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PHY422/820: Classical Mechanics

FS 2021

Worksheet #3 (Sep 13 – 17)

October 5, 2021

1 Preparation

- Lemos, Section 1.4-1.5, 2.1–2.4
- Goldstein, Sections 2.1–2.4

2 Notes

2.1 Geometric Derivation of the Lagrange Equations

A drawback of Newton's Second Law is that the form of the resulting equations of motion for each coordinate of a particle are in general *not invariant* under *point transformations* between two sets of independent coordinates,

$$q_i\left(s_1,\ldots,s_n\right) \quad \rightarrow \quad s_i\left(q_1,\ldots,q_n\right)\,, \quad i=1,\ldots,n\,. \tag{1}$$

For instance, the term $m\ddot{\vec{r}}$ already becomes much more complicated if we switch to curvilinear coordinates to describe motion in an inertial frame, and in non-inertial frames *fictitious* terms like the centrifugal or Coriolis forces appear (see Exercise 2.1). D'Alembert's principle

$$\left(\vec{F}-m\ddot{\vec{r}}\right)\cdot\delta\vec{r}=0\,, \tag{2}$$

on the other hand, can be rewritten in a way that makes it form-invariant under point transformations between generalized coordinates (see Exercise 2.2). For simplicity, we will consider a holonomic system with one generalized coordinate first, and generalize our result afterwards.

2.1.1 The Lagrange Equations for One Degree of Freedom

Let us denote the generalized coordinate describing the single degree of freedom of our system by q. We start our derivation by considering the force term in d'Alembert's principle (2):

$$\vec{F}\cdot\delta\vec{r}=\vec{F}\cdot\frac{\partial\vec{r}}{\partial q}\delta q\equiv Q\delta q\,. \tag{3}$$

Here, we have defined the **generalized force** Q associated with the coordinate q. Just like a generalized coordinate need not have the dimensions of a length, Q need not have the dimensions of a force, but $Q\delta q$ will *always* have the dimensions of an energy.

Exercise 2.1: Newton's Second Law in Different Coordinate Systems

Consider a mass moving in a two-dimensional plane, In Cartesian coordinates, Newton's Second Law yields the following equations of motion:

$$\ddot{x} = \frac{F_x}{m}\,, \tag{E2.1-1}$$

$$\ddot{y} = \frac{F_y}{m}\,. \tag{E2.1-2}$$

Show that in polar coordinates

$$x = r\cos\phi, \quad y = r\sin\phi\,, \tag{E2.1-3}$$

Newton's Second Law takes the form

$$\ddot{r} - r\dot{\phi}^2 = \frac{F_r}{m}\,, \tag{E2.1-4}$$

$$2\dot{r}\dot{\phi} + r\ddot{\phi} = \frac{F_\phi}{m}\,, \tag{E2.1-5}$$

with $F_r = \vec{F}\cdot\vec{e}_r$ and $F_\phi = \vec{F}\cdot\vec{e}_\phi$. Thus, the form of Newton's Second Law depends on the choice of coordinates.

Next, we rewrite the inertial term. Using the product rule, we have

$$m\ddot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial q} = \frac{d}{dt}\left(m\dot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial q}\right) - m\dot{\vec{r}}\cdot\frac{d}{dt}\frac{\partial\vec{r}}{\partial q}\,. \tag{4}$$

Consider the time derivative of $\vec{r}$, which we can evaluate using the chain rule:

$$\dot{\vec{r}} = \frac{\partial\vec{r}}{\partial q}\dot{q}\,. \tag{5}$$

Since we have holonomic constraints, $\vec{r}$ does not depend on $\dot{q}$, and $\dot{\vec{r}}$ is a function that only depends on $\dot{q}$ *linearly*. If we take the partial derivative with respect to $\dot{q}$ on both sides of Eq. (5), we obtain

$$\frac{\partial\dot{\vec{r}}}{\partial\dot{q}} = \frac{\partial\vec{r}}{\partial q}\,. \tag{6}$$

This identity is often referred to as a *cancellation of dots*, but keep in mind that it only applies under certain circumstances. Now consider the second term on the right-hand side of Eq. (4). The derivatives commute (even for non-holonomic constraints and explicitly time dependent $\vec{r}$), so we have

$$\frac{d}{dt}\frac{\partial\vec{r}}{\partial q} = \frac{\partial\dot{\vec{r}}}{\partial q}\,. \tag{7}$$

Using identities (6) and (7), we can rewrite Eq. (4) as

$$m\ddot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial q} = \frac{d}{dt}\left(m\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial\dot{q}}\right) - m\dot{\vec{r}}\cdot\frac{\partial\dot{\vec{r}}}{\partial q}\,. \tag{8}$$

If we now introduce the kinetic energy $T(q,\dot{q}) = \frac{1}{2}m\dot{\vec{r}}^2$, we recognize its partial derivatives with respect to q and $\dot{q}$ on the right-hand side:

$$\begin{aligned} m\ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q} &= \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}m\dot{\vec{r}}^2\right) - \frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}m\dot{\vec{r}}^2\right) \\ &= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q}\,. \end{aligned} \tag{9}$$

