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PHY422/820: Classical Mechanics

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Worksheet #9 (Oct 25 – Oct 29)

November 3, 2021

1 Preparation

- Lemos, Chapter 3
- Goldstein, Section 3.12, Chapter 4 (skip 4.5)

2 Orbital Dynamics

In the following we want to discuss two important orbital maneuvers based on our solutions for the Kepler problem: The transfer of a spacecraft between two Kepler orbits, and the gravitational assist or slingshot.

2.1 Transfer Orbits

An important problem in orbital dynamics is the determination of optimal ways to send a spacecraft to another planet. Since craft can only carry limited fuel, this usually means to find a so-called **minimum-energy transfer orbit**, or **Hohmann transfer orbit**, named for its discoverer. Such an orbit is shown schematically in Fig. 1.

A spacecraft that is starting from a circular low-Earth orbit burns fuel to achieve a velocity boost $\Delta\vec{v}$ at the right time to lift it out of Earth's gravity well and place it on an elliptical orbit around the sun. Eventually, it will reach the orbit of the destination planet around the sun, and a second boost $\Delta\vec{v}$ will "insert" the craft in a circular orbit in that planet's gravity well. It is clear that such a transfer orbit will crucially depend on the locations of the planets' relative to each other, which defines the so-called **launch window**: The destination planet better be near the rendezvous point when the spacecraft arrives on its transfer orbit. Figure 1 also suggests that the timing matters for the velocities: The transfer will be most efficient if the spacecraft is launched from its Earth orbit with a velocity $\vec{v}_\infty$ that has the same direction as Earths velocity around the Sun, because then the velocity of the spacecraft in the Sun's reference frame will have its largest possible magnitude, $v_E + v_\infty$. Likewise, the rendezvous should be timed such that the relative velocity of the craft and the destination planet is small, to allow a controlled orbital insertion.

Before we discuss the necessary steps of the transfer maneuver, we can determine how long it will take the spacecraft to reach its destination using Kepler's Third Law (cf. worksheet #8). We recall that

$$T = 2\pi\sqrt{\frac{a^3\mu_S}{k_S}}\,, \quad k_S = GM_Sm\,, \quad \mu_S \equiv \frac{M_Sm}{m+M_S}\,, \tag{1}$$

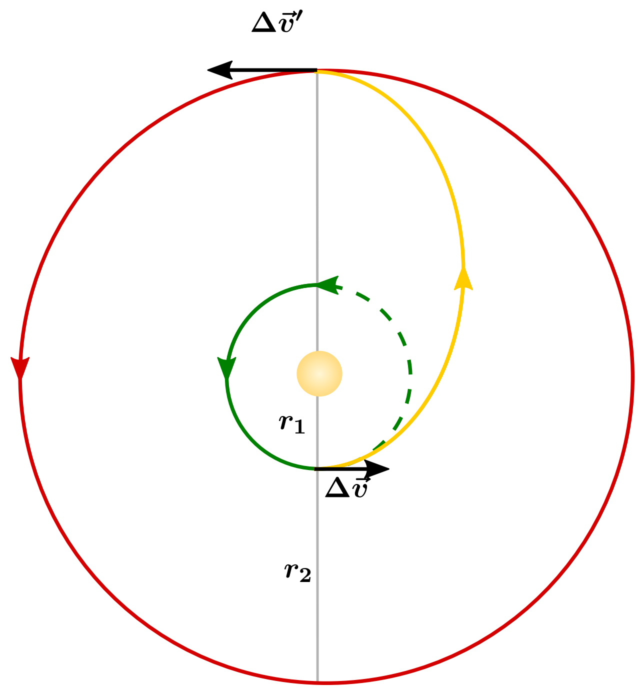

Figure 1: A Hohmann transfer orbit between low-eccentricity orbits with radii r_1 and r_2.

where the potential strength and reduced mass refer to the system consisting of the sun and the object in question. Expressing the period of the craft's transfer orbit in terms of the Earth's orbital period T_E, we have

$$\frac{T}{T_E} = \sqrt{\frac{a^3}{r_1^3}}\,, \tag{2}$$

Figure 1 also indicates that the semi-major axis of the transfer orbit is

$$a = \frac{r_1 + r_2}{2}\,, \tag{3}$$

as a consequence of the optimal launch window discussed before. Thus, the orbital period of the transfer orbit is

$$T = \left(\frac{r_1 + r_2}{2r_1}\right)^{3/2} T_E\,, \tag{4}$$

and the craft's travel time is $\Delta t = T/2$, since it will only travel the outgoing leg of the orbit.

