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PHY422/820: Classical Mechanics

FS 2021

Worksheet #8 (Oct 18 – Oct 22)

October 20, 2021

1 Preparation

- Goldstein, Sections 3.7–3.11

2 The Kepler Problem

2.1 Determination of the Trajectories

Let us consider a central potential of the form

$$V(r) = -\frac{k}{r} \tag{1}$$

with a (not necessarily positive) constant k. The effective potential is given by

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} - \frac{k}{r} \tag{2}$$

and we can perform the general classification of orbits based on its properties. In fact, we used the attractive case $k > 0$ as the primary example on worksheet #7, so we can refer to these results. In the repulsive case $k < 0$, we will obviously not have any bound orbits, but only scattering trajectories.

Circular Trajectory, Minimal Energy and Turning Points

For further use, we determine the circular trajectories and turning points, if they exist. From

$$V'_{\text{eff}}(r) = -\frac{l^2}{mr^3} + \frac{k}{r^2} \stackrel{!}{=} 0\,, \tag{3}$$

we find

$$\frac{k}{R^2} = \frac{l^2}{mR^3} \quad \Rightarrow \quad R = \frac{l^2}{mk}\,. \tag{4}$$

Since radii must always be positive, we only find a physical solution in the attractive case $k > 0$; no circular trajectories are possible if the potential is repulsive. The energy of the circular trajectory is given by

$$E_{\min} = V_{\text{eff}}(R) = \frac{l^2}{2mR^2} - \frac{k}{R} = \frac{l^2m^2k^2}{2ml^4} - \frac{mk^2}{l^2} = -\frac{mk^2}{2l^2}\,, \tag{5}$$

and since $V_{\text{eff}}(r) \to 0$ for $r \to \infty$, we see that we will have bound orbits for $E_{\text{min}} \leq E < 0$. For non-circular bound orbits, the turning points can be determined by solving

$$E = V_{\text{eff}}(r) \quad \Rightarrow \quad Er^2 + kr - \frac{l^2}{2m} = 0\,. \tag{6}$$

We find

$$r_{\mp} = \frac{-k \pm \sqrt{k^2 + \frac{2El^2}{m}}}{2E} = -\frac{k}{2E}\left(1 \mp \sqrt{1 + \frac{2El^2}{mk^2}}\right) = \frac{k}{2|E|}\left(1 \mp \sqrt{1 - \frac{2|E|l^2}{mk^2}}\right)\,, \tag{7}$$

where we have made the signs more explicit in the final step. Here, r_- and r_+ are the minimal and maximal distances from the center of the potential, respectively. If we plug in $E = E_{\text{min}}$ from above, we see that the discriminant vanishes, and we obtain $r_+ = r_- = R$.

General Solution

As discussed previously, the general trajectory for an object of mass m in a central potential can be obtained from

$$\phi - \phi_0 = \pm\frac{l}{\sqrt{2m}} \int dr \frac{1}{r^2\sqrt{E - V_{\text{eff}}(r)}}\,, \tag{8}$$

where any integration constants from the right-hand side have been absorbed into ϕ_0. Let us now plug in $V_{\text{eff}}(r)$:

$$\phi - \phi_0 = \pm\frac{l}{\sqrt{2m}} \int dr \frac{1}{r^2\sqrt{E - \frac{l^2}{2mr^2} + \frac{k}{r}}}\,, \tag{9}$$

To solve this integral, we make the substitution

$$u = \frac{1}{r}\,, \quad du = -\frac{1}{r^2}dr\,, \quad dr = -\frac{1}{u^2}du\,, \tag{10}$$

which brings the integral to the form

$$\phi - \phi_0 = \mp \int du \frac{1}{\sqrt{\frac{2mE}{l^2} + \frac{2mk}{l^2}u - u^2}}\,. \tag{11}$$

We can use

$$\int du \frac{1}{\sqrt{a + bu - u^2}} = -\arccos\frac{2u - b}{\sqrt{4a + b^2}} + c\,, \quad \text{if } 4a + b^2 > 0\,. \tag{12}$$

Identifying the constants, the condition $4a + b^2 > 0$ turns into

$$\frac{8mE}{l^2} + \frac{4m^2k^2}{l^4} > 0 \quad \Rightarrow \quad E > -\frac{mk^2}{2l^2} = E_{\text{min}}\,, \tag{13}$$

which is clearly satisfied. Carrying out the integration, we have

$$\begin{aligned} \phi - \phi_0 &= \pm\arccos\frac{2u - \frac{2mk}{l^2}}{\sqrt{\frac{8mE}{l^2} + \frac{4m^2k^2}{l^4}}} = \pm\arccos\frac{2u - \frac{2mk}{l^2}}{\frac{2mk}{l^2}\sqrt{\frac{2El^2}{mk^2} + 1}} \\ &= \pm\arccos\frac{\frac{l^2}{mk}u - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}\,. \end{aligned} \tag{14}$$

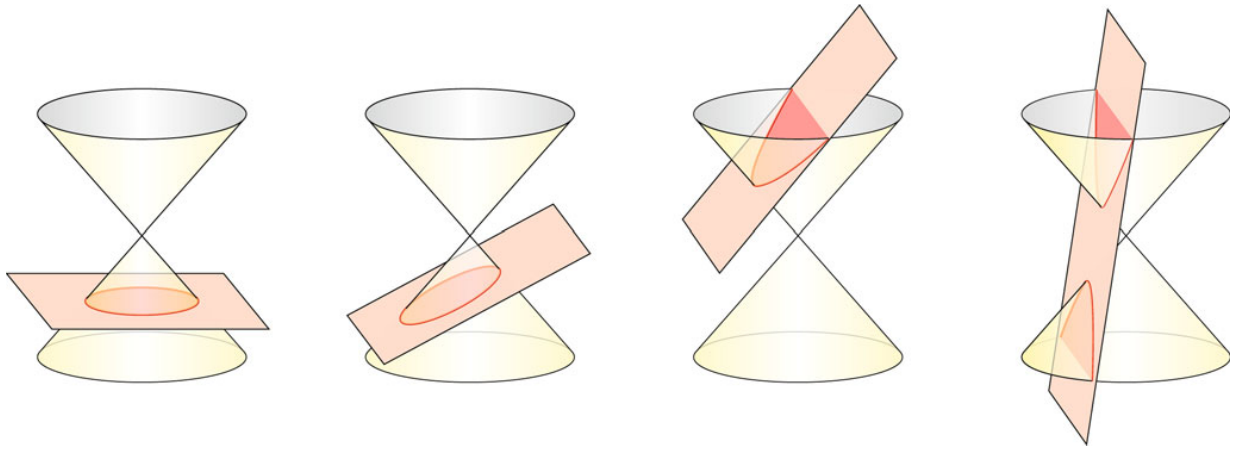

Figure 1: Trajectories of the Kepler problem corresponding to conic sections. We chose $\phi_0 = 0$ in Eq. (18).

Taking the cosine on both sides and using that its symmetry means that the overall sign of the right-hand side does not matter, we obtain

$$\cos(\phi - \phi_0) = \frac{\frac{l^2}{mk}u - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}} . \tag{15}$$

Introducing

$$\lambda \equiv \frac{l^2}{mk} , \quad \epsilon \equiv \sqrt{1 + \frac{2El^2}{mk^2}} , \tag{16}$$

and reverting back to our original variable r, we have

$$\cos(\phi - \phi_0) = \frac{\frac{\lambda}{r} - 1}{\epsilon} , \tag{17}$$

and therefore (cf. homework problem H11):

$$\boxed{r(\phi) = \frac{\lambda}{1 + \epsilon \cos(\phi - \phi_0)} .} \tag{18}$$

The trajectories (18) are so-called **conic sections**, because they can be obtained by considering the intersections of a cone with a plane, as shown in Fig. 1:

- For $\epsilon = 0$, the intersecting plane is parallel to the top or bottom of the cone, hence the intersection curve of the plane with the cone's mantle is a circle.
- For $0 < \epsilon < 1$, the intersection occurs at an angle and yields an ellipse.
- For $\epsilon = 1$, the angle at which the plane intersects with the cone is equal to the opening angle of the cone, and the intersecution curve is a parabola.
- Last but not least, $\epsilon > 1$ yields an hyperbola, with the two distinct branches corresponding to the solution for an attractive or repulsive potential (see below.)

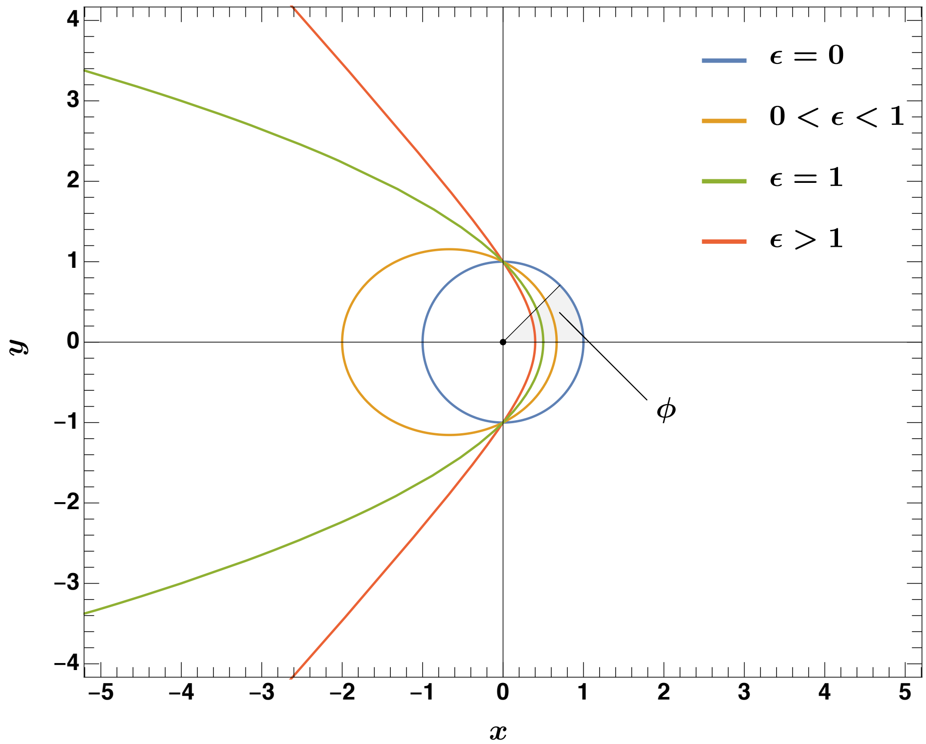

Figure 2: Trajectories for the force field $f(r) = -\frac{k}{r^2}$ for different values of ϵ.