Putting everything together, we see that d'Alembert's principle satisfies

$$\begin{aligned} \left(m\ddot{\vec{r}} - \vec{F}\right) \cdot \delta\vec{r} &= \left(m\ddot{\vec{r}} - \vec{F}\right) \cdot \frac{\partial \vec{r}}{\partial q}\delta q \\ &= \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} - Q\right)\delta q = 0\,. \end{aligned} \tag{10}$$

Since this equation has to hold for *arbitrary* virtual displacements δq, the expression in the parenthesis must vanish, which leads us to the **Lagrange equation** for a system with one degree of freedom:

$$\boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} - Q = 0\,.} \tag{11}$$

If the forces acting on the system are *conservative*, Q can be derived from a potential $V(q)$

$$Q = -\frac{\partial V}{\partial q}\,. \tag{12}$$

In this case, we can define the **Lagrangian** L of the system,

$$L(q,\dot{q}) \equiv T(q,\dot{q}) - V(q)\,, \tag{13}$$

and rewrite Eq. (10) as

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0\,.} \tag{14}$$

2.1.2 Lagrange Equations for Multiple Degrees of Freedom

The derivations of the previous section can be readily generalized to systems with multiple degrees of freedom — that is, multiple particles and/or multiple degrees of freedom per particle — without going through the math in detail again. In an N-particle system, we obviously have positions and velocities for each particle, corresponding to $3N$ coordinates:

$$\vec{r}, \dot{\vec{r}} \quad \longrightarrow \quad \{\vec{r}_i, \dot{\vec{r}}_i\}_{i=1,\ldots,N}\,. \tag{15}$$

If the system is subject to k holonomic constraints, the total number of degrees of freedom is $n = 3N - k$, and to each of them we associate an an independent generalized coordinate and velocity:

$$q, \dot{q} \quad \longrightarrow \quad \{q_j, \dot{q}_j\}_{j=1,\ldots,3N-k}\,. \tag{16}$$

In general, each $\vec{r}_i$ is a function of *all* generalized coordinates, because the motion of each particle could be related to that of all others by the constraint — just think of the case of the rigid body. As a consequence, differentials and virtual displacements will have the generalization

$$\delta\vec{r} = \frac{\partial\vec{r}}{\partial q}\delta q \quad \longrightarrow \quad \delta\vec{r}_i = \sum_{j=1}^{3N-k}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j\,, \tag{17}$$

$$d\vec{r} = \frac{\partial\vec{r}}{\partial q}dq + \frac{\partial\vec{r}}{\partial t}\,dt \quad \longrightarrow \quad d\vec{r}_i = \sum_{j=1}^{3N-k}\frac{\partial\vec{r}_i}{\partial q_j}dq_j + \frac{\partial\vec{r}}{\partial t}dt\,. \tag{18}$$

Using these rules for the coordinates, we can generalize the kinetic and potential energies as

$$T = \frac{1}{2}m\dot{\vec{r}}^{\,2} \quad \longrightarrow \quad \frac{1}{2}\sum_i m_i\dot{\vec{r}}_i^{\,2} \tag{19}$$

and

$$V(\vec{r}) \quad \longrightarrow \quad V(\vec{r}_1,\ldots,\vec{r}_N)\,, \tag{20}$$

respectively. Here we have assumed that the potential does not depend on the velocity: This is the case for most of our applications, although we will discuss an important counter-example later in the course. Since the particle coordinates depend on the q_i, we can also express the potential energy in terms of the generalized coordinates instead:

$$V(q) \quad \longrightarrow \quad V(q_1,\ldots,q_{3N-k})\,. \tag{21}$$

The generalized forces are extended via

$$Q = \vec{F}\cdot\frac{\partial\vec{r}}{\partial q} \quad \longrightarrow \quad Q_j = \sum_{i=1}^{N}\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\,. \tag{22}$$

For conservative forces, we have

$$\vec{F}_i = -\vec{\nabla}_i V(\vec{r}_1,\ldots,\vec{r}_N)\,, \tag{23}$$

where $\vec{\nabla}_i$ acts on the coordinates of particle i. Plugging this into the definition of the generalized force, we obtain

$$Q_j = -\sum_{i=1}^{N}\left(\vec{\nabla}_i V\right)\cdot\frac{\partial\vec{r}_i}{\partial q_j} = -\sum_{i=1}^{N}\sum_{k=1}^{3}\frac{\partial V}{\partial x_{ik}}\frac{\partial x_{ik}}{\partial q_j} = -\frac{\partial V}{\partial q_j}\,, \tag{24}$$

where we have written out the scalar product in components, and used the chain rule in the final step (noting again that V cannot depend on $\dot{q}_j$ or t).