The orbital transfer now proceeds in the following steps:

- The spacecraft is lifted into a parking orbit with radius R_0 and a speed $v_0 = \sqrt{\frac{GM_E}{R_0}}$ (cf. problem G21).

- At the appropriate time, the craft receives a boost $\Delta\vec{v}$ in the same direction as $\vec{v}_0$ and the Earth's velocity $\vec{v}_E$ in order to allow an efficient escape from the Earth's gravity well. Energy conservation implies that

$$\frac{1}{2}\mu_E v_\infty^2 = \frac{1}{2}\mu_E (v_0 + \Delta v)^2 - \frac{GM_E m}{R_0}\,, \tag{5}$$

where $\mu_E \approx m$ is the reduced mass of the spacecraft and the Earth. Therefore,

$$v_\infty = \sqrt{(v_0 + \Delta v)^2 - \frac{2GM_E m}{\mu_E R_0}} \approx \sqrt{(v_0 + \Delta v)^2 - 2v_0^2}\,, \tag{6}$$

The craft is now traveling on a *hyperbolic* relative to the Earth, but is still close to Earth's orbit around the sun.

- Assuming we timed the launch properly, the spacecraft's velocity in the reference frame of the sun will now be

$$v = v_\infty + v_E\,, \tag{7}$$

and it is moving under the influence of the sun's gravity alone. As indicated in Fig. 1, we consider the craft to be at the perihelion of the transfer orbit now. This implies

$$E = \frac{1}{2}\mu_S v^2 - \frac{GM_S m}{r_1} = -\frac{GM_S m}{2a}\,, \tag{8}$$

where we have used that

$$a = -\frac{k}{2E} \tag{9}$$

for Kepler ellipses (cf. worksheet #8). We can once again approximate the reduced mass of spacecraft and sun, $\mu_S \approx m$. Solving for v, we obtain

$$v^2 = GM_S\left(\frac{2}{r_1} - \frac{1}{a}\right) = 2GM_S\left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right)\,, \tag{10}$$

and we can put everything together. Equation (7) yields

$$\begin{aligned} v_\infty^2 &= (v - v_E)^2 = v^2 - 2vv_E + v_E^2 \\ &= GM_S\left(\frac{2}{r_1} - \frac{1}{a}\right) - 2v_E\sqrt{GM_S\left(\frac{2}{r_1} - \frac{1}{a}\right)} + v_E^2 \end{aligned} \tag{11}$$

and combining this with Eq. (6), we have

$$\Delta v^2 + 2v_0\Delta v = GM_S\left(\frac{2}{r_1} - \frac{1}{a}\right) - 2v_E\sqrt{GM_S\left(\frac{2}{r_1} - \frac{1}{a}\right)} + v_E^2 + v_0^2\,, \tag{12}$$

whose solutions determine the required Δv.

2.2 Gravitational Assist

The second example we want to discuss is a simplified version of the **gravitational assist** or **gravitational slingshot**, which is used in interplanetary travel to increase the velocity of a spacecraft relative to the Sun, without burning fuel. Our particular example is inspired by the trajectory of the Voyager probes: We will consider a small spacecraft that is scattered by Jupiter's gravitational potential to boost its speed and take it from one heliocentric orbit to another.

The first step is to set the spacecraft on a transfer orbit whose perihelion is near Earth, and whose aphelion is just *beyond* Jupiter's orbit around the sun. The scattering process will occur in a sufficiently short time and spatial region to allow us to model it as an elastic scattering event in an inertial reference frame that is moving with Jupiters orbital velocity.

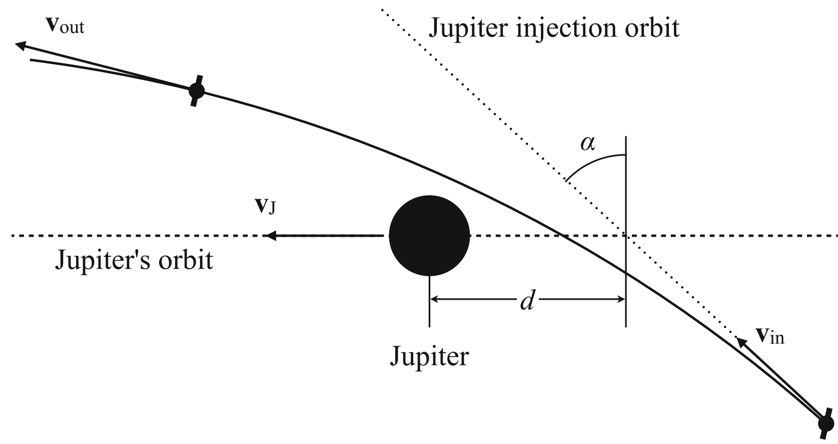

Figure 2: Gravitational scattering of a spacecraft with incoming velocity $\vec{v}_{\rm in}$ off Jupiter's potential.

