\vec{r}} = \frac{\partial^2 \vec{r}}{\partial \phi^2}\,\dot{\phi}^2 + \frac{\partial \vec{r}}{\partial \phi}\,\ddot{\phi}\,. \tag{32}$$

The vector in the first term can be written as

$$\frac{\partial^2 \vec{r}}{\partial \phi^2} = (-a\cos\phi, -a\sin\phi, 0)^T = -\vec{e}_\rho\,, \tag{33}$$

so we see that

$$\frac{\partial^2 \vec{r}}{\partial \phi^2} \cdot \frac{\partial \vec{r}}{\partial \phi} = -\vec{e}_\rho \cdot (a\vec{e}_\phi + b\vec{e}_z) = 0 \tag{34}$$

because of the orthogonality of the unit vectors. Consequently, we obtain

$$m\ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial \phi} = \ddot{\phi}\left|\frac{\partial \vec{r}}{\partial \phi}\right|^2 = \ddot{\phi}(a^2 + b^2)\,. \tag{35}$$

The right-hand side is obtained easily:

$$\vec{F} \cdot \frac{\partial \vec{r}}{\partial \phi} = -mg\vec{e}_z \cdot \frac{\partial \vec{r}}{\partial \phi} = -mgb\,. \tag{36}$$

Thus, the **equation of motion** for the bead becomes

$$\begin{aligned} & m(a^2+b^2)\ddot{\phi} = -mgb \\ \Rightarrow \quad & \ddot{\phi} = -\frac{b}{a^2+b^2}\, g\,, \end{aligned} \tag{37}$$

which has the general solution

$$\phi(t) = \phi_0 + \omega_0\, t - \frac{1}{2}\frac{bg}{a^2+b^2}t^2\,. \tag{38}$$

When we multiply $\phi(t)$ with b (cf. Eq. (21), we essentially obtain a "free" fall in z with a reduced acceleration.

D'Alembert's Principle

We can rewrite our projected form of Newton's Second Law, Eq. (30), as

$$\left(\vec{F} - m\ddot{\vec{r}}\right)\cdot\frac{\partial\vec{r}}{\partial q} = 0\,, \tag{39}$$

where we switched to an arbitrary generalized coordinate q. Now we can recall the definition of the virtual displacements along the constraint manifolds from Sec. 2.2, which states

$$\delta\vec{r} = \frac{\partial\vec{r}}{\partial q}\,\delta q\,. \tag{40}$$

Thus, we can simply multiply the geometric condition (39) by δq on both sides, and obtain

$$\boxed{\left(\vec{F} - m\ddot{\vec{r}}\right)\cdot\delta\vec{r} = 0\,.} \tag{41}$$

This is **d'Alembert's principle** for a single particle in the form that is usually found in textbooks. It is statement of the fundamental laws of classical motion, and while it is equivalent to Newton's Laws for essentially all intents and purposes, *it cannot be derived from them.* It is also fundamentally related to the Principle of Least Action that we will discuss later, but in fact more general since it applies to systems with non-holonomic constraints.

2.4 Examples

In this section, we discuss the application of d'Alembert's principle to additional examples.

Planar Pendulum

Figure 5 shows a planar pendulum, consisting of a mass m that swings on a string of length l under the influence of gravity. The pendulum motion is most efficiently described using polar coordinates, but with some alterations: For instance, it is much more appropriate to measure the polar angle with respect to vertical instead of the horizontal axis, because the vertical axis defines the pendulum at rest. Also, we can avoid carrying negative signs and awkward angular offsets in the coordinates if we let the y axis point downward, but we must properly account for that choice when we define the gravitational force and potential, then.

Box 2.2: Virtual Work

Traditionally, d'Alemberts principle has been stated in the following form:

The constraint forces perform no virtual work.