Finally, we can state the many-particle version of d'Alembert's principle,

$$\sum_i\left(\vec{F}_i - \dot{\vec{p}}_i\right)\cdot\delta\vec{r}_i = 0\,, \tag{25}$$

as well as the Lagrange equations for each generalized coordinate:

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j = 0\,. \quad j = 1,\ldots,3N-k\,, \tag{26}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_j} - \frac{\partial L}{\partial q_j} = 0\,. \quad j = 1,\ldots,3N-k \tag{27}$$

Exercise 2.2: Form-Invariance of the Lagrange Equations

Let $q_1, \dots q_n$ be a set of independent generalized coordinates for a system of n degrees of freedom, with a Lagrangian $L(q, \dot{q}, t)$. Suppose we transform to another set of independent coordinates $s_1, \dots, s_n$ by means of transformation equations

$$q_i = q_i(s_1, \dots, s_n, t)\,, \qquad i = 1, \dots, n\,. \tag{E2.2-1}$$

Show that if the Lagrangian is expressed as a function of s_j, $\dot{s}_j$, and t through the equations of transformation, then L satisfies Lagrange's equations with respect to the s coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \frac{\partial L}{\partial s_j} = 0\,. \tag{E2.2-2}$$

In other words, the *form* of Lagrange's equations is invariant.

2.1.3 Examples

Let us now demonstrate the Lagrange formalism in action by working through some examples.

Bead on a Spiral Wire

In our first example, we revisit the problem of a bead on a spiral wire, which we used to derive D'Alembert's principle. We will, however, make one small alteration: Instead of starting from Cartesian coordinates, we will take the *symmetries* of the system into account and work in cylindrical coordinates $\{r, \phi, z\}$ instead (cf. worksheet #2).

With this coordinate choice, the constraints of the motion can be expressed as

$$\rho - a = 0\,, \tag{28}$$

$$z - b\phi = 0\,, \tag{29}$$

and the polar angle ϕ is the generalized coordinate. We recall that can let ϕ perform an arbitrary amount of revolutions, so that we can cover the full height of the spiral wire, i.e., the range of z coordinates the spiral wire encompasses.

The trajectory of the bead can be written as

$$\vec{r} = \rho\vec{e}_\rho + z\vec{e}_z = a\vec{e}_\rho + b\phi\vec{e}_z\,, \tag{30}$$

which leads to the following expression for the velocity:

$$\dot{\vec{r}} = a\dot{\vec{e}}_\rho + b\dot{\phi}\vec{e}_z = a\dot{\phi}\vec{e}_\phi + b\dot{\phi}\vec{e}_z\,. \tag{31}$$

We can easily compute the square of the velocity vector, exploiting the orthonormality of the unit vectors,

$$\left(a\dot{\phi}\vec{e}_\phi + b\dot{\phi}\vec{e}_z\right)^2 = (a^2 + b^2)\dot{\phi}^2\,. \tag{32}$$

In this way, we obtain the kinetic energy

$$T = \frac{1}{2}m(a^2 + b^2)\dot{\phi}^2\,. \tag{33}$$

The potential energy is given by

$$V = mgz = mgb\phi \,, \tag{34}$$

so our Lagrangian is

$$L = T - V = \frac{1}{2}m(a^2 + b^2)\dot{\phi}^2 - mgb\phi \,. \tag{35}$$

Next, we compute the Lagrangian's partial derivatives with respect to ϕ and ϕ,

$$\frac{\partial L}{\partial \phi} = -mgb \,, \tag{36}$$

$$\frac{\partial L}{\partial \dot{\phi}} = m(a^2 + b^2)\dot{\phi} \,, \tag{37}$$

and plugging these into the Lagrange equation, we obtain

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = m(a^2 + b^2)\ddot{\phi} + mgb = 0 \,. \tag{38}$$

The equation of motion for ϕ can be rearranged in the form

$$\ddot{\phi} = -\frac{gb}{a^2 + b^2} \,, \tag{39}$$

which is of course our result from earlier.

Block Sliding on a Gliding Wedge

As a second example, we consider a block of mass m sliding without friction on a wedge with inclination α that can itself glide on a frictionless plane (Fig. 1). We can consider the motion of block and wedge in two dimensions if neither of them starts spinning while it moves. The coordinates for the wedge are X, its distance from the origin in the horizontal plane, and

$$Z = \text{const.} \,, \tag{40}$$

which is defined by our choice of coordinate system and acts as a constraint of the motion. The coordinates of the block are

$$x = X + s\cos\alpha \,, z = h - s\sin\alpha \,, \tag{41}$$

where h is the height of the wedge. (We could have eliminated this constant by shifting the coordinate system in z direction — this will only give an offset to the potential energy that has no consequences for the dynamics.)