In Fig. 2, we show the different classes of trajectories in a coordinate system whose center is in the focus of the curve. We have chosen $\phi_0 = 0$ so that $\phi = 0$ is the axis through the origin and the so-called **pericenter** or **periapsis**, which is the point of the objects closest approach to the center of the potential. Other choices merely correspond to a rotation of the coordinate system around the axis defined by the conserved angular momentum (usually chosen to be the z axis of the coordinate system).

Let us consider the geometry of the elliptical orbit, shown in Fig. 3, in a bit more detail. We can relate its characteristic lengths to the turning points of the motion (Eq. (7) as well as the parameters λ and ϵ we introduced above (see Eq. (16)):

- The axis through the points of smallest and greatest distance from the center is aligned with the large major axis of the ellipse. We clearly have

$$2a = r_+ + r_- \,, \tag{19}$$

  where a is the semi-major axis of the ellipse. Plugging in our expressions for the turning points, we have

$$a = \frac{1}{2}\left(r_+ + r_-\right) = -\frac{k}{2E} = \frac{k}{2|E|} \tag{20}$$

  since $E < 0$ for bound orbits.

- The points of smallest and largest distance are the aforementioned **periapsis** and the **apoapsis** of the orbit, respectively[1], for which we will also use the notation

$$r_p = r_- \,, \quad r_a = r_+ \,. \tag{21}$$

[1] From Greek *apsis*, pl. apsides, meaning "orbit", and the prefixes *peri-* and *ap(o)-* meaning "near" and "far".

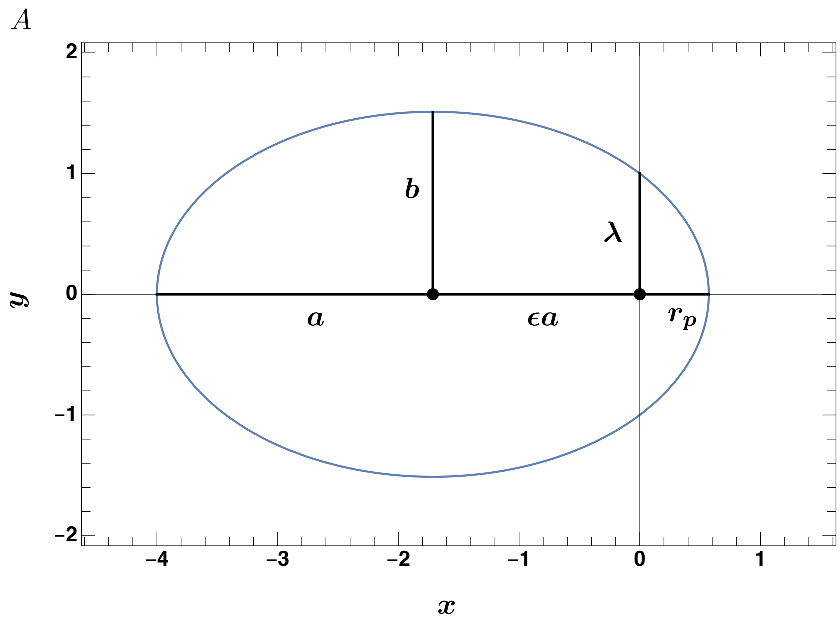

Figure 3: Definition of lengths in an ellipse.

For Earth orbits, one specifically uses the terms **perigee** and **apogee**, and similar terminology is used for the other planets of the solar system, albeit less frequently. For orbits around the Sun, we refer to the **perihelion** and **aphelion**, and for other solar systems to **periastron** and **apastron**.

- ϵ is the **eccentricity**[2] of the ellipse. Geometrically,

$$\epsilon = \sqrt{1 - \frac{b^2}{a^2}} \,. \tag{22}$$

It parameterizes the deviation of the ellipse from a circle, which is obtained in the limit $a = b = R$, meaning $\epsilon = 0$. For proper ellipses $b < a$ and we have $0 < \epsilon < 1$.

The length ϵa is the **linear eccentricity**. We see that

$$\epsilon^2 a^2 = a^2 - b^2 \,, \tag{23}$$

and using Eqs. (16) and (22), we see that the semi-minor axis is

$$b^2 = a^2(1 - \epsilon^2) = \frac{k^2}{4E^2}\left(-\frac{2El}{mk^2}\right) = \frac{l^2}{2|E|m} = a\lambda \,. \tag{24}$$

- λ is the so-called **semi-latus rectum** of the ellipse, which intersects with the major axis in the focus and forms a right-angled triangle with the periapsis. It can be expressed in terms of the semi-major and semi-minor axes as

$$\lambda = \frac{b^2}{a} = a(1 - \epsilon^2) = -\frac{k}{2E}\left(-\frac{2El^2}{mk^2}\right) = \frac{l^2}{mk^2} \,. \tag{25}$$

which is consistent with Eqs. (24) and (16).

Looking over the discussion, we see that the shape of the ellipse is completely determined by the parameters of the particle and the potential, i.e., m and k, as well as the conserved quantities E and l.

[2] From Greek *ekkentros*, meaning out of center, adopted in Latin as "eccentricus".

2.2 Kepler's Laws

Using the general solution for bound trajectories, we can readily obtain **Kepler's laws**, which we state in a more general fashion:

1. *The orbit of an object is a circle or an ellipse with the center of the gravitational potential at one of the foci.*

   This is immediately clear form Eq. (18) for $0 \leq \epsilon < 1$. For a gravitational two-body problem, we simply replace m by the reduced mass μ in all relevant quantities, and consider the center-of-mass of the two-body system as the origin of the gravitational potential.

2. *A line segment joining the orbiting object to the center of the gravitational potential sweeps out equal areas during equal intervals of time.*

   Mathematically, we can express this law as

$$\frac{dA}{dt} = \frac{l}{2m} = \text{const.}\,, \tag{26}$$

   which immediately reveals it to be a direct result of angular momentum conservation (see group exercise G16).

3. *The square of the orbiting object's orbital period — defined as the time between successive passings of the periapsis — is proportional to the cube of the length of the semi-major axis of its orbit.*

   Rearranging Eq. (26), we have

$$dt = \frac{2m}{l} dA\,. \tag{27}$$

   Over a full period T, the line segment from the origin to the object's position sweeps out an ellipse of area $A = \pi ab$. Using the relations between the principal axes and the physical quantities we found in the previous section, we obtain

$$T = \frac{2m}{l}\pi ab = \frac{2m}{l}\pi a\sqrt{a\lambda} = \frac{2m}{l}\pi\sqrt{\frac{a^3 l^2}{mk}} = 2\pi\sqrt{\frac{a^3 m}{k}}\,, \tag{28}$$

   or when squared

$$T^2 = \frac{4\pi^2 m}{k}\,a^3\,. \tag{29}$$

2.3 The Laplace-Runge-Lenz Vector

2.3.1 General Considerations

As we have seen in exercise G18, the **Laplace-Runge-Lenz vector**

$$\vec{A} = \frac{\vec{p}\times\vec{l}}{mk} - \vec{e}_r \tag{30}$$

is also a constant of the motion for the Kepler problem. Together with E and $\vec{l}$, it can be used to determine the trajectories (18) without integrating the equation of motion, simply by evaluating the scalar products $\vec{l}\cdot\vec{A}$ and $\vec{r}\cdot\vec{A}$ (cf. worksheet #7). From this calculation, we obtain

$$r(\phi) = \frac{l^2/mk}{1 + |\vec{A}|\cos\phi}\,, \tag{31}$$

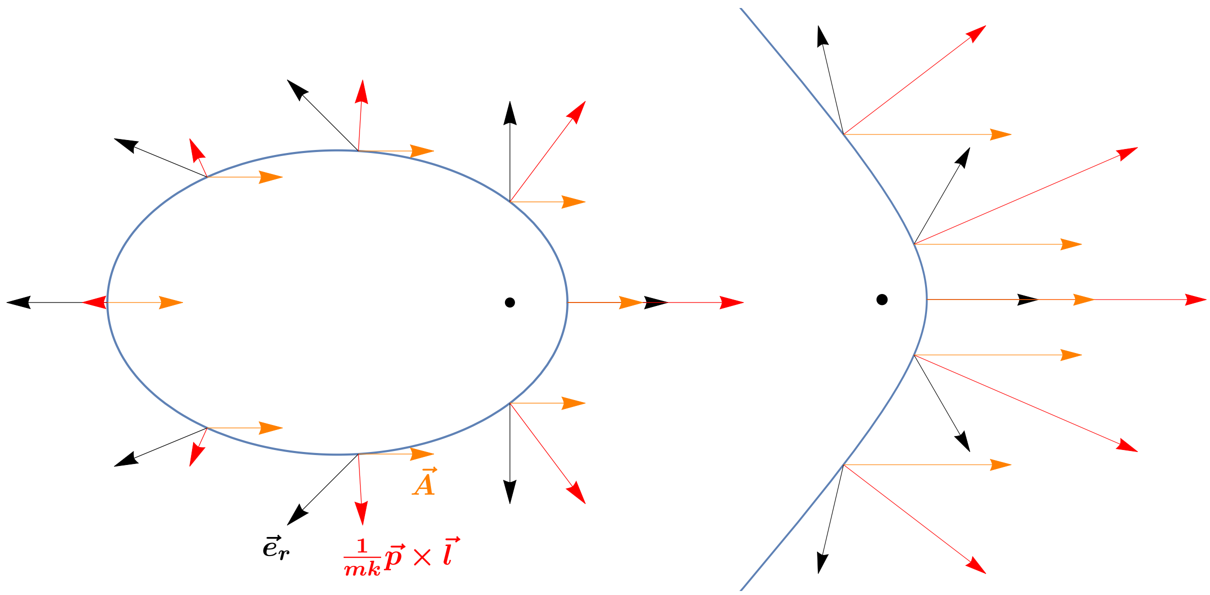

Figure 4: The Laplace-Runge-Lenz vector $\vec{A}$. The color coding of the various vectors is the same in each point of the ellipse and hyperbola, respectively.

which shows that the magnitude of $\vec{A}$ is identical to the eccentricity ϵ in Eq. (18). Comparing Eqs. (18) and (31), we see that the angle ϕ between $\vec{r}$ and $\vec{A}$ is identical to the angle between the axis through the focal points and the periapsis. As we can see in Fig. 4, $\vec{A}$ is always parallel to this axis, and it points in the general direction of the periapsis. This is true for all types of Kepler trajectory with the exception of the circle, for which $\epsilon = |\vec{A}| = 0$.