Figure 2.2 show the process in the Sun's reference frame. The incoming trajectory of the spacecraft — tangential to the Hohmann transfer orbit that took it to Jupiter — will intersect Jupiter's orbit around the sun with an angle α, at some distance d from the planet itself. Using

$$r(\alpha) = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\alpha}\,, \qquad \left.\frac{dr}{d\phi}\right|_{\phi=\alpha} = \frac{\epsilon\sin\alpha}{a(1-\epsilon^2)}r^2\,, \tag{13}$$

(cf. homework H12), we can determine α from

$$\tan\alpha = \frac{v_\phi}{v_r} = \frac{r\dot{\phi}}{\dot{r}} = r\left(\frac{dr}{d\phi}\right)^{-1} = \sqrt{\frac{1-\epsilon^2}{\epsilon^2-\left(1-\frac{r}{a}\right)^2}}\,. \tag{14}$$

Next, we need to find the impact parameter b and the scattering angle θ in Jupiter's reference frame (cf. Fig. 2.2. We can change frames by subtracting $\vec{v}_J$ from all velocities in the problem (cf. worksheet #8), which immediately yields

$$\vec{v}_i = \vec{v}_{\rm in} - \vec{v}_J\,. \tag{15}$$

Referring to the inset in Fig. 2.2, we see that

$$v_i^2 = v_{\rm in}^2 - 2v_{\rm in}v_J\sin\alpha + v_J^2\,, \tag{16}$$

and that the angle between $\vec{v}_i$ and the axis defined by Jupiter's orbit is given by

$$\tan\beta = \frac{v_{\rm in}\cos\alpha}{v_J - v_{\rm in}\sin\alpha}\,. \tag{17}$$

This allows us to determine the impact parameter using

$$b = d\sin\beta\,. \tag{18}$$

With this impact parameter b, we can determine the scattering angle in Jupiter's reference frame from

$$\cot\frac{\theta}{2} = \frac{bv_i^2}{2GM_J}\,, \tag{19}$$

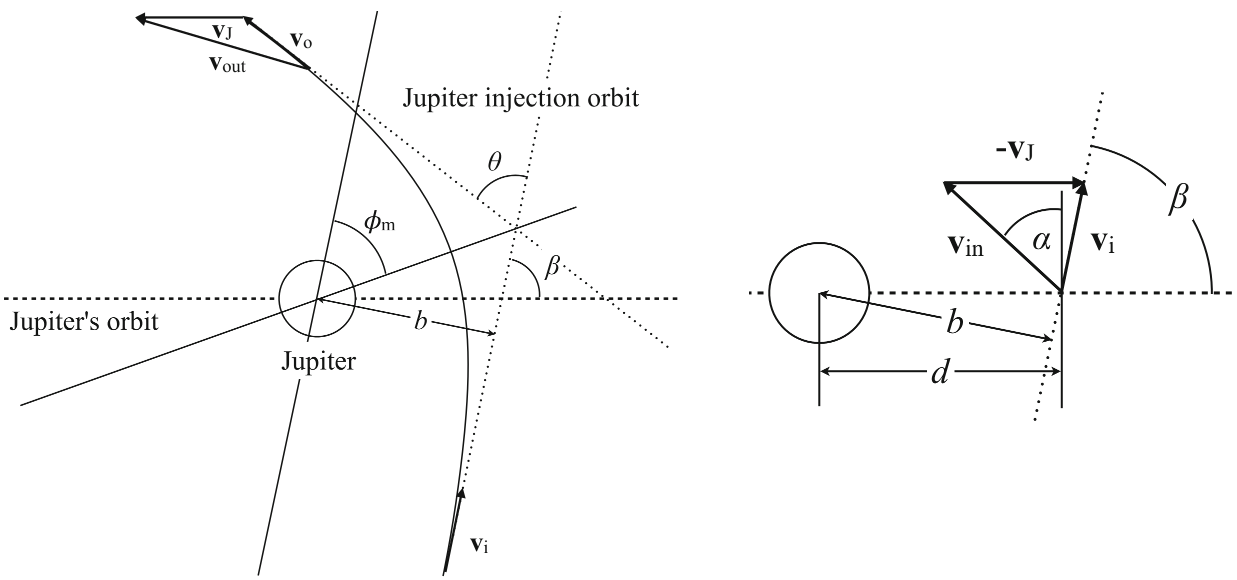

Figure 3: Gravitational scattering of the spacecraft with incoming velocity $\vec{v}_\mathrm{i}$ in Jupiter's reference frame.