The time derivatives of the coordinates are

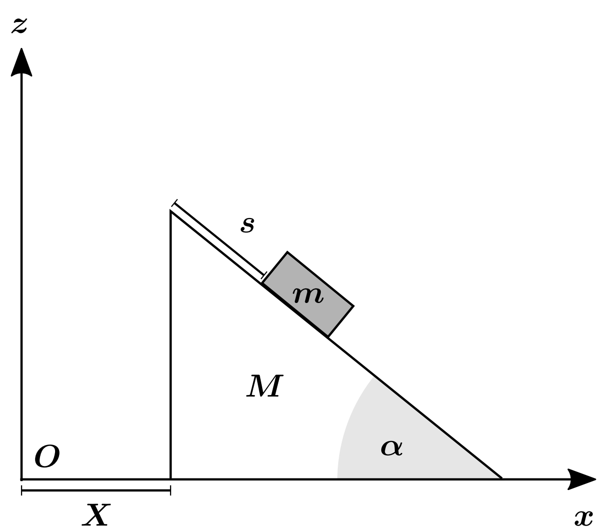

Figure 1: A block sliding without friction on a wedge that can itself glide on a frictionless plane.

$$\dot{x} = \dot{X} + \dot{s}\cos\alpha \,, \tag{42}$$

$$\dot{z} = -\dot{s}\sin\alpha\,, \tag{43}$$

so the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) \\ &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{X}^2 + \dot{s}^2\cos^2\alpha + 2\dot{X}\dot{s}\cos\alpha + \dot{s}^2\sin^2\alpha\right) \\ &= \frac{1}{2}\left(m+M\right)\dot{X}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{X}\dot{s}\cos\alpha\,. \end{aligned} \tag{44}$$

The potential energy is given by

$$V = mg\left(h - s\sin\alpha\right)\,, \tag{45}$$

so we obtain the Lagrangian

$$L = \frac{1}{2}\left(m+M\right)\dot{X}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{X}\dot{s}\cos\alpha - mg\left(h - s\sin\alpha\right)\,. \tag{46}$$

Now let us derive the Lagrange equations. We immediately notice that L does not explicitly depend on X, so we have

$$\frac{\partial L}{\partial X} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{X}}\,. \tag{47}$$

This means that

$$\frac{\partial L}{\partial \dot{X}} = (m+M)\dot{X} + m\dot{s}\cos\alpha \tag{48}$$

is a conserved quantity. It is easy to see that Eq. (48) is the total momentum in the horizontal direction, which is conserved because there is *no external force* acting on the system in x direction. (The internal forces between the block and the wedge cancel because of Newton's Third Law.)

The second Lagrange equation is obtained from

$$\frac{\partial L}{\partial s} = mg\sin\alpha\,, \tag{49}$$

$$\frac{\partial L}{\partial \dot{s}} = m\dot{s} + m\dot{X}\cos\alpha\,, \tag{50}$$

which yields

$$\ddot{s} + \ddot{X}\cos\alpha = g\sin\alpha \tag{51}$$

(s increases as the block slides down the slope). The conservation law (48) can be used to eliminate $\ddot{X}$:

$$(m+M)\ddot{X} = -m\ddot{s}\cos\alpha \quad \Rightarrow \quad \ddot{X} = -\frac{m}{m+M}\ddot{s}\cos\alpha\,, \tag{52}$$

so

$$\ddot{s} - \frac{m}{m+M}\ddot{s}\cos^2\alpha = g\sin\alpha\,. \tag{53}$$

Rearranging, we obtain the equation of motion

$$\ddot{s} = \frac{(m+M)\sin\alpha}{m\sin^2\alpha + M}g\,. \tag{54}$$

Box 2.1: Recipe for Solving Problems in Lagrangian Mechanics

As we have seen from our discussion of the examples in Sec. 2.1.3, the general procedure for solving problems in Lagrangian mechanics consists of the following steps:

1. Choose **convenient coordinates** for your problem, e.g., by exploiting **symmetries**.
2. Formulate the **constraints**.
3. Construct the **Lagrangian**.
4. Use the **Lagrange equations** to derive the **equations of motion** and identify **conserved quantities**.
5. Solve the equations of motion, and **analyze your solutions**.

2.2 The Principle of Least Action

2.2.1 Elements of Variational Calculus

Varying the Functional of a Curve

The calculus of variations aims to determine the function $y(x)$ for which the integral

$$\mathcal{I}[y] \equiv \int_{x_1}^{x_2} dx\, f\left(x, y(x), y'(x)\right) \tag{55}$$

becomes stationary, $\delta I = 0$. The integral $\mathcal{I}[y]$ is also referred to as a **functional** on the space of curves $y(x)$ that are compatible with the boundary conditions, i.e., that have the same values at x_1 and x_2.

Let us assume we already know the solution. We can define variations of this curve in the vicinity of the solution by defining

$$y(x,\epsilon) = y(x,0) + \epsilon\eta(x), \qquad \epsilon \ll 1, \tag{56}$$

where we choose an auxiliary function $\eta(x)$ that is twice continuously derivable, to avoid singularities and general pathological behavior. We also demand that

$$\eta(x_1) = \eta(x_2) = 0\,, \tag{57}$$

so that the boundary conditions are automatically satisfied.