For any object in a three-dimensional conservative potential, we can find six constants of the motion, which are related to the components of the intitial position and velocity vectors. Since the total energy E and three components each of $\vec{l}$ and $\vec{A}$ would yield seven constants of the motion, these quantities must be related in some way. Indeed, we have

$$\vec{A}^2 = \epsilon^2 = 1 + \frac{2El^2}{mk^2}\,, \tag{32}$$

and the aforementioned orthogonality of $\vec{l}$ and $\vec{A}$,

$$\vec{l} \cdot \vec{A} = 0\,, \tag{33}$$

which provide two conditions that reduce the number of constants of the motion to five. The additional sixth constant must involve the time, since the quantities discussed here are purely geometrical; a natural candidate is the orbital period T (Eq. (28)).

2.3.2 Derivation from Noether's Theorem

It is possible to derive the Laplace-Runge-Lenz vector using Noether's theorem [1], although the necessary symmetry transformation is not intuitive unless one delves into more advanced topics of Hamiltonian mechanics that are beyond the scope of this course. Let us choose Cartesian coordinates, and consider the following transformation:

$$x_i \to x_i' = x_i + \frac{1}{2}\epsilon\left(2p_i x_s - x_i p_s - \delta_{is}(\vec{r}\cdot\vec{p})\right)\,, \tag{34}$$

where $s = 1, 2, 3$ is arbitrary, but fixed throughout the calculation. Expressing x'_i in terms of the velocities and taking its time derivative, we have

$$x'_i = x_i + \frac{1}{2}m\epsilon\left(2\dot{x}_i x_s - x_i\dot{x}_s - \delta_{is}(\vec{r}\cdot\dot{\vec{r}})\right)\,, \tag{35}$$

$$\dot{x}'_i = \dot{x}_i + \frac{1}{2}m\epsilon\left(2\ddot{x}_i x_s + 2\dot{x}_i\dot{x}_s - \dot{x}_i\dot{x}_s - x_i\ddot{x}_s - \delta_{is}(\dot{\vec{r}}\cdot\dot{\vec{r}} + \vec{r}\cdot\ddot{\vec{r}})\right)\,. \tag{36}$$

Next, we derive the equations of motion of the Kepler problem. In Cartesian coordinates, we can write the Lagrangian as

$$L = \frac{1}{2}m\dot{\vec{r}}^2 + \frac{k}{r}\,, \qquad r = |\vec{r}| = \sqrt{\sum_j x_j^2}\,. \tag{37}$$

The partial derivatives are

$$\frac{\partial L}{\partial x_i} = -\frac{k}{2r^3}\cdot 2\sum_j \delta_{ij}x_j = -\frac{k}{r^3}x_i\,, \qquad \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i\,, \tag{38}$$

hence

$$\ddot{x}_i = -\frac{k}{mr^3}x_i\,. \tag{39}$$

Using the equations of motion, we can eliminate $\ddot{x}_i$ in Eq. (36):

$$\begin{aligned}\dot{x}'_i &= \dot{x}_i + \frac{1}{2}m\epsilon\left(-\frac{2k}{mr^3}x_i x_s + \dot{x}_i\dot{x}_s + \frac{k}{mr^3}x_i x_s - \delta_{is}\dot{\vec{r}}^2 + \delta_{is}\frac{\kappa}{mr^3}\vec{r}^2\right)\\ &= \dot{x}_i + \frac{1}{2}m\epsilon\left(-\frac{k}{mr^3}x_i x_s + \dot{x}_i\dot{x}_s - \delta_{is}\dot{\vec{r}}^2 + \delta_{is}\frac{k}{mr}\right)\,.\end{aligned} \tag{40}$$

Let us now show that the transformation $x_i \to x_i + \delta x_i$ leaves L invariant up to a total time derivative, which means that the equations of motion are preserved. For the kinetic energy, we have

$$\begin{aligned}\sum_i \dot{x}'_i\dot{x}'_i &= \sum_i \dot{x}_i\dot{x}_i + 2\cdot\frac{1}{2}m\epsilon\left(-\frac{k}{mr^3}\sum_i \dot{x}_i x_i x_s + \sum_i \dot{x}_i\dot{x}_i\dot{x}_s - \dot{x}_s\sum_i \dot{x}_i\dot{x}_i + \dot{x}_s\frac{k}{mr}\right) + O\left(\epsilon^2\right)\\ &= \dot{\vec{r}}^2 + m\epsilon\left(-\frac{k}{mr^3}(\dot{\vec{r}}\cdot\vec{r})x_s + \dot{x}_s\frac{k}{mr}\right) + O\left(\epsilon^2\right)\,.\end{aligned} \tag{41}$$

The potential energy requires an additional expansion:

$$\begin{aligned}\frac{1}{r'} &= \frac{1}{\sqrt{\sum_i x'_i x'_i}} = \frac{1}{\sqrt{\sum_i x_i x_i}}\left(1 + m\epsilon\frac{2\sum_i x_i\dot{x}_i x_s - \sum_i x_i x_i\dot{x}_s - x_s\sum_i x_i\dot{x}_i}{\sum_i x_i x_i}\right)^{-1/2}\\ &= \frac{1}{r}\left(1 - \frac{1}{2}m\epsilon\frac{(\vec{r}\cdot\dot{\vec{r}})x_s - \vec{r}^2\dot{x}_s}{r^2} + O\left(\epsilon^2\right)\right)\,.\end{aligned} \tag{42}$$

Putting everything together and dropping higher-order terms in ϵ, we have

$$L' = \frac{1}{2}m\sum_i \dot{x}'_i\dot{x}'_i + \frac{k}{r'}$$

$$= \frac{1}{2}m\sum_i \dot{x}_i\dot{x}_i + \frac{1}{2}m^2\epsilon\left(-\frac{k}{mr^3}\sum_i \dot{x}_i x_i x_s + \dot{x}_s\frac{k}{mr}\right) + \frac{k}{r} - \frac{1}{2}m\epsilon\left(\frac{k}{r^3}x_j\dot{x}_j x_s - \frac{k}{r}\dot{x}_s\right) + O(\epsilon^2)$$
$$= L + \epsilon\left(-\frac{mk}{r^3}(\vec{r}\cdot\dot{\vec{r}})x_s + \frac{mk}{r}\dot{x}_s\right) . \tag{43}$$

Defining

$$F(\vec{r}) \equiv mk\frac{x_s}{r} , \tag{44}$$

we see that

$$\frac{dF}{dt} = mk\left(\frac{d}{dt}\frac{1}{r}\right)x_s + mk\frac{\dot{x}_s}{r} = mk\left(-\frac{1}{r^2}\sum_i\frac{\partial r}{\partial x_i}\dot{x}_i x_s + \frac{\dot{x}_s}{r}\right)$$
$$= mk\left(-\frac{1}{r^2}\frac{\vec{r}\cdot\dot{\vec{r}}}{r}x_s + \frac{\dot{x}_s}{r}\right) , \tag{45}$$

which allows us to write

$$L' = L + \epsilon\frac{dF}{dt} = L + \epsilon\frac{d}{dt}\left(mk\frac{x_s}{r}\right) . \tag{46}$$

Identifying

$$\eta_i \equiv \frac{1}{2}m\left(2\dot{x}_i x_s - x_i\dot{x}_s - \delta_{is}(\vec{r}\cdot\dot{\vec{r}})\right) , \quad \tau = 0 , \tag{47}$$

we can apply Noether's theorem:

$$J_s = -\sum_i \frac{\partial L}{\partial \dot{x}_i}\cdot\eta_i + F$$
$$= -\frac{1}{2}m^2\left(2\dot{\vec{r}}^2 x_s - (\dot{\vec{r}}\cdot\vec{r})\dot{x}_s - \dot{x}_s(\vec{r}\cdot\dot{\vec{r}})\right) + mk\frac{x_s}{r}$$
$$= -\left(\vec{p}^2 x_s - (\vec{p}\cdot\vec{r})p_s\right) + mk\frac{x_s}{r} = \text{const.} \tag{48}$$

Now we can use

$$\left[\vec{a}\times(\vec{b}\times\vec{c})\right]_s = b_s(\vec{a}\cdot\vec{c}) - c_s(\vec{a}\cdot\vec{b}) \tag{49}$$

with $\vec{a} = \vec{c} = \vec{p}$ und $\vec{b} = \vec{x}$:

$$x_s\vec{p}^2 - (\vec{r}\cdot\vec{p})p_s = [\vec{p}\times(\vec{r}\times\vec{p})]_s = \left[\vec{p}\times\vec{l}\right]_s . \tag{50}$$

Thus,

$$A_s \equiv -J_s = \left[\vec{p}\times\vec{l}\right]_s - mk\frac{x_s}{r} = \text{const.} \tag{51}$$

and since s was arbitrary, all three components of the Laplace-Runge-Lenz vector are conserved separately:

$$\vec{A} = \frac{\vec{p}\times\vec{l}}{mk} - \frac{\vec{r}}{r} = \text{const.} \tag{52}$$

The deeper origin of this conservation law is the invariance of the Kepler trajectories under certain types of rotations in a *four-dimensional* space [Add some good refs.]. The group of such transformation is the so-called **special orthogonal group** in four dimensions, or $SO(4)$ for short. In quantum mechanics, this additional SO(4) symmetry is the reason why the energies of the hydrogen atom are *independent of the electron's orbital angular momentum quantum number,*

$$E_n = -\frac{13.6\,\text{eV}}{n^2} . \tag{53}$$

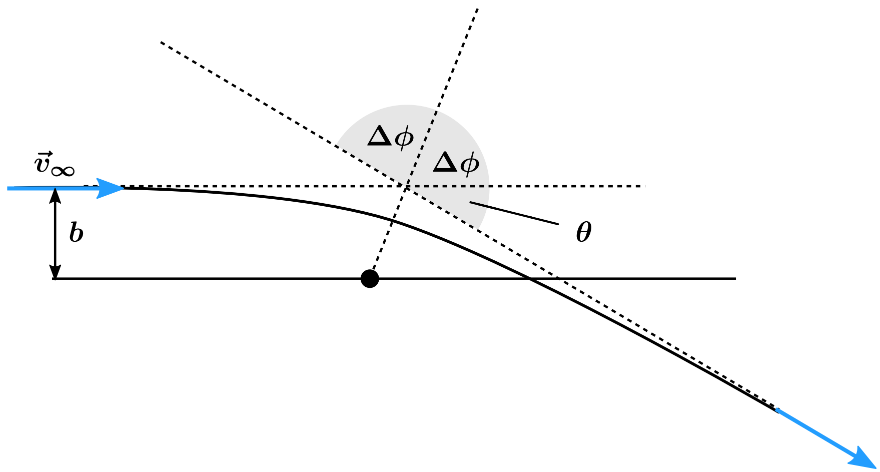

Figure 5: Scattering off an attractive central potential.

