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PHY422/820: Classical Mechanics

FS 2021

Worksheet #6 (Oct 4 – Oct 8)

September 30, 2021

1 Plan for the Week

- **Midterm #1** on **Oct 7/8**
- Finish discussion of dissipation (cf. worksheet #5).
- A brief discussion of nonstandard Lagrangians.
- Recap and Q&A.

2 Nonstandard Lagrangians

In our applications of variational calculus, we have constructed a Lagrangian and derived equations of motion that yield the extrema of the associated functional, action or otherwise. The so-called **inverse problem of variational calculus** aims to reverse-engineer a Lagrangian that will reproduce a given set of known equations of motion (see, e.g., Ref. [1]). You can find several examples in the textbook exercises.

Example: Dissipative Systems

Using inverse-problem techniques, various authors have constructed nonstandard Lagrangians for dissipative systems. Here we want to consider projectile motion under a linear drag force (cf. worksheet #5), using a combination of a standard Lagrangian and a dissipation force,

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mgy\,, \quad D = \frac{1}{2}\beta\left(\dot{x}^2 + \dot{y}^2\right)\,, \tag{1}$$

and the nonstandard Lagrangian

$$L' = e^{\beta t/m}\left[\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mgy\right]\,. \tag{2}$$

For the combination of L and D, we obtain

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = -\frac{\partial D}{\partial \dot{x}} \qquad \Rightarrow \quad m\ddot{x} = -\beta\dot{x}\,, \tag{3}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = -\frac{\partial D}{\partial \dot{y}} \qquad \Rightarrow \quad m\ddot{y} - mg = -\beta\dot{y}\,. \tag{4}$$

Starting from the nonstandard Lagrangian, we have

$$\begin{aligned}\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} - \frac{\partial L'}{\partial x} &= \frac{d}{dt}\left(e^{\beta t/m} m\dot{x}\right) = e^{\beta t/m}\frac{\beta}{m} m\dot{x} + e^{\beta t/m} m\ddot{x} \\ &= e^{\beta t/m}\left(m\ddot{x} + \beta\dot{x}\right) = 0\,, \end{aligned} \tag{5}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial L'}{\partial \dot{y}} - \frac{\partial L'}{\partial y} &= \frac{d}{dt}\left(e^{\beta t/m} m\dot{y}\right) - e^{\beta t/m} mg \\ &= e^{\beta t/m}\left(m\ddot{y} + \beta\dot{y} - mg\right) = 0\,, \end{aligned} \tag{6}$$

so we obtain the same equations of motion.

Note that Eq. (5) implies that the canonical momentum

$$p_{0x} \equiv e^{\beta t/m} m\dot{x} \tag{7}$$

is a constant. Clearly, this is *not* the mechanical momentum, so let us try and interpret it. Rearranging the equation, we obtain the mass' velocity in x direction,

$$\dot{x} = \frac{p_{0x}}{m} e^{-\beta t/m} \equiv v_{0x} e^{-\beta t/m}\,, \tag{8}$$

which is decaying exponentially in time due to the action of the drag force, as expected. Thus, we see that p_{0x} is nothing but the initial momentum of the mass in x direction. While it is constant, it merely characterizes the initial conditions of the system, and does not contain useful information about the state of the system at times $t > 0$. This is inherently different from the conserved quantities like the total energy or conserved (angular) momenta, which *are* characterizing the system at *all* times.

References

[1] J. Douglas, Trans. Amer. Math. Soc. **50**, 71 (1941).

3 Group Exercises

Problem G14 – The Cycloidal Pendulum

An ideal cycloidal pendulum consists of a mass that oscillates under gravity along a frictionless cycloidal track that is parameterized by the following expressions:

$$x = R(\theta - \sin\theta)\,, \quad y = R(1 - \cos\theta)\,, \tag{9}$$

where the vertical y-axis points downward.

1. Show that the Lagrangian for this system is given by

$$L = 2mR^2\dot{\theta}^2 \sin^2\left(\frac{\theta}{2}\right) + mgR(1 - \cos\theta)\,. \tag{10}$$

   HINT:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

2. Make a point transformation to the new generalized coordinate $u = \cos\left(\frac{\theta}{2}\right)$ and derive the Lagrangian in u.

3. Derive the Lagrange equations and show that the period of oscillation is

$$\mathcal{T} = 4\pi\sqrt{\frac{R}{g}}\,, \tag{11}$$

   independent of the amplitude. C. Huygens recognized this property of the cycloid in 1659 in his attempt to come up with an improved design for a pendulum clock.

A Jupyter notebook (`w06_cycloidal_pendulum.ipynb`) that visualizes the oscillations of a cycloidal pendulum as a function of the amplitude has been posted to the repository and the course website.

Problem G15 – Solving the Dynamics Using Constants of the Motion

[**cf. Lemos, problem 2.23**] The Lagrangian for a one-dimensional mechanical system is

$$L = \frac{1}{2}\dot{x}^2 - \frac{g}{x^2}\,, \tag{12}$$

where g is a constant.

1. Show that the action is invariant under the finite transformations

$$x'(t') = e^{\alpha}x(t)\,, \quad t' = e^{2\alpha}t\,, \tag{13}$$

   where α is a constant. Use Noether's theorem to conclude that

$$I = x\dot{x} - 2Et \tag{14}$$

   is a constant of the motion, where E is the total energy.

2. Show that the action is *quasi-invariant* (i.e., invariant up to the addition of a total time derivative $\dot{F}$ to the Lagrangian) under the infinitesimal transformation

$$x'(t') = x(t) - \epsilon t x(t)\,, \quad t' = t + \epsilon t^2\,. \tag{15}$$

   Use the equation of motion to prove that

$$F = \frac{1}{2}x^2 - 2tx\dot{x}\,, \tag{16}$$

   and conclude that

$$K = Et^2 - tx\dot{x} + \frac{1}{2}x^2 \tag{17}$$

   is a constant of the motion.

3. Combine your previous results to find the solution $x(t)$ by purely algebraic means (i.e., without solving differential or integral equations).