In this way, $\mathcal{I}[y]$ becomes a function of ϵ,

$$\mathcal{I}(\epsilon) = \int_{x_1}^{x_2} dx f\left(x, y(x,\epsilon), y'(x,\epsilon)\right)\,, \tag{58}$$

and since $y(x,0)$ is supposed to make the functional stationary, we can perform a Taylor expansion around $\epsilon = 0$:

$$\mathcal{I}(\epsilon) = \int_{x_1}^{x_2} dx \left(f(x, y(x,0), y'(x,0)) + \epsilon \left.\frac{\partial f}{\partial y}\right|_{\epsilon=0} \eta(x) + \epsilon \left.\frac{\partial f}{\partial y'}\right|_{\epsilon=0} \eta'(x) + O(\epsilon^2) \right)\,. \tag{59}$$

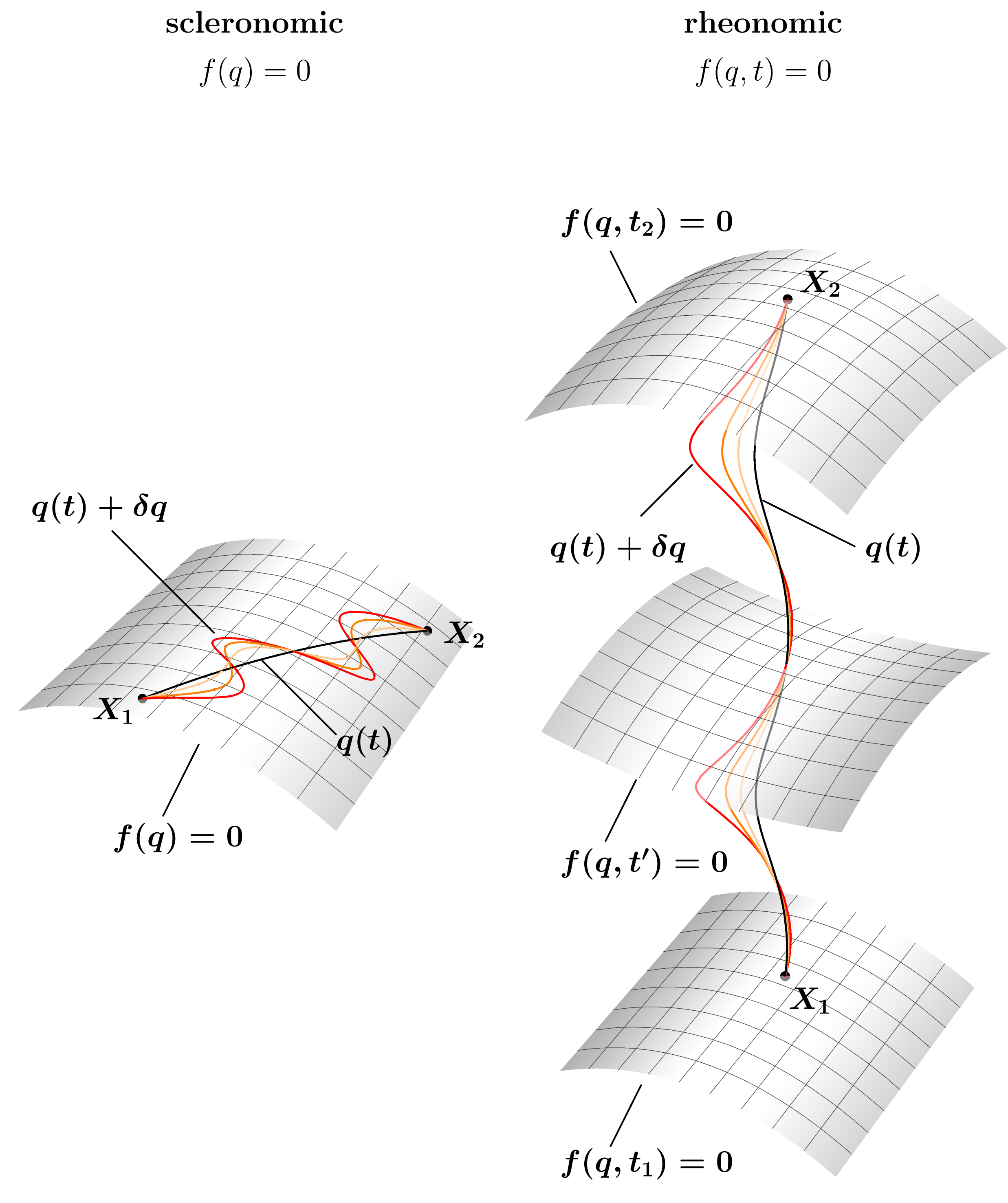

Figure 2: Variation of trajectories with fixed endpoints in configuration manifolds defined by holonomic constrains. We continue to use the notation $q = (q_1, \ldots, q_n)$, $\delta q = (\delta q_1, \ldots, \delta q_n)$ for *points* (not vectors!) in the configuration manifold.