References

[1] J.-M. Lévy-Leblond, American Journal of Physics **39**, 502 (1971).

3 Scattering

3.1 General Considerations

After our extensive discussion of bounded trajectories in central potentials, we now consider scattering processes. Figures 5 and 6 show typical scenarios for a particle of mass m scattering off attractive and repulsive potentials $V(r)$, respectively. As in the bound-state case, the following discussion carries over to the scattering of two particles by a pairwise interaction $V(|\vec{r}_1 - \vec{r}_2|)$ in the effective one-body frame if we simply replace $m \to \mu$ in our expressions and interpret the angular momentum and energy as those of the relative motion (see worksheet #7 and Sec. 3.7).

We choose the potential center to be the origin of our coordinate system. The particle's initial velocity vector $\vec{v}_\infty$ defines a preferred direction, breaking the rotational symmetry of the potential to a mere symmetry for rotations around the axis through the origin that is parallel to $\vec{v}_\infty$.

Assuming that the potentials vanish for $r \to \infty$, the momentum and energy of the incoming particle are

$$\vec{p} = m\vec{v}_\infty \,, \quad E = \frac{1}{2} m v_\infty^2 \,. \tag{54}$$

Its position vector $\vec{r}$ in the chosen coordinate system can be decomposed into components that are parallel and orthogonal to the axis we defined above:

$$\vec{r} = r_\parallel \vec{e}_\parallel + b\vec{e}_\perp \,, \quad r = \sqrt{r_\parallel^2 + b^2} \,. \tag{55}$$

This implies that the angular momentum is given by

$$\vec{l} = \vec{r} \times \vec{p} = m \left(r_\parallel \vec{e}_\parallel + b\vec{e}_\perp \right) \times v_\infty \vec{e}_\parallel = m v_\infty b \left(\vec{e}_\parallel \times \vec{e}_\perp \right) \tag{56}$$

which is a vector pointing out of the shown scattering plane. Since we are dealing with a central potential $\vec{l}$ is conserved. Furthermore, we note that it only depends on the initial velocity and the

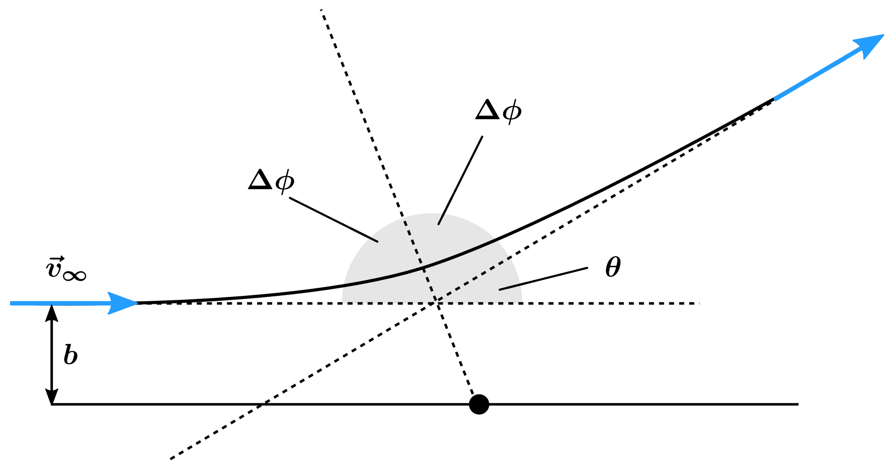

Figure 6: Scattering off a repulsive central potential.

so-called **impact parameter** b, the distance between the particle trajectory and the axis through the scattering center. For future use, we note that

$$l = mv_\infty b = \sqrt{2mE}b\,. \tag{57}$$

Let us now assume that the scattering process is elastic, so that the (mechanical) energy is going to be conserved at all times. Then the scattered object will move from infinity to a distance of closest approach, i.e., the pericenter r_p, and back to infinity. This point can be found by solving

$$E = V_{\text{eff}}(r_p) = \frac{l^2}{2mr_p^2} + V(r_p) = \frac{Eb^2}{r_p^2} + V(r_p) \tag{58}$$

i.e.,

$$1 - \frac{b^2}{r_p^2} - \frac{V(r_p)}{E} = 0\,. \tag{59}$$

Energy conservation also implies that the scattering trajectories will be symmetric with respect to r_p, and that the asymptotes to the incoming and outgoing trajectories form the same angle $\Delta\phi$ with an axis through the pericenter and the origin. This is illustrated in Figs. 6 and 5.

The angle $\Delta\phi$ can be determined from the integral equation for general trajectories in a central potential, Eq. (8). Rewriting the equation in terms of the impact parameter, we have

$$\begin{aligned}\Delta\phi &= \frac{l}{\sqrt{2m}}\int_{r_p}^{\infty} dr \frac{1}{r^2\sqrt{E - \frac{l^2}{2mR^2} - V(r)}} = \sqrt{E}b\int_{r_p}^{\infty} dr \frac{1}{r^2\sqrt{E - \frac{Eb^2}{r^2} - V(r)}} \\ &= \int_{r_p}^{\infty} dr \frac{b}{r^2}\frac{1}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\,. \end{aligned} \tag{60}$$

Referring to the figures again, we can define the **scattering angle** θ, which measures the scattered object's deflection from its original trajectory, as

$$\theta = \pi - 2\Delta\phi\,. \tag{61}$$

Using $\Delta\phi$ from above, we have

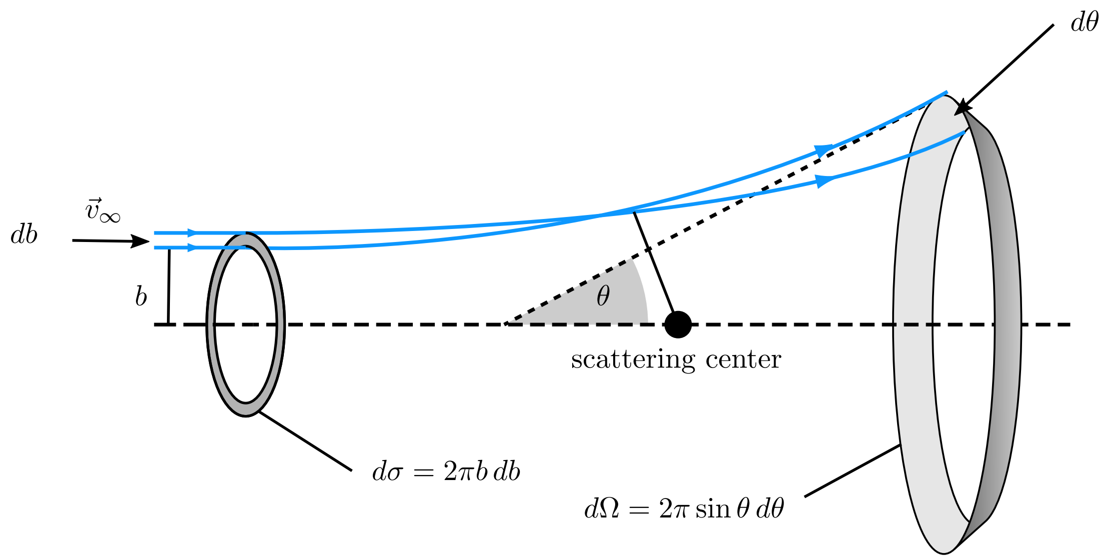

Figure 7: Differential cross section for a repulsive potential.

$$\theta(b) = \pi - 2\int_{r_p}^{\infty} dr \frac{b}{r^2} \frac{1}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\,. \tag{62}$$

At this point, a word of caution is in order: In defining θ in Eq.(61), we treated all angles as positive, but when we compute integrals like (60) the direction of the trajectories and the angles may matter. For instance, we compute the integral from r_p to infinity for the outgoing branch of the trajectory. This would mean that $\Delta\phi$ increases *clockwise* instead of counterclockwise, which implies that it is negative under typical conventions. We may have to judiciously apply absolute values when we work with Eqs. (60)–(62).

3.2 The Differential Cross Section

While the scattering of a single incident particle off a given potential can be described by computing its trajectory and the associated scattering angle using Eq. (62), such scenarios tend to be the exception rather than the rule. In the more common cases, we might only know the impact parameter b with some uncertainty db, or we might be studying the scattering of a beam of a large number of incident particles, as in a nuclear or particle physics experiment.

Consider the setup shown in Fig. 7. An incident particle with an impact parameter between b and $b + db$ will emerge from the ring of area

$$d\sigma = 2\pi b db\,, \tag{63}$$

and it will be scattered by the potential into a segment of the solid angle

$$d\Omega = 2\pi \sin\theta d\theta\,. \tag{64}$$

Inspecting Fig. 7 more closely, we note that the scattering trajectories for incoming particles from the inner and outer boundaries of the *intersect*. The reason is that physical scattering potentials get weaker with growing distance, which implies that particles with small impact parameter b experience a greater deflection than particles that approach with large b. Thus, we have

$$db \sim -d\theta \quad \Rightarrow \quad \frac{db}{d\theta} < 0\,, \tag{65}$$

which must be taken into account when we compare the areas $d\sigma$ and $d\Omega$. Taking the ratio of the two quantities, we obtain the **differential cross section**

$$\frac{d\sigma}{d\Omega} = -\frac{b}{\sin\theta}\frac{db}{d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|\,, \tag{66}$$

where we have inserted the explicit negative sign or the absolute values to ensure that the differential cross section is always positive (since it is a ratio of infinitesimal areas). It is hardly surprising that the **total cross section** can be found by integrating Eq. (66) over the full solid angle,

$$\sigma = \oint_{\partial S} d\Omega\,\frac{\partial\sigma}{\partial\Omega} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,\sin\theta\,\frac{d\sigma}{d\Omega}\,. \tag{67}$$