where we have used $v_\infty = v_i$ and M_J is the mass of Jupiter (cf. worksheet #8). Since the scattering process is elastic, we know that the outgoing speed will be identical to the incoming speed, $v_o = v_i$, but as we can see from Fig. 2.2 the angle the velocity vector makes with Jupiter's orbit changes from β to $\theta + \beta$. This change in angle leads to an increase of $\vec{v}_\mathrm{out}$ in the Sun's reference frame: This is already indicated by the figure, but we can explicitly derive the change between v_in and v_out via

$$\begin{aligned} v_\mathrm{out}^2 - v_\mathrm{in}^2 &= (\vec{v}_o + \vec{v}_J)^2 - (\vec{v}_i + \vec{v}_J)^2 \\ &= 2 v_i v_J \left(\cos\left(\pi - (\theta + \beta)\right) - \cos\left(\pi - \beta\right)\right) \\ &= 2 v_i v_J \left(-\cos\left(\theta + \beta\right) + \cos\beta\right) = 2 v_i v_J \left(\cos\beta - \cos\left(\theta + \beta\right)\right) > 0\,. \end{aligned} \tag{20}$$

Here, we have used that $v_o = v_i$ and that the cosine decreases monotonically from 0 to π.

In general, the expressions for $\vec{v}_\mathrm{out}$ are rather complicated and not particularly insightful, so we will just summarize the steps of the general calculation:

1. Determine $\vec{v}_\mathrm{in}$ from the properties of the transfer orbit, and transform it to the "target" planet's reference frame to obtain $\vec{v}_i$.
2. Find α using Eq. (14), where r is the radius of the target planet's orbit.
3. Determine β from α.
4. Calculate b from β and the distance d at which the spacecraft intersects the planet's orbit (Eq. (18)).
5. Compute the gravitational scattering angle θ.
6. Compute the outgoing velocity v_o and transform it back to the Sun's reference frame.

Exercise 2.1: Gravitational Slingshot

Verify Eqs. (14), (16), (17).

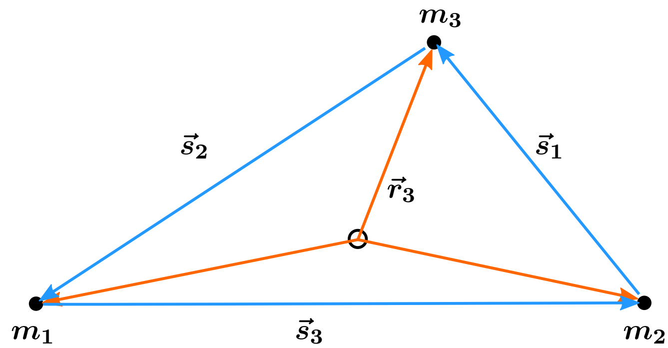

Figure 4: Three-body problem in the center-of-mass system.

3 Three-Body Systems

While we have extensively discussed the exact solutions of the gravitational two-body problem in recent weeks, the addition of an additional mass will, in general, prevent analytic solutions. Here, we still want to discuss general aspects of the problem, as well as some special cases in which an analytic treatment is possible.

3.1 The General Case

Figure 4 shows a system of three masses in their center-of-mass system. The equations of motion read

$$m_1\ddot{\vec{r}}_1 = -Gm_1m_2\frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|^3} - Gm_1m_3\frac{\vec{r}_1-\vec{r}_3}{|\vec{r}_1-\vec{r}_3|^3}\,, \tag{21}$$

$$m_2\ddot{\vec{r}}_2 = -Gm_2m_1\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_2-\vec{r}_1|^3} - Gm_2m_3\frac{\vec{r}_2-\vec{r}_3}{|\vec{r}_2-\vec{r}_3|^3}\,, \tag{22}$$

$$m_3\ddot{\vec{r}}_3 = -Gm_3m_1\frac{\vec{r}_3-\vec{r}_1}{|\vec{r}_3-\vec{r}_1|^3} - Gm_3m_2\frac{\vec{r}_3-\vec{r}_2}{|\vec{r}_3-\vec{r}_2|^3}\,. \tag{23}$$