The stationarity condition $\delta\mathcal{I} = 0$ implies that

$$0 = \frac{d\mathcal{I}(\epsilon)}{d\epsilon} = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y}\bigg|_{\epsilon=0} \eta(x) + \frac{\partial f}{\partial y'}\bigg|_{\epsilon=0} \eta'(x) \right) . \tag{60}$$

The second term in the integrand can be rewritten using integration by parts, leading to

$$\begin{aligned} 0 &= \frac{\partial f}{\partial y'}\eta(x)\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} + \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \eta(x) \\ &= \int_{x_1}^{x_2} dx \left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} \right) \eta(x) \,, \end{aligned} \tag{61}$$

where we have used that the first term (sometimes referred to as the *boundary term*) vanishes at the boundaries, i.e., the starting and end points of the curve, because of the condition (57). Since Eq. (61) must hold for arbitrary $\eta(x)$, the expression in the parenthesis must vanish[1], and we obtain the **Euler-Lagrange equation**

$$\boxed{\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \;,} \tag{62}$$

This is both a *necessary and sufficient* condition that a curve $y(x)$ must satify in order to make $\mathcal{I}[y]$ stationary. The left-hand side of the Euler-Lagrange equation can also be used to define the **functional derivative**

$$\frac{\delta\mathcal{I}}{\delta y} \equiv \frac{d}{dx}\frac{\partial f}{\partial y} - \frac{\partial f}{\partial y} \,. \tag{63}$$

Euler-Lagrange Equations for Multiple Degrees of Freedom and Variables

The extension of the Euler-Lagrange equations to multiple variables — i.e., multiple particles and degrees of freedom — is straightforward. A general curve will be characterized by the values of all coordinates $y(x) = (y_1(x), \ldots, y_n(x))$ as a function of the variable x that is used to parameterize it, and the functional generalizes to

$$\mathcal{I}[y] = \int_1^2 dx\, f(x, y_1, \ldots, y_n, y_1', \ldots, y_n') \,. \tag{64}$$

Variations of a curve that makes $\mathcal{I}$ stationary are written as

$$y_i(x, \epsilon_i) = y_i(x) + \epsilon_i \eta_i(x) \equiv y_i(x, 0) + \delta y_i(x, \epsilon) \,, \tag{65}$$

The δy_i must vanish at the start and end points of the curves, i.e.,

$$\delta y_i(x_1) = \delta y_i(x_2) = 0 \,. \tag{66}$$

The stationarity condition can be expressed for independent (but infinitesimal) variations by introducing $\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)^T$, and we obtain (cf. Eq. 59):

$$0 = \delta\mathcal{I} = \nabla\mathcal{I}(\vec{\epsilon}) \cdot \vec{\epsilon} = \int_{x_1}^{x_2} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i}\epsilon_i \eta_i + \frac{\partial f}{\partial y_i'}\epsilon_i \eta_i' \right) = \int_{x_1}^{x_2} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i}\delta y_i + \frac{\partial f}{\partial y_i'}\delta y_i' \right) \;, \tag{67}$$

[1] This is properly proven in the so-called fundamental lemma of the calculus of variations.

where $\delta y_i' = \epsilon_i \eta_i'$. Partially integrating as in the one-dimensional case, we get

$$0 = \int_{x_1}^{x_2} dx \sum_{i=1}^{n} \left(\frac{d}{dx}\frac{\partial f}{\partial y_i'} - \frac{\partial f}{\partial y_i} \right) \delta y_i \,, \tag{68}$$

and since the δy_i are independent, *all* parentheses must vanish separately, leading to the **Euler-Lagrange equations**

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0 \quad , \qquad i = 1, \ldots, n \ .} \tag{69}$$

2.2.2 Examples

Brachistochrone

We parameterize the trajectory using the arc length s, which is the variable of choice if we are interested in the shape of a curve. Since $ds = vdt$, the time required to move from the start to the end of the curve is given by the functional

$$\mathcal{T} = \int_{t_1}^{t_2} dt = \int_1^2 ds\, \frac{1}{v} \ . \tag{70}$$

Energy conservation implies

$$E = \frac{mv^2}{2} + mgy = mgy_1 \ , \tag{71}$$

so we can solve for v and obtain

$$v = \sqrt{2g(y_1 - y)}\,. \tag{72}$$

The differential can be rewritten as

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + y'(x)^2}, \qquad y' \equiv \frac{dy}{dx}\,, \tag{73}$$

and plugging in our expression for the velocity, the functional $\mathcal{T}$ becomes

$$\mathcal{T} = \int_1^2 ds \frac{1}{v} = \int_{x_1}^{x_2} dx \frac{\sqrt{1 + y'(x)^2)}}{\sqrt{2g(y_1 - y(x))}} \ . \tag{74}$$

We need to find the trajectory that minimizes this integral, so we set

$$f(x, y, y') = \sqrt{\frac{1 + y'^2}{2g(y_1 - y)}}\,. \tag{75}$$

First, we note that f does not explicitly depend on x, which implies the so-called **Beltrami identity**:

$$f - \frac{\partial f}{\partial y'} y' = \text{const.}\,. \tag{76}$$

The proof is straightforward:

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial y'} y' \right) = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' - \frac{\partial f}{\partial y'} y'' - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) y'$$

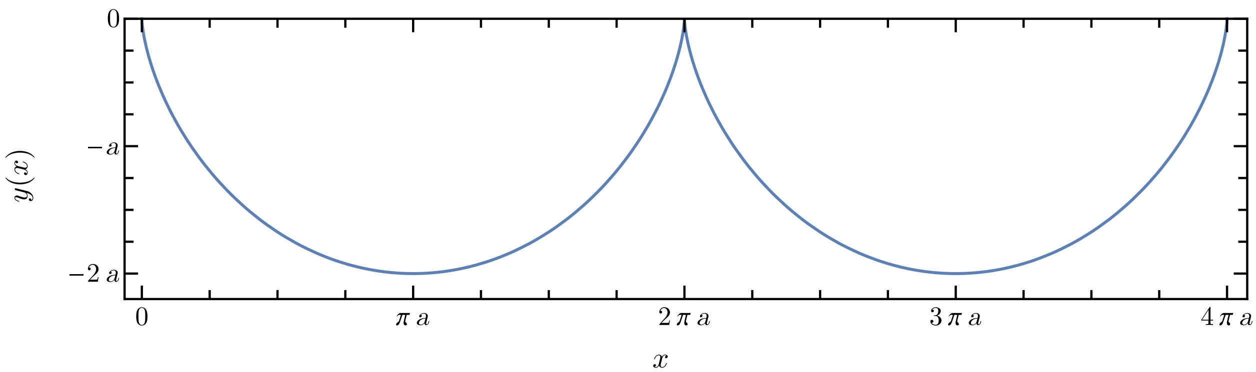

Figure 3: The cycloid defined by Eqs. (80), (81).

$$= \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) y' = 0\,. \tag{77}$$

Using the relation (76), we have

$$\begin{aligned} \text{const.} &= \sqrt{2g}\left(f - \frac{\partial f}{\partial y'}y'\right) \\ &= \sqrt{\frac{1+y'^2}{y_1 - y}} - \frac{y'^2}{\sqrt{y_1-y}\sqrt{1+y'^2}} \\ &= \frac{1}{\sqrt{y_1-y}\sqrt{1+y'^2}}\underbrace{\left(1+y'^2-y'^2\right)}_{=1} \end{aligned} \tag{78}$$

Squaring both sides, the equation can be rearranged as

$$(y_1 - y)\left(1+y'^2\right) = \text{const.} \tag{79}$$

This differential equation is solved by the following *cycloid* trajectory (see Fig. 3):

$$x = a(t - \sin t) \tag{80}$$
$$y = a(\cos t - 1)\,. \tag{81}$$

We can show that the cycloid satisfies Eq. (79) by plugging in Eqs. (80) and (81):

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-a\sin t}{a(1-\cos t)}\,, \tag{82}$$

hence

$$1 + y'^2 = 1 + \frac{a^2\sin^2 t}{a^2(1-\cos t)^2} = \frac{a^2(1 - 2\cos t + \cos^2 t + \sin^2 t)}{y^2} \tag{83}$$
$$= \frac{2a^2}{y^2}(1-\cos t) = -\frac{2a}{y}\,. \tag{84}$$

Noting that $y_1 = 0$, Eq. (79) now reads

$$(y_1 - y)\left(1+y'^2\right) = (-y)\left(\frac{-2a}{y}\right) = 2a = \text{const.}\,, \tag{85}$$

as required.

Shortest Line Connecting Two Points

We again start from the line element $ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + y'^2}$, which defines

$$f = \sqrt{1 + y'^2} \; . \tag{86}$$

Plugging this into the Euler-Lagrange equation (62) yields

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0 \quad \Longrightarrow \quad \frac{y'}{\sqrt{1+y'^2}} = c \; . \tag{87}$$

We square both sides and rearrange the equation, obtaining

$$y'^2 = \frac{c^2}{1-c^2} \quad \Rightarrow \quad y' = \sqrt{\frac{c^2}{1-c^2}} = \text{const.} \tag{88}$$

This implies

$$y(x) = ax + b \, , \tag{89}$$

where $a = c/\sqrt{1-c^2}$ and b is a constant obtained upon integration of the differential equation. Thus, the shortest trajectory connecting two points in a plane is a line. The same procedure can be used to compute the shortest connections — the so-called **geodesics** — between two points in arbitray smooth manifolds.