3.3 Gravitational Scattering

Let us now apply the scattering formalism developed in the previous sections to the gravitational scattering of a mass m, i.e., the scattering off the attractive potential

$$V(r) = -\frac{k}{r}\,, \quad k > 0 \tag{68}$$

(see Figs. 5 and 8). We plug the potential into Eq. (62), which takes the form

$$\theta(b) = \pi - 2\Delta\phi = \pi - 2\int_{r_p}^{\infty} dr\,\frac{b}{r^2}\frac{1}{\sqrt{1-\frac{b^2}{r^2}+\frac{k}{rE}}}\,. \tag{69}$$

Since we already solved this integral in our discussion of the Kepler trajectories, we can immediately use the general solution (18) to obtain

$$\Delta\phi = \arccos\frac{\lambda - r}{\epsilon r}\bigg|_{r_p}^{\infty} = \arccos\frac{-1}{\epsilon} - \arccos\frac{\lambda - r_p}{\epsilon r_p}\,. \tag{70}$$

Rearranging Eq. (59), we obtain

$$r_p^2 + \frac{kr_p}{E} - b^2 = 0\,, \quad \Rightarrow \tag{71}$$

and the solution

$$\begin{aligned} r_p &= -\frac{k}{2E} + \sqrt{\frac{k^2}{4E^2}+b^2} = -\frac{k}{2E} + \sqrt{\frac{k^2}{4E^2}+\frac{l^2}{2mE}} = -\frac{k}{2E} + \frac{k}{2E}\sqrt{1+\frac{2El^2}{mk^2}} \\ &= \frac{k}{2E}(\epsilon - 1)\,. \end{aligned} \tag{72}$$

The negative branch of the solution, $-\frac{k}{2E}(1+\epsilon)$, would be unphysical for $E > 0$. This result is consistent with the existence of a single turning point for a hyperbola. Plugging r_p into our expression for r_p, we have

$$\Delta\phi = \arccos\frac{-1}{\epsilon} - \arccos\frac{\frac{l^2}{mk}+\frac{k}{2E}(1-\epsilon)}{\frac{k}{2E}(\epsilon^2-\epsilon)} = \arccos\frac{-1}{\epsilon} - \arccos\frac{\frac{2El^2}{mk^2}+(1-\epsilon)}{\epsilon^2-\epsilon}$$

$$
\begin{aligned}
&= \arccos\frac{-1}{\epsilon} - \arccos\frac{\epsilon^2 - 1 + (1-\epsilon)}{\epsilon^2 - \epsilon} = \arccos\frac{-1}{\epsilon} - \underbrace{\arccos 1}_{=0} \\
&= \arccos\frac{-1}{\epsilon}\,. \qquad (73)
\end{aligned}
$$

It makes sense that the contribution of r_p to the integral vanishes because we measure ϕ counter-clockwise from the pericenter.

Plugging our $\Delta\phi$ into Eq. (69), we find

$$
\theta = \pi - 2\arccos\frac{-1}{\epsilon}\,, \qquad (74)
$$

and after rearranging and taking the cosine on both sides, we obtain

$$
-\frac{1}{\epsilon} = \cos\frac{\pi-\theta}{2} = \cos\frac{\pi}{2}\cos\frac{\theta}{2} + \sin\frac{\pi}{2}\sin\frac{\theta}{2} = \sin\frac{\theta}{2}\,. \qquad (75)
$$

Squaring and inverting both sides, we have

$$
\epsilon^2 = 1 + \frac{2El^2}{mk^2} = 1 + \frac{2E\cdot 2mEb^2}{mk^2} = 1 + \frac{4E^2b^2}{k^2} = \frac{1}{\sin^2\frac{\theta}{2}}\,, \qquad (76)
$$

and rearranging, we find

$$
b^2 = \frac{k^2}{4E^2}\frac{1-\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \frac{k^2}{4E^2}\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \frac{k^2}{4E^2}\cot^2\frac{\theta}{2}\,. \qquad (77)
$$

Since $0 \le \theta \le \pi$, we see that $\cot\theta/2$ is always positive, and since k and E are both positive as well, the function $b(\theta)$ is given by

$$
b = \frac{k}{2E}\cot\frac{\theta}{2}\,. \qquad (78)
$$

Taking the derivative with respect to θ, we have

$$
\frac{db}{d\theta} = \frac{k}{2E}\frac{-\frac{1}{2}\sin\frac{\theta}{2}\cdot\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\cdot\frac{1}{2}\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = -\frac{k}{4E}\frac{1}{\sin^2\frac{\theta}{2}}\,. \qquad (79)
$$

Note that the derivative is indeed negative, as discussed above (see Eq. (65)). We can now combine all our results and compute the differential cross section:

$$
\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{k^2}{8E^2}\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\frac{1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\frac{1}{\sin^2\frac{\theta}{2}} = \frac{k^2}{16E^2\sin^4\frac{\theta}{2}}\,. \qquad (80)
$$

For the gravitational potential, we have $k = GmM$, where G is the gravitational constant, M is the mass generating the potential, and m is the mass of the scattered object. Then

$$
\boxed{\frac{d\sigma}{d\Omega} = \frac{(GMm)^2}{16E^2\sin^4\frac{\theta}{2}} = \frac{(GM)^2}{4v_\infty^2\sin^4\frac{\theta}{2}}\,. \qquad (81)}
$$

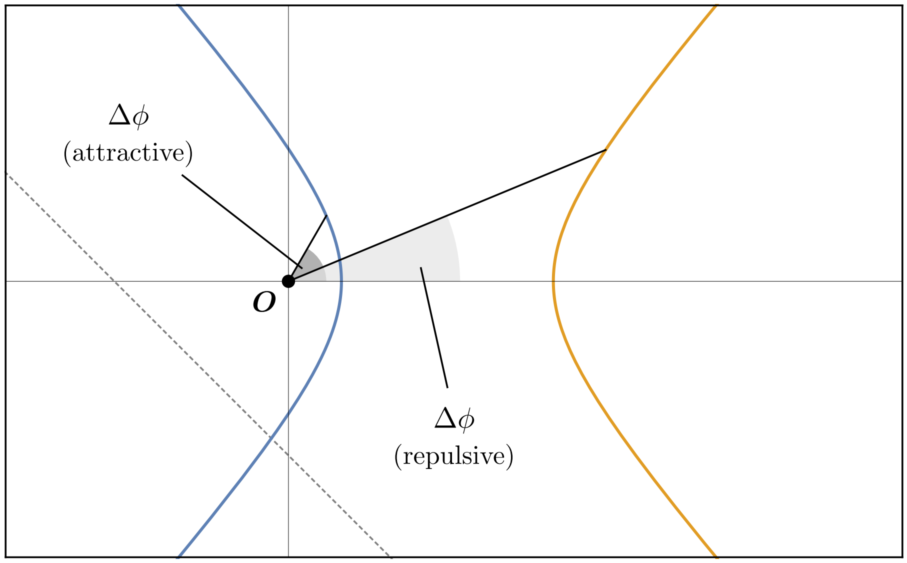

Figure 8: Hyperbolic trajectories for scattering off attractive (blue) and repulsive (orange) $1/r$ potentials.

3.4 Rutherford Scattering

Next, we discuss **Coulomb** or **Rutherford scattering** of a particle in the potential

$$V(r) = -\frac{k}{r}\,, \tag{82}$$

where k can be either positive or negative. For $k > 0$, the potential is attractive and we obtain the same hyperbolic trajectories and cross sections as in the gravitational case. The case of a repulsive potential with $k < 0$, shown in Figs. 6 and 8, deserves separate consideration, since the origin of the coordinate system now lies in the exterior focus of the hyperbola (see Fig. 8).

The integral for $\Delta\phi$ now reads

$$\Delta\phi = \int_{r_p}^{\infty} dr \frac{b}{r^2} \frac{1}{\sqrt{1 - \frac{b^2}{r^2} - \frac{|k|}{rE}}}\,. \tag{83}$$

Substituting

$$u = \frac{b}{r}\,, \quad dr = \frac{b}{u^2} du\,, \tag{84}$$

we obtain

$$\Delta\phi = -\int_{b/r_p}^{0} du \frac{1}{\sqrt{1 - \frac{|k|}{Eb}u - u^2}} = \arcsin \frac{-2u - \frac{|k|}{Eb}}{\sqrt{4 + \left(\frac{|k|}{Eb}\right)^2}}\Bigg|_{b/r_p}^{0}$$

$$= \arcsin \frac{-\frac{|k|}{Eb}}{\sqrt{4 + \left(\frac{|k|}{Eb}\right)^2}} - \arcsin \frac{-2\frac{b}{r_p} - \frac{|k|}{Eb}}{\sqrt{4 + \left(\frac{|k|}{Eb}\right)^2}} = \arcsin \frac{-1}{\sqrt{1 + \frac{4E^2b^2}{|k|^2}}} - \arcsin \frac{-2\frac{Eb}{|k|}\frac{b}{r_p} - 1}{\sqrt{1 + \frac{4E^2b^2}{|k|^2}}}$$

$$= \arcsin\frac{-1}{\epsilon} - \arcsin\frac{-2\frac{Eb}{|k|}\frac{b}{r_p} - 1}{\epsilon} \tag{85}$$

We use Eq. (59) to determine r_p:

$$r_p = \frac{|k|}{2E} + \sqrt{\frac{|k|^2}{4E^2} + b^2} = \frac{|k|}{2E} + \frac{|k|}{2E}\sqrt{1 + \frac{4E^2b^2}{|k|^2}} = \frac{|k|}{2E}(1+\epsilon)\,, \tag{86}$$

where the negative branch will again yield an unphysical negative radius because the square root term has a greater magnitude than the first term. Plugging this into the result for the integral and using Eq. (76), we find

$$\begin{aligned}\Delta\phi &= \arcsin\frac{-1}{\epsilon} - \arcsin\frac{-\frac{4E^2b^2}{|k|^2}\frac{1}{1+\epsilon} - 1}{\epsilon} = \arcsin\frac{-1}{\epsilon} - \arcsin\frac{-\epsilon^2 + 1 - 1 - \epsilon}{\epsilon^2 + \epsilon} \\ &= \arcsin\frac{-1}{\epsilon} - \arcsin(-1) = \arcsin\frac{-1}{\epsilon} - \left(-\frac{\pi}{2}\right) \\ &= \arcsin\frac{-1}{\epsilon} + \frac{\pi}{2}\,.\end{aligned} \tag{87}$$

The scattering angle now becomes

$$\theta = \pi - 2\Delta\phi = \pi - 2\arcsin\frac{-1}{\epsilon} - 2\frac{\pi}{2} = -2\arcsin\frac{-1}{\epsilon}\,, \tag{88}$$

and we have

$$-\sin\frac{\theta}{2} = -\frac{1}{\epsilon}\,. \tag{89}$$

Squaring both sides, we get

$$\epsilon^2 = 1 + \frac{4E^2b^2}{|k|^2} = \frac{1}{\sin^2\frac{\theta}{2}}\,, \tag{90}$$

which matches Eq. (77). Thus, the remaining steps proceed as in the gravitational case: The relationship between θ and b is given by

$$b = \frac{|k|}{2E}\cot\frac{\theta}{2}\,, \tag{91}$$

and the differential cross section once again becomes

$$\frac{d\sigma}{d\Omega} = \frac{k^2}{16E^2\sin^4\frac{\theta}{2}}\,. \tag{92}$$

If we now plug in the potential strength for like charges,

$$k = \frac{qQ}{4\pi\epsilon_0}\,, \tag{93}$$

we obtain

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{Qq}{4\pi\epsilon_0}\right)^2 \frac{1}{16E^2\sin^4\frac{\theta}{2}}\,.} \tag{94}$$

3.5 Divergent Total Cross Sections

In problem G20, we show that the total cross section of a *finite-range* potential is always finite. However, we are often dealing with potentials that are *not* strictly finite-range as well, but only decay according to some functional form at large distances, e.g., via an inverse power law or an exponential.