Now we introduce the relative vectors

$$\vec{s}_1 = \vec{r}_3 - \vec{r}_2\,, \quad \vec{s}_2 = \vec{r}_1 - \vec{r}_3\,, \quad \vec{s}_3 = \vec{r}_2 - \vec{r}_1\,. \tag{24}$$

Noting that

$$\vec{s}_1 + \vec{s}_2 + \vec{s}_3 = 0\,, \tag{25}$$

and using

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 = 0\,, \tag{26}$$

we see that they are related to the $\vec{r}_i$ by

$$\vec{r}_1 = \frac{1}{M}\left(m_3\vec{s}_2 - m_2\vec{s}_3\right)\,, \quad \vec{r}_2 = \frac{1}{M}\left(m_1\vec{s}_3 - m_3\vec{s}_1\right)\,, \quad \vec{r}_3 = \frac{1}{M}\left(m_2\vec{s}_1 - m_1\vec{s}_2\right)\,. \tag{27}$$

Using the $\vec{s}_i$, we can rewrite the equations of motion as

$$\ddot{\vec{s}}_i = m_i\vec{G} - GM\frac{\vec{s}_i}{s_i^3}\,, \qquad s_i = |\vec{s}_i|\,, \tag{28}$$

where

$$\vec{G} \equiv G\left(\frac{\vec{s}_1}{s_1^3} + \frac{\vec{s}_2}{s_2^3} + \frac{\vec{s}_3}{s_3^3}\right)\,. \tag{29}$$

Let us now discuss some of the special solutions to these equations.

Euler's Collinear Solutions

Euler discovered a class of solutions in which the masses are collinear and their distances are in a fixed ratio. Assuming that m_2 lies between the other masses for specificity, we have

$$\vec{s}_1 = \lambda\vec{s}_3\,, \quad \vec{s}_2 = -(1+\lambda)\vec{s}_3\,. \tag{30}$$

From the equation of motion for $\vec{s}_3$, we have

$$\vec{G} = \frac{1}{m_3}\left(\ddot{\vec{s}}_3 + GM\frac{\vec{s}_3}{s_3^3}\right)\,, \tag{31}$$

which we can use to eliminate $\vec{G}$ in the remaining equations of motion:

$$\ddot{\vec{s}}_1 + GM\frac{\vec{s}_1}{s_1^3} = \frac{m_1}{m_3}\left(\ddot{\vec{s}}_3 + GM\frac{\vec{s}_3}{s_3^3}\right)\,, \tag{32}$$

$$\ddot{\vec{s}}_2 + GM\frac{\vec{s}_2}{s_2^3} = \frac{m_2}{m_3}\left(\ddot{\vec{s}}_3 + GM\frac{\vec{s}_3}{s_3^3}\right)\,. \tag{33}$$

Plugging in the relations (30) and rearranging, we obtain

$$\left(m_2 + m_3\,(1+\lambda)\right)\ddot{\vec{s}}_3 = -GM\left(m_2 + \frac{m_3}{(1+\lambda)^2}\right)\frac{\vec{s}_3}{s_3^3}\,, \tag{34}$$

$$\left(m_1 - m_3\lambda\right)\ddot{\vec{s}}_3 = -GM\left(m_1 - \frac{m_3}{\lambda^2}\right)\frac{\vec{s}_3}{s_3^3}\,. \tag{35}$$

Thus,

$$\frac{m_2 + m_3\,(1+\lambda)}{m_1 - m_3\lambda} = \frac{m_2 + \frac{m_3}{(1+\lambda)^2}}{m_1 - \frac{m_3}{\lambda^2}} \tag{36}$$

and we find

$$\begin{aligned} P(\lambda) = (m_1+m_2)\lambda^5 + (3m_1+2m_2)\lambda^4 + (3m_1+m_2)\lambda^3 \\ - (m_2+3m_3)\lambda^2 - (2m_2+3m_3)\lambda - (m_2+m_3) = 0\,. \end{aligned} \tag{37}$$

We can use Descartes' sign rule for polynomials to determine that $P(\lambda)$, whose coefficients have the signature $(+,+,+,-,-,-)$, has at most one positive real root. We also note that

$$P(0) < 0\,, \qquad \lim_{\lambda\to+\infty} P(\lambda) = +\infty\,, \tag{38}$$

so a positive root must exist. This implies the existence of a unique solution λ for given mass ratios $m_1 : m_2 : m_3$. We can then plug this solution either into Eq. (34) or (35) to determine $\vec{s}_3(t)$, and from this the $\vec{r}_i(t)$. A (schematic) solution is shown in Fig. 5. Additional solutions can be found by swapping the mass in the center.