Virtual work is defined as the work done by the forces along virtual displacements that are compatible with the constraints:

$$\delta W \equiv \vec{F} \cdot \delta\vec{r} = \vec{F}_A \cdot \delta\vec{r}\,. \tag{42}$$

It can be a useful concept in the determination and analysis of *static equilibrium* configurations (with $\ddot{\vec{r}} = 0$) of mechanical systems, which are defined by

$$\delta W = 0\,. \tag{43}$$

In his original work, d'Alembert generalized this idea by including the inertial force $-m\ddot{\vec{r}}$ (or $-\dot{\vec{p}}$ if the mass is allowed to change) in his balancing equation for the virtual work, giving rise to Eq. (41). Mathematically, this moves the inertial term in Newton's Law from one side of the equation to the other, but philosophically, it removes the special role that the inertial term plays in Newtonian mechanics and instead treats it on an equal footing as all other forces.

Starting from Cartesian coordinates, we have

$$x = r\sin\phi\,, \tag{44}$$
$$y = r\cos\phi\,, \tag{45}$$

with the constraints

$$\phi = \arctan\frac{x}{y} \quad \in \quad \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[\tag{46}$$

and

$$f(r,\phi) = r - l = 0\,. \tag{47}$$

Thus, the motion is confined to the usual circular arc shown in Fig. 5, which serves as the configuration manifold for the pendulum. The angle ϕ serves as our generalized coordinate.

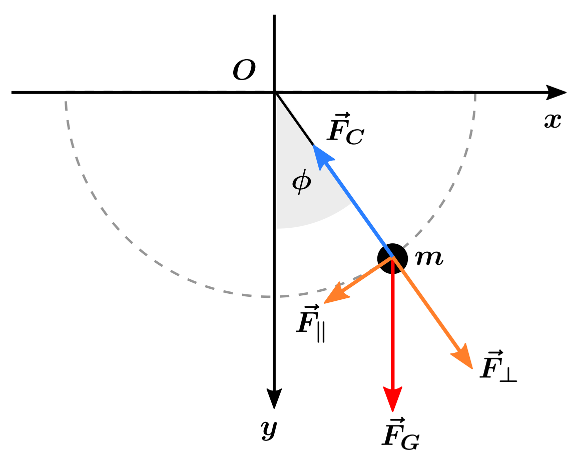

Figure 5: A planar pendulum.

The basis vectors for the coordinate system are

$$\vec{e}_r = \left|\frac{\partial\vec{r}}{\partial r}\right|^{-1} \frac{\partial\vec{r}}{\partial r} = (\sin\phi, \cos\phi)^T\,, \tag{48}$$

$$\vec{e}_\phi = \left|\frac{\partial\vec{r}}{\partial \phi}\right|^{-1} \frac{\partial\vec{r}}{\partial \phi} = (\cos\phi, -\sin\phi)^T\,, \tag{49}$$

(notice the difference from the usual definition). It is clear from the figure that $\vec{e}_\phi$ is tangential to the configuration manifold, and we therefore have

$$\delta\vec{r} = \frac{\partial\vec{r}}{\partial\phi}\,\delta\phi = (r\,\delta\phi)\;\vec{e}_\phi\,. \tag{50}$$

The constraint force $\vec{F}_C$ — i.e., the tension in the pendulum string — is pointing in negative radial direction

$$\vec{F}_C \sim -\vec{e}_r \,, \tag{51}$$

as expected from the gradient of Eq. (47). This immediately yields

$$\vec{F}_C \cdot \delta\vec{r} = 0 \,, \tag{52}$$

i.e., the tension does not perform any work on the pendulum and can be dropped from consideration.