If we try to compute the total cross section by integrating the differential cross section of the $1/r$ potentials over the full solid angle, we find that the integral diverges due to the contribution from the forward scattering angle $\theta = 0$:

$$\sigma \sim 2\pi \int_0^\pi d\theta \, \sin\theta \frac{1}{\sin^4 \frac{\theta}{2}} = -4\pi \frac{1}{\sin^2 \frac{\theta}{2}}\bigg|_0^\pi = \lim_{\theta \to 0} 4\pi \cot^2 \frac{\theta}{2} = \infty \,. \tag{95}$$

The reason for this divergence is that the $1/r$ potential does not decay rapidly enough at large distances: If we disregard relativity, the potential extends through all of space, and objects *everywhere* in space would undergo scattering processes under the $1/r$ potential of the tiniest mass element or charge. In the integral for the total cross section, all of these processes contribute.

In the Rutherford case, we are "saved" by noticing that the universe appears to be neutral as a whole, which means that the potentials of positive and negative charges cancel out in the vast majority of space. Therefore, the electromagnetic interactions are screened, and effectively decay more rapidly than $1/r$. Unfortunately, that same logic does not work for masses, since masses only attract each other and no cancellation between potentials can occur. One could expect that a more fundamental theory of gravity at large distances, i.e., General Relativity, might resolve this issue, but to the best of my knowledge, this is presently not the case.

3.6 The Inverse Scattering Problem

As discussed above, many modern experiments measure cross sections in order to extract information about the fundamental forces governing a system. Instead of trying to model the cross section through simulations of the system based on the best available (effective) theory of these forces, we may ask whether it is possible to extract them *directly* from a measured cross section, provided the target and probes are sufficiently simple, so we do not have to worry about their intrinsic structure. As the name suggests, this **inverse scattering problem** mainly consists of reversing the steps we took to derive the differential cross section. The following discussion is based on [1, 2], stripping away details that are related to the applications in semi-classical approximations to quantum mechanics.

The starting point is Eq. (62), which we used to derive $b(\theta)$ and then $\frac{d\sigma}{d\Omega}$ via Eq. (66). Now, we will do the opposite. Let us assume we have measured the cross section, and we have used Eq. (66) to extract $\theta(b)$. We introduce the new function

$$y(r) = r\sqrt{1 - \frac{V(r)}{E}} \tag{96}$$

and write the integral in Eq. (62) as

$$I = 2b \int_{r_p}^\infty dr \, \frac{1}{r\sqrt{y^2 - b^2}} \,. \tag{97}$$

Now we assume that the function (96) is invertible, so that $r(y)$ exists, and we formally change variables from r to y. Since scattering implies that an object is able to escape from the potential,

Exercise 3.1: An Integral for the Inverse Scattering Problem

Prove

$$\pi = 2b\int_b^\infty dy\frac{1}{y\sqrt{y^2-b^2}} = 2b\int_b^\infty \frac{dy}{\sqrt{y^2-b^2}}\frac{d}{dy}\ln y\,. \tag{E3.1-1}$$

we can assume that $V(r)$ vanishes at large distances in realistic applications, hence the upper limit of the integral remains unchanged. For the lower limit, Eq. (59) implies

$$1-\frac{V(r_p)}{E}=\frac{b^2}{r_p^2}\,, \tag{98}$$

hence

$$y(r_p)=r_p\sqrt{1-\frac{V(r_p)}{E}}=r_p\sqrt{\frac{b^2}{r_p^2}}=b\,. \tag{99}$$

Noting that

$$dr=r'(y)dy\,,\quad \frac{r'(y)}{r(y)}=\frac{d}{dy}\ln r(y)\,, \tag{100}$$

the integral first takes the form

$$I=2b\int_b^\infty dy\frac{r'(y)}{r(y)\sqrt{y^2-b^2}}=2b\int_b^\infty\frac{dy}{\sqrt{y^2-b^2}}\frac{d}{dy}\ln r(y)\,. \tag{101}$$

Using the identity (E3.1-1), we can combine both terms on the right-hand side of Eq. (62) into a single integral,

$$\theta(b)=\pi-I=2b\int_b^\infty\frac{dy}{\sqrt{y^2-b^2}}\frac{d}{dy}\ln\frac{y}{r(y)}\,. \tag{102}$$

Instead of calculating this integral directly, we define a new function $T(y)$ that will eventually allow us to find $y(r)$ from a given $\theta(b)$:

$$T(y)\equiv\frac{1}{\pi}\int_y^\infty db\,\frac{\theta(b)}{\sqrt{b^2-y^2}}\,. \tag{103}$$

This integral diverges if $\theta(b)$ is a nonzero constant, but this would only occur if the scattering angle θ were independent of b, which only happens if there is no scattering at all. Thus, $\theta(b)=$ const. would imply $\theta(b)=0$, which in turn means that $T(y)$ vanishes as well. When scattering does occur,

$$\lim_{b\to\infty}\theta(b)=0 \tag{104}$$

and if $\theta(b)$ approaches zero as $b^{-\lambda},\lambda>0$, the integral converges. This will be the case under physically reasonable assumptions.

Let us now show the relation between $T(y)$ and $y(r)$. Plugging Eq. (102) into the definition of $T(y)$ (Eq. (103)), we obtain

$$T(y)=\frac{1}{\pi}\int_y^\infty db\,\frac{2b}{\sqrt{b^2-y^2}}\left[\int_b^\infty\frac{dy'}{\sqrt{y'^2-b^2}}\frac{d}{dy'}\ln\frac{y'}{r(y')}\right]\,. \tag{105}$$

The double integral can be performed if we switch the order of integration, but that requires care with the limits. In the integral over b, the limits imply that $b > y$, while the limits of the integral over y' imply that $y' > b$, and therefore $y < b < y'$. The double integral can then be written as

$$T(y) = \frac{1}{\pi} \int_y^\infty dy' \left[\frac{d}{dy'} \ln \frac{y'}{r(y')} \right] \int_y^{y'} db \, \frac{2b}{\sqrt{(b^2 - y^2)\,(y'^2 - b^2)}} \tag{106}$$

The integral over b yields

$$\begin{aligned} \int_y^{y'} db \, \frac{2b}{\sqrt{(b^2 - y^2)\,(y'^2 - b^2)}} &= 2 \arctan \frac{\sqrt{b^2 - y^2}}{\sqrt{y'^2 - b^2}} \bigg|_y^{y'} \\ &= 2\,(\arctan \infty - \arctan 0) = 2 \left(\frac{\pi}{2} - 0 \right) = \pi \,, \end{aligned} \tag{107}$$

so we find

$$T(y) = \int_y^\infty dy' \left[\frac{d}{dy'} \ln \frac{y'}{r(y')} \right] = \ln \frac{y'}{r(y')} \bigg|_y^\infty = 0 - \ln \frac{y}{r(y)} = \ln \frac{r(y)}{y} \,. \tag{108}$$

To see that there is no contribution from the upper limit, we have used that for large distances

$$\lim_{r \to \infty} \frac{y(r)}{r} = \lim_{r \to \infty} \frac{r \sqrt{1 - \frac{V(r)}{E}}}{r} = 1 \,. \tag{109}$$

Taking the exponential on both sides, we finally obtain

$$r(y) = y \exp\left[T(y)\right] \,, \tag{110}$$

and we can obtain the potential by plugging this solution into a rearranged version of Eq. (96):

$$V(r) = E \frac{r^2 - y^2}{r^2} \,. \tag{111}$$

Example: Rutherford Scattering

Let us assume that we have measured the differential cross section for Rutherford scattering and extracted its functional form, Eq. (80). Using this in eq. (66), we just reverse the steps that produce $\frac{d\sigma}{d\Omega}$ from a given $b(\theta)$

$$\begin{aligned} b \frac{db}{d\theta} &= -\sin\theta \frac{d\sigma}{d\Omega} = -\sin\theta \frac{k^2}{16E^2} \frac{1}{\sin^4 \frac{\theta}{2}} = -\frac{k^2}{16E^2} \frac{2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}}{\sin^4\frac{\theta}{2}} \\ &= -\frac{k^2}{8E^2} \frac{\cos\frac{\theta}{2}}{\sin^3\frac{\theta}{2}} = -\frac{k^2}{8E^2} \frac{\cot\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = -\frac{k^2}{8E^2} \cot\frac{\theta}{2} \frac{d}{d\theta} \left(-2 \cot\frac{\theta}{2} \right) \\ &= \frac{k^2}{4E^2} \cot\frac{\theta}{2} \frac{d}{d\theta} \cot\frac{\theta}{2} = \frac{k}{2E} \cot\frac{\theta}{2} \frac{d}{d\theta} \left(\frac{k}{2E} \cot\frac{\theta}{2} \right) \,. \end{aligned} \tag{112}$$