2.2.3 The Principle of Least Action for Mechanical Systems

We can now apply the variational discussed in the previous sections to the *action functional*

$$S = \int_{t_1}^{t_2} dt \, L(q, \dot{q}, t) \, , \tag{90}$$

where $q = (q_1, \ldots, q_n)$, $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_n)$, as usual. The *extrema* of the action functional are determined by finding the zeroes of the functional derivatives,

$$\frac{\delta S}{\delta q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \, . \tag{91}$$

In practice, we are interested in the solutions that minimize the action, which gives rise to **Hamilton's principle**, which is a way of stating the **Principle of Least Action**, or more accurately, the **Principle of Stationary Action**. For **holonomic** systems, it provides us with a formulation of dynamical laws of motion that is equivalent to D'Alembert's principle. However, as stated above, the latter is more general because it can accommodate generalized forces associated with dissipation or nonholonomic constraints. In recent years, there has been renewed interest to extend Hamilton's principle to such forces, in part driven by the needs of modern robotics research[2] [1, 2, 3, 4, 5].

From the principle of least action we immediately see that the dynamics of a holonomic mechanical system remain invariant under the addition of a total time derivative to the Lagrangian: If we introduce

$$\widetilde{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF(q,t)}{dt} \, , \tag{92}$$

the corresponding action functional reads

$$\widetilde{S} = S + \int_{t_1}^{t_2} dt \, \frac{dF(q,t)}{dt} = S + F_2 - F_1 \, , \tag{93}$$

[2] These research papers can be found on the coruse website and in the course repository.

where F_1, F_2 are constants. Thus, the variation of the action $\widetilde{S}$ is identical to that of the original action S:

$$\delta\widetilde{S} = \delta S + \underbrace{\delta\left(F_2 - F_1\right)}_{=0} . \tag{94}$$

This means that the Lagrange equations of a holonomic system will remain invariant as well[3].

References

[1] M. R. Flannery, American Journal of Physics **73**, 265 (2005).

[2] M. R. Flannery, American Journal of Physics **79**, 932 (2011).

[3] M. R. Flannery, Journal of Mathematical Physics **52**, 032705 (2011).

[4] C. R. Galley, Phys. Rev. Lett. **110**, 174301 (2013).

[5] C. R. Galley, D. Tsang, and L. C. Stein, (2014), 1412.3082.

[3] In a homework problem, this invariance was proven directly using the Lagrange equations. This is actually a stronger statement, because it allows the extension to systems with nonholonomic constraints.

3 Group Exercises

Problem G6 – Lagrangian Treatment of an Atwood Machine

[**cf. Lemos, example 1.15**] Consider an Atwood machine consisting of masses m_1 and m_2, as shown in the figure.

1. Formulate the constraint that links the coordinates x_1 and x_2

   Hint: It is useful to start from a nonholonomic form that relates the changes in the coordinates, and integrate it to obtain a holonomic constraint that can be used to eliminate one of them.

2. Construct the Lagrangian.

3. Derive the Lagrange equations.

4. State the general solutions of the equations of motion for x_1 and x_2.

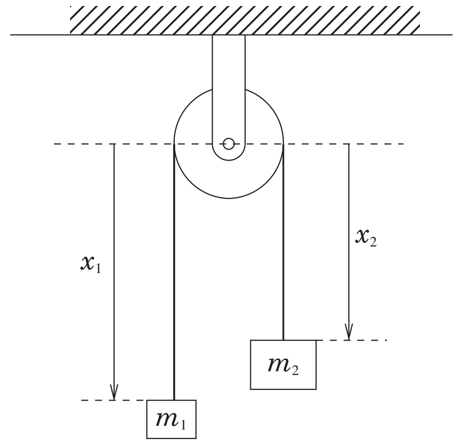

Figure 4: An Atwood machine.

Problem G7 – Geodesics on a Cylinder

Use variational calculus to show that the geodesics on a cylinder of radius R, i.e., the shortest paths between points $P_1 = (\phi_1, z_1)$ and $P_2(\phi_2, z_2)$ are *helices* (spiral trajectories with constant radius) of the form

$$z(\phi) = a\phi + b\,, \tag{95}$$

where the constants a, b are determined by the boundary conditions (i.e., the starting and end points of the curve).

Problem G8 – Geodesics on a Sphere

Determine the geodesics on a sphere of constant radius R.

1. Show that the geodesic on the sphere is given by the equation

   $$\cos(\phi - \phi_0) = k \cot\theta\,, \tag{96}$$

   where the constants k and ϕ_0 are determined by the boundary conditions, i.e., the points we are connecting.

   Hint: Use θ as the curve parameter in your functional (Why?), and derive a first-order differential equation for $\phi(\theta)$ from the Euler-Lagrange equation. To integrate it, you will find the substitution $u = a\cot\theta$ useful. What does this substitution imply for $\sin\theta$?

   $$-\int dx \frac{1}{\sqrt{b^2 - x^2}} = \arccos\frac{x}{b} + c \tag{97}$$

2. Show that Eq. (96) defines an arc on a *great circle* on the sphere.

   HINT: Great circles are the intersections of the sphere with a plane through the sphere's origin, parameterized by

$$Ax + By + Cz = 0\,. \tag{98}$$

   You will find the ansätze

$$\cos\phi_0 = \frac{A}{\sqrt{A^2+B^2}}\,, \quad \sin\phi_0 = \frac{B}{\sqrt{A^2+B^2}}\,, \tag{99}$$

   useful.