Let us now apply Eq. (39), using $\vec{F} = mg\vec{e}_y$ as the force acting on the pendulum. To evaluate the inertial term, we note that

$$\vec{r} = r\vec{e}_r \,, \tag{53}$$

$$\dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\phi}\,\vec{e}_\phi \,, \tag{54}$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\,\vec{e}_r + (2\dot{r}\dot{\phi} + r\ddot{\phi})\,\vec{e}_\phi \,, \tag{55}$$

and therefore

$$m\ddot{\vec{r}} \cdot \delta\vec{r} = m(2\dot{r}\dot{\phi} + r\ddot{\phi}) \cdot r\delta\phi \,. \tag{56}$$

For the gravitational force, we obtain

$$\vec{F}_G \cdot \delta\vec{r} = mg\vec{e}_y \cdot (r\delta\phi)\vec{e}_\phi = -mgr\sin\phi\,\delta\phi \,. \tag{57}$$

D'Alembert's principle now reads

$$m\left(2r\dot{r}\dot{\phi} + r^2\ddot{\phi} + gr\sin\phi\right)\delta\phi = 0 \,, \tag{58}$$

and after using Eq. (47) and simplifying, we find

$$\left(\ddot{\phi} + \frac{g}{l}\sin\phi\right)\delta\phi = 0 \,. \tag{59}$$

Since this expression must hold for arbitrary $\delta\phi$, the term in parentheses must vanish, and this condition is the usual equation of motion for the pendulum:

$$\ddot{\phi} + \frac{g}{l}\sin\phi = 0 \,. \tag{60}$$

For small angles, we can expand

$$\sin\phi = \phi + \mathcal{O}(\phi^3) \approx \phi \,, \tag{61}$$

which turns the equation of motion into that of a harmonic oscillator with frequency $\omega = \sqrt{g/l}$:

$$\ddot{\phi} + \omega^2\phi \equiv \ddot{\phi} + \frac{g}{l}\phi = 0 \,. \tag{62}$$

For comparison, let us also consider solving the problem in the Newtonian approach. To apply Newton's Second Law, we would split gravity into components that are perpendicular and tangential to the arc of the pendulum. They can be determined by projecting on $\vec{e}_r$ and $\vec{e}_\phi$:

$$\vec{F}_\parallel \equiv (\vec{F}_G \cdot \vec{e}_\phi)\,\vec{e}_\phi = mg(\vec{e}_y \cdot \vec{e}_\phi)\,\vec{e}_\phi = -mg\sin\phi\,\vec{e}_\phi \,, \tag{63}$$

$$\vec{F}_\perp \equiv (\vec{F}_G \cdot \vec{e}_r)\,\vec{e}_r = mg(\vec{e}_y \cdot \vec{e}_r)\,\vec{e}_r = mg\cos\phi\,\vec{e}_r \,. \tag{64}$$

Note that we essentially determined $\vec{F}_{\parallel}$ when applying d'Alembert's principle. To prevent any acceleration in radial direction, the tension must compensate $\vec{F}_{\perp}$, which implies

$$\vec{F}_C = -mg\cos\phi\,\vec{e}_r\,, \tag{65}$$

in accordance with our earlier consideration. Using the expression for the acceleration in polar coordinates above and comparing coefficients on both sides, Newton's Second Law yields the following system of equations for the radial and angular directions

$$m(\ddot{r} - r\dot{\phi}^2) = F_C + F_{\perp} = 0 \tag{66}$$

$$m(2\dot{r}\dot{\phi} + r\ddot{\phi}) = F_{\parallel} = -mg\sin\phi\,, \tag{67}$$

which leads to the same equation of motion after some minor rearranging. This particular problem is easy enough so that we could have skipped the calculation of $\vec{F}_C$ and $\vec{F}_{\perp}$ while determining the pendulum equation, which means the effort is more or less the same as for applying d'Alembert's principle. In complex systems, however, d'Alembert's principle offers a more efficient approach.

3 Group Exercises

Problem G4 – Constraints and Constraint Forces

Figures 1 and 2 show physical systems that are subject to constraints:

- a *spherical pendulum*, consisting of a mass m that is suspended from the ceiling by a string of fixed length l, and
- a bead of mass m that can slide without friction along a wire that rotates with constant angular velocity ω around an axis through the origin, with a **fixed angle** $\boldsymbol{\alpha}$ between the wire and the axis.