Thus, we can directly read off

$$b = \frac{k}{2E} \cot\frac{\theta}{2} \,. \tag{113}$$

Inverting the function, we have

$$\theta(b) = 2 \, \mathrm{arccot} \, \frac{2Eb}{k} \,, \tag{114}$$

and Eq. (103) becomes

$$T(y;E) = \frac{2}{\pi}\int_y^\infty db\,\frac{\operatorname{arccot}\frac{2Eb}{k}}{\sqrt{b^2-y^2}}\,, \tag{115}$$

where we have added E as an argument to T to indicate that it depends *smoothly* on the energy. This allows us to apply a useful trick to evaluate the integral: Taking the partial derivative with respect to E, we have

$$\frac{\partial}{\partial E}T(y;E) = \frac{2}{\pi}\int_y^\infty db\,\frac{\operatorname{arccot}\frac{2Eb}{k}}{\sqrt{b^2-y^2}} = \frac{2}{\pi}\int_y^\infty db\,\frac{-2bk}{4b^2E^2+k^2}\frac{1}{\sqrt{b^2-y^2}}\,. \tag{116}$$

We introduce the substitution

$$x = \sqrt{b^2-y^2}\,, \qquad \frac{dx}{db} = \frac{b}{\sqrt{b^2-y^2}}\,, \tag{117}$$

which allows us to write

$$\begin{aligned}\frac{\partial}{\partial E}T(y;E) &= -\frac{2}{\pi}\int_0^\infty dx\,\frac{2k}{4(x^2+y^2)E^2+k^2} = -\frac{2}{\pi}\frac{k}{E\sqrt{k^2+4E^2y^2}}\arctan\frac{2Ex}{k^2+4E^2y^2}\bigg|_0^\infty \\ &= -\frac{2}{\pi}\frac{k}{E\sqrt{k^2+4E^2y^2}}\left(\frac{\pi}{2}-0\right) = -\frac{k}{E\sqrt{k^2+4E^2y^2}}\,.\end{aligned} \tag{118}$$

We can integrate this expression in turn:

$$T(y;E) = -\int_E^\infty dE'\,\frac{k}{E'\sqrt{k^2+4E'^2y^2}} = \ln\frac{k^2\left(1+\sqrt{1+\frac{4E'^2y^2}{k^2}}\right)}{E'}\Bigg|_E^\infty \tag{119}$$

The limits of the integration were chosen on physical grounds: In a realistic scattering process, we must have $E>0$ because the particle would not be moving towards the potential otherwise. For $E\to\infty$, the particle should not be scattered at all since

$$\lim_{E\to\infty}\operatorname{arccot}\frac{2Eb}{k} = 0\,. \tag{120}$$

Thus,

$$\begin{aligned}T(y;E) &= -\ln 2ky + \ln\frac{k^2\left(1+\sqrt{1+\frac{4E^2y^2}{k^2}}\right)}{E} = \ln\frac{k^2\left(1+\sqrt{1+\frac{4E'^2y^2}{k^2}}\right)}{2kyE} \\ &= \ln\left(\frac{k}{2yE}+\sqrt{\frac{k^2}{4E^2y^2}+1}\right)\,,\end{aligned} \tag{121}$$

which has the desired property that

$$\lim_{E\to\infty}T(y;E) = \lim_{E\to\infty}\ln\left(\frac{k}{2yE}+\sqrt{\frac{k^2}{4E^2y^2}+1}\right) = \ln 1 = 0\,. \tag{122}$$

Inserting $T(y;E)$ into Eq. (110), we have

$$r(y) = y \exp\left[T(y;E)\right] = \frac{k}{2E} + \sqrt{\frac{k^2}{4E^2} + y^2} \tag{123}$$

and squaring, we find

$$r^2 = \frac{k^2}{4E} + 2\frac{k}{2E}\sqrt{\frac{k^2}{4E^2} + y^2} + \frac{k^2}{4E^2} + y^2 = 2\frac{k}{2E}\left(\frac{k}{2E} + \sqrt{\frac{k^2}{4E^2} + y^2}\right) + y^2 = \frac{k}{E}r + y^2\,, \tag{124}$$

which implies

$$y^2 = r^2 - \frac{k}{E}r\,. \tag{125}$$

Squaring Eq. (96) and plugging in this result, we see that

$$\begin{aligned} r^2\left(1 - \frac{V(r)}{E}\right) &= r^2 - \frac{k}{E}r \\ \Leftrightarrow \qquad -r^2\frac{V(r)}{E} &= -\frac{k}{E}r\,, \end{aligned} \tag{126}$$

and we finally obtain

$$V(r) = \frac{k}{r}\,. \tag{127}$$

3.7 Two-Body Scattering

3.7.1 Transformation of the Differential Cross Section

As mentioned above, the scattering of two particles that have an interaction of the form $V(|\vec{r}_1 - \vec{r}_2|)$ in the center-of-mass frame is formally equivalent to the scattering of a particle off the central potential $V(r)$. The expressions we derived for the scattering angles and cross sections apply if we replace $m \to \mu$ in the *kinematic quantities* l and E (but *not* in the interaction strength $k = GMm$ entering the gravitational scattering cross section (80), for instance).

In contrast, scattering experiments are usually performed in the **laboratory frame**, where the target is at rest and the center of mass and projectile and target is therefore moving, as show in Fig. 9. Thus, we need to translate our scattering expressions to this frame by finding an analytic relation between ψ and θ.

Using the scattering angle ψ, we can define the differential cross section in the laboratory frame analogous to Eq. (66):

$$\frac{d\sigma}{d\Omega_\psi} = \frac{d\sigma}{2\pi \sin\psi d\psi}\,, \tag{128}$$

where we have used that the switch between the frames does not change the area from which the incoming particles approach the target's potential, nor does it break the azimuthal symmetry with respect to the $b = 0$ axis. This also implies

$$d\sigma = \frac{d\sigma}{d\Omega_\psi}d\Omega_\psi = \frac{d\sigma}{d\Omega}d\Omega\,, \tag{129}$$

which is a way of stating that the same total number of particles entering from the ring $d\sigma$ must be detected in the solid angles $d\Omega$ and $d\Omega_\psi$ in the center-of-mass and laboratory systems, respectively. Thus, we see that

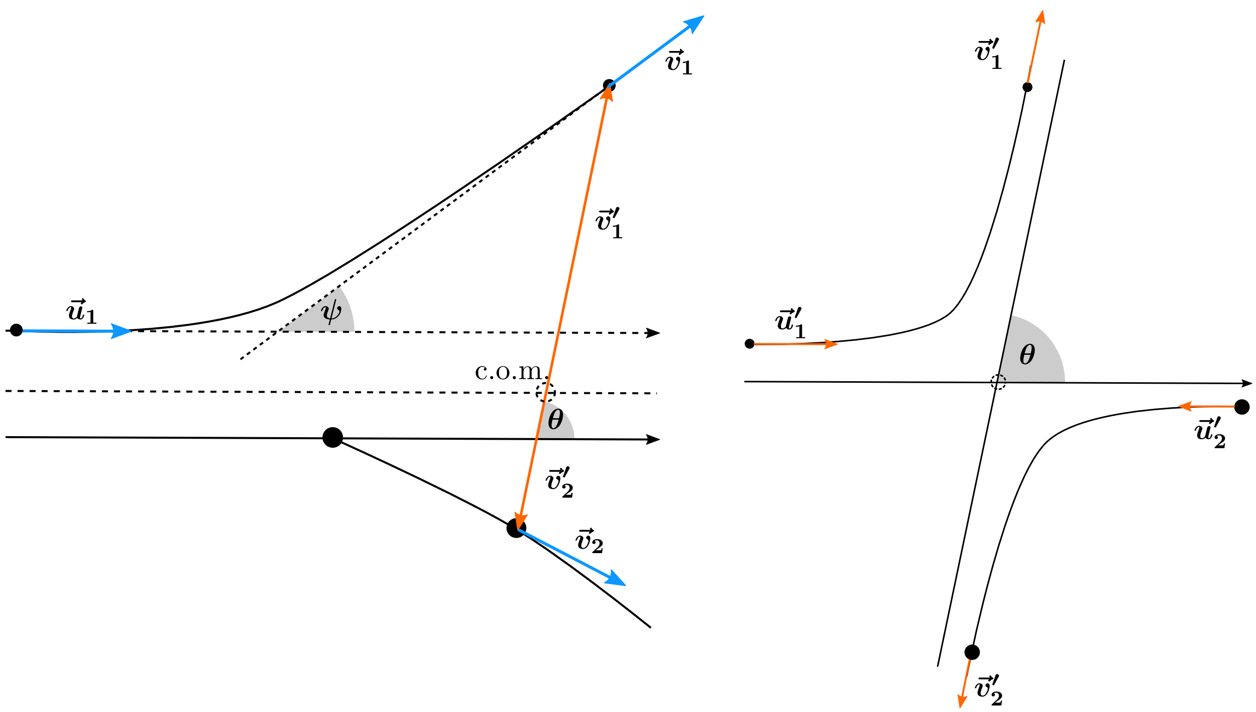

Figure 9: Scattering in the laboratory (left) and center-of-mass frame (right).

$$\frac{d\sigma}{d\Omega_\psi} = \frac{d\sigma}{d\Omega} \cdot \frac{d\Omega}{d\Omega_\psi} = \frac{d\sigma}{d\Omega} \cdot \frac{\sin\theta\, d\theta}{\sin\psi\, d\psi}\,, \tag{130}$$

where $\frac{d\sigma}{d\Omega}$ should now be understood as a function of the angle ψ.

3.7.2 Elastic Scattering

Let us briefly recall the treatment of elastic *collisions*, as shown in Fig. 10 — in the present context, we can view it as the scattering of hard spheres (cf. problem G19). We denote the initial and final vectors in the laboratory frame by $\vec{u}_i$ and $\vec{v}_i$, respectively, and we use primes to indicate the corresponding vectors in the center-of-mass frame (also cf. Fig. 9).

In the center-of-mass frame, the initial and final total momenta are zero, hence

$$m_1\vec{u}'_1 = -m_2\vec{u}'_2\,, \tag{131}$$
$$m_1\vec{v}'_1 = -m_2\vec{v}'_2 \tag{132}$$

and the magnitudes of the velocites satisfy

$$m_1 u'_1 = m_2 u'_2\,, \tag{133}$$
$$m_1 u'_1 = m_2 v'_2\,. \tag{134}$$

The total kinetic energy is conserved, so

$$\frac{1}{2}m_1 u'^2_1 + \frac{1}{2}m_2 u'^2_2 = \frac{1}{2}m_1 v'^2_1 + \frac{1}{2}m_2 v'^2_2\,. \tag{135}$$

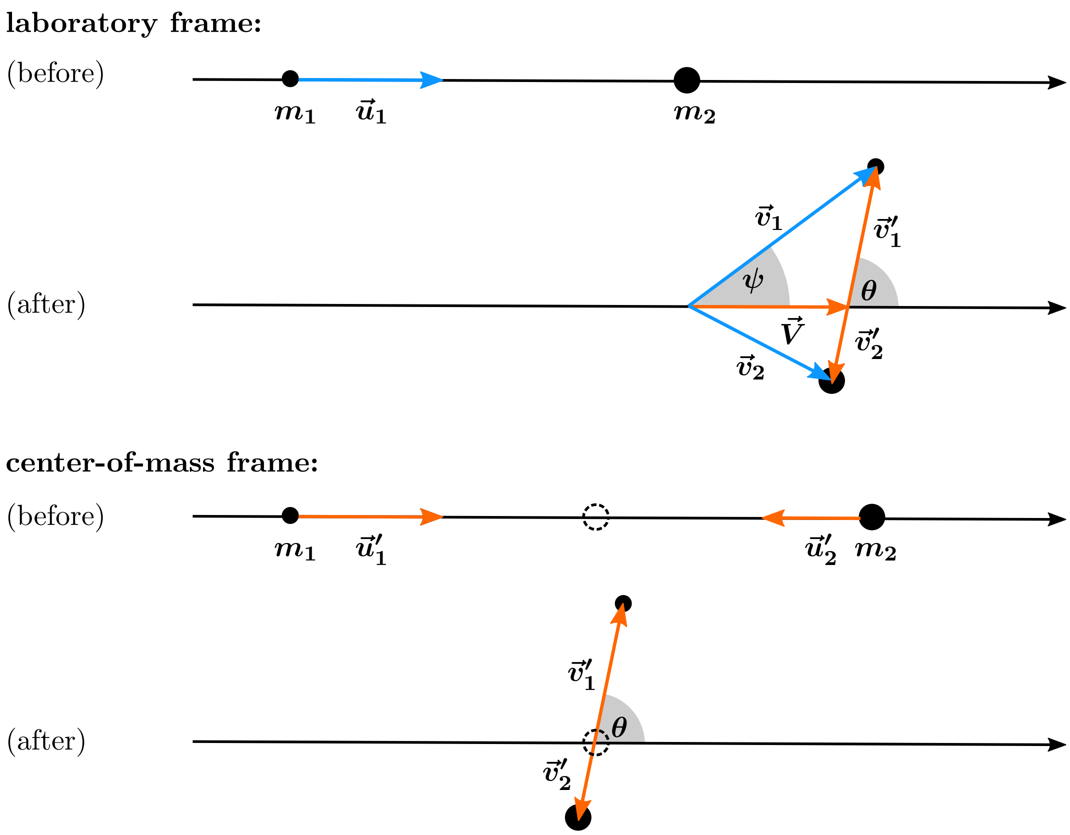

Figure 10: Elastic collisions in the laboratory (top) and center-of-mass frames (bottom).

In the collision shown in Fig. 10, particle 1 is deflected by the scattering angle θ to the upper right, and particle 2 by the same angle to the lower left. In the laboratory frame, particle 2 is at rest before the collision, and consequently, the center of mass is moving with the velocity

$$\vec{V} = \frac{m_1}{M}\vec{u}_1 \,. \tag{136}$$

Since the total momentum is conserved, $\vec{V}$ is the same before and after the collision, and we can subtract it from the velocities $\vec{u}_i'$, $\vec{v}_i'$ to translate them to the laboratory frame:

$$v_1' = u_1' = u_1 - V = \left(1 - \frac{m_1}{M}\right) u_1 = \frac{m_2}{M} u_1 \,, \tag{137}$$

$$v_2' = u_2' = V = \frac{m_1}{M} u_1 \,. \tag{138}$$

Next, we relate the laboratory-frame scattering angle for particle 1, ψ, to the scattering angle θ in the center-of-mass frame. In components, we have

$$v_{1x} = v_1 \cos\psi = v_{1x}' + V = u_1 \left(\frac{m_2}{M}\cos\theta + \frac{m_1}{M}\right) \,, \tag{139}$$

$$v_{1y} = v_1 \sin\psi = v_{1y}' = u_1 \frac{m_2}{M} \sin\theta \,. \tag{140}$$

The final speed of particle 1 in the lab frame is then given by

$$v_1 = \frac{u_1}{M}\sqrt{m_1^2 + m_2^2 + 2 m_1 m_2 \cos\theta} \,, \tag{141}$$

Exercise 3.2: Recoil Angle in the Laboratory Frame

Show that the **recoil angle** ζ of the target is related to the scattering angle in the center-of-mass frame by

$$\cos\zeta = \sqrt{\frac{1-\sin\theta}{2}} = \sin\frac{\theta}{2}\,. \tag{E3.2-1}$$

and dividing the x and y components, we have

$$\tan\psi = \frac{m_2\sin\theta}{m_2\cos\theta + m_1} = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}\,. \tag{142}$$

Let us consider a few special cases:

- If the target is much heavier than the projectile, $m_2 \gg m_1$, the scattering angles and differential cross sections in the laboratory and center-of-mass frames are the same:

$$\psi = \theta\,, \quad \frac{d\sigma}{d\Omega_\psi} = \frac{d\sigma}{d\Omega}\,. \tag{143}$$

  This makes sense, because the center of mass of m_1 and m_2 will essentially coincide with the center of mass of m_2.

- If $m_1 = m_2$, we have

$$\tan\psi = \frac{\sin\theta}{\cos\theta + 1} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + 1} = \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}\,, \tag{144}$$

  i.e., $\psi = \frac{\theta}{2}$, and therefore

$$\frac{d\sigma}{d\Omega_\psi} = \frac{d\sigma}{d\Omega}\cdot\frac{\sin\theta}{\sin\frac{\theta}{2}}\frac{d\theta}{\frac{d\theta}{2}} = 2\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\frac{d\sigma}{d\Omega} = 4\cos\frac{\theta}{2}\cdot\frac{d\sigma}{d\Omega} = 4\cos\psi\cdot\frac{d\sigma}{d\Omega}\,. \tag{145}$$

3.7.3 Inelastic Scattering

The geometry of inelastic scattering processes is essentially the same as in Fig. 9, but since the kinetic energy is not conserved, the magnitudes of the initial and final velocities will differ. For particle 1, we have

$$\vec{v}_1 = \vec{v}_1{}' + \vec{V}\,. \tag{146}$$

Decomposing the vector in Cartesian components, we have

$$v_1\cos\psi = v_1'\cos\theta + V\,, \tag{147}$$

$$v_1\sin\psi = v_1'\sin\theta\,, \tag{148}$$

and squaring, we have

$$v_1^2 = v_1'^2 + 2v_1'V\cos\theta + V^2\,. \tag{149}$$

Taking the ratio of the components, we have

$$\tan\psi = \frac{\sin\theta}{\cos\theta + \frac{V}{v_1'}}\,, \tag{150}$$

analogous to Eq. (142). A similar analysis for particle 2 yields (cf. Fig. 9)

$$v_2 \cos\zeta = -v_2' \cos\theta + V\,, \tag{151}$$
$$-v_2 \sin\zeta = -v_2' \sin\theta\,, \tag{152}$$

and

$$v_2^2 = v_2'^2 - 2v_2' V \cos\theta + V^2\,. \tag{153}$$

Taking the ratio once more, we see that the **recoil angle** (cf. Eq. (E3.2-1)) is given by

$$\tan\zeta = \frac{\sin\theta}{\cos\theta - \frac{V}{v_2'}}\,. \tag{154}$$

As in the discussion of the elastic scattering, we have

$$V = \frac{m_1}{M} u_1 = \frac{m_1}{M} v_\infty\,. \tag{155}$$

In the center-of-mass frame, momentum conservation implies

$$m_1 \vec{v}_1' + m_2 \vec{v}_2' = 0 \tag{156}$$

and introducing the relative velocity $\vec{v}' = \vec{v}_1' - \vec{v}_2'$,

$$\vec{v}_1' = \frac{m_1}{M}\vec{v}'\,, \quad \vec{v}_2' = -\frac{m_2}{M}\vec{v}'\,. \tag{157}$$

The magnitudes of these vectors are given by

$$v_1' = \frac{m_1}{M} v'\,, \quad v_2' = \frac{m_2}{M} v'\,, \tag{158}$$

and putting everything together, we have

$$\rho_1 \equiv \frac{V}{v_1'} = \frac{m_1}{m_2}\frac{v_\infty}{v'}\,, \qquad \rho_2 \equiv \frac{V}{v_2'} = \frac{v_\infty}{v'}\,. \tag{159}$$

Since the target was initially at rest, v_∞ corresponds to the initial relative velocity of target and projectile. Thus, v_∞/v' is the ratio of the initial and final relative velocities, which is a measure of the loss of kinetic energy. In the limit of elastic scattering, $v_\infty = v'$, ρ_1 reduces to the familiar mass ratio m_1/m_2. Summarizing, we have

$$\boxed{\tan\psi = \frac{\sin\theta}{\cos\theta + \rho_1}\,, \qquad \tan\zeta = \frac{\sin\theta}{\cos\theta - \rho_2}\,.} \tag{160}$$

References

[1] W. H. Miller, The Journal of Chemical Physics **51**, 3631 (1969).

[2] J. V. José and E. J. Saletan, *Classical Dynamics: A Contemporary Approach* (Cambridge University Press, 1998).

4 Group Exercises

Problem G19 – Hard-Sphere Scattering

Consider the collision of two billard balls with radius $\frac{R}{2}$ in the rest frame of the target ball. We assume that the balls are not deformed in the collision, and can therefore be treated as hard spheres. Mathematically, this is equivalent to the scattering off a potential of the form

$$V(r) = \begin{cases} 0 & \text{for} \quad r > R\,, \\ \infty & \text{for} \quad r \leq R\,. \end{cases} \tag{161}$$

Show that the differential and total cross sections are

$$\frac{d\sigma}{d\Omega} = \frac{R^2}{4}\,, \tag{162}$$

and

$$\sigma = \pi R^2\,, \tag{163}$$

respectively.
HINT:

$$\int du \frac{1}{\sqrt{1-u^2}} = \arcsin u + c \tag{164}$$

Problem G20 – Total Cross Sections for Finite-Range Potentials

Show that the total cross-section for an arbitrary finite-range potential

$$V(r) = \begin{cases} 0 & \text{for} \quad r > R\,, \\ v(r) & \text{for} \quad r \leq R \end{cases} \tag{165}$$

is

$$\sigma = \pi R^2\,. \tag{166}$$

HINT: Start from Eq. (67) and cosonder appropriate integration limits at all stages of your calculation.