1. Formulate *all* constraints on the motion, both in Cartesian and spherical coordinates, and state what kind of constraint each of them is (e.g., holonomic-scleronomic, holonomic-rheonomic, nonholonomic, ...).

2. Compute the constraint forces for the Cartesian and spherical-coordinate versions of the constraints, and give a physical interpretation.

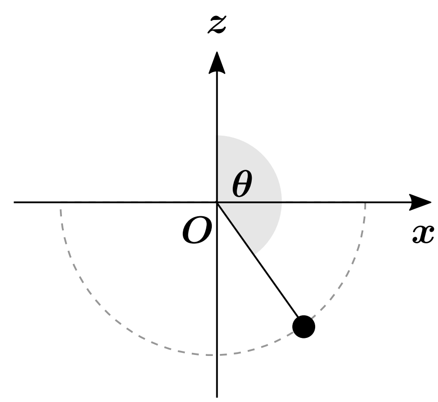

Figure 6: Spherical pendulum, suspended from a ceiling (side view).

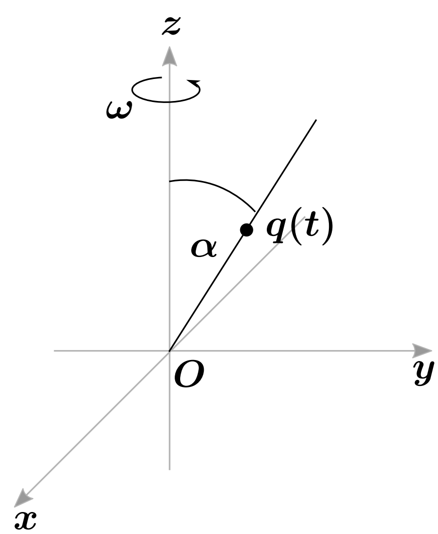

Figure 7: Bead on a rotating wire.

Problem G5 – Motion on a Torus

A torus in three-dimensional space can be parameterized as

$$x = (R + r\cos\theta)\cos\phi\,, \tag{68}$$
$$y = (R + r\cos\theta)\sin\phi\,, \tag{69}$$
$$z = r\sin\theta\,, \tag{70}$$

where $\theta \in [0, 2\pi]\,, \phi \in [0, 2\pi]$, r is the *minor radius*, i.e., the radius of the tube, and R the *major radius*, measured from the origin to the center of tube.

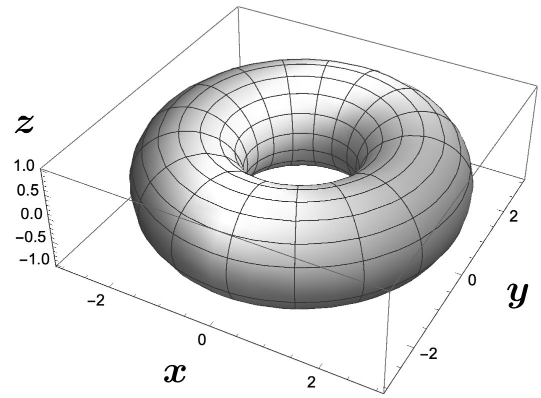

Figure 8: A torus with $R = 2, r = 1$.

1. Show that for $R =$ const., $r =$ const., the coordinates satisfy the equation

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2\,. \tag{71}$$

2. Construct the basis vectors $\{\vec{e}_\theta, \vec{e}_\phi\}$ that span the tangent space at each point of the torus' surface. Express the vectors in terms of the Cartesian basis.

3. Construct the force that will constrain the motion of a point mass to the torus. (Recall that it will only be defined up to a factor λ, whose sign, size and dimensions depend on the other forces acting on the system.)

4. Show explicitly that the constraint force does not perform any work on a mass that moves on the torus.