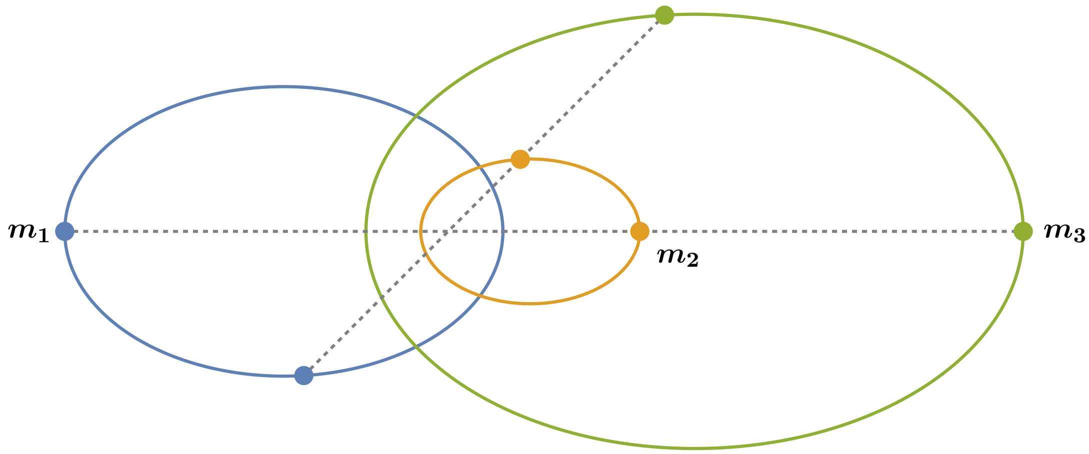

Figure 5: Euler's collinear solution for the three-body problem.

Lagrange's Triangular Solutions

Equation (28) implies that the equations of motion decouple if $\vec{G} = 0$, which turns the three-body problem in to a set of three separate gravitational problems which we can solve with the usual techniques. Lagrange proved that this condition can be met if the interacting objects form an equilateral triangle at all times, regardless of their masses, with trajectories given by

$$\vec{r}_1 = -\frac{2m_2 + m_3}{2M}\vec{s}_3 + \frac{\sqrt{3}}{2}\frac{m_3}{M}\vec{n} \times \vec{s}_3 \,, \tag{39}$$

$$\vec{r}_2 = \frac{2m_1 + m_3}{2M}\vec{s}_3 + \frac{\sqrt{3}}{2}\frac{m_3}{M}\vec{n} \times \vec{s}_3 \,, \tag{40}$$

$$\vec{r}_3 = \frac{m_1 - m_2}{2M}\vec{s}_3 + \frac{\sqrt{3}}{2}\frac{m_1 + m_2}{M}\vec{n} \times \vec{s}_3 \,. \tag{41}$$

Here, $\vec{n}$ is a unit vector that is normal to the plane of the triangle. Its size and orientation will change during the masses' motion, as illustrated in Fig. 6. We will work through the derivation of Lagrange's triangular solutions in problem G22.

3.2 The Restricted Three-Body Problem

3.2.1 The Synodic Frame

The **restricted three-body problem** refers to the case where one of the masses is much smaller than the other two, $m \ll M_1, M_2$. Common examples are spacecraft that move within the gravitational fields of two primary bodies, e.g., the Sun and a planet, or the Earth and the Moon, as well as objects in the asteroid belt that are subject to the gravity of the Sun and Jupiter. To describe the motion of the mass m, we introduce the so-called **synodic** coordinate system shown in Fig. 7. Its origin lies in the center of mass of the two heavy masses, which is essentially the center of mass of the three-body system as well, and we assume that it *rotates* with the vector connecting the primary masses, which is $\vec{s} = \vec{s}_3$ in the conventions from before. Let us define

$$\alpha \equiv \frac{M_2}{M_1 + M_2}\,, \quad \beta \equiv 1 - \alpha = \frac{M_1}{M_1 + M_2}\,, \tag{42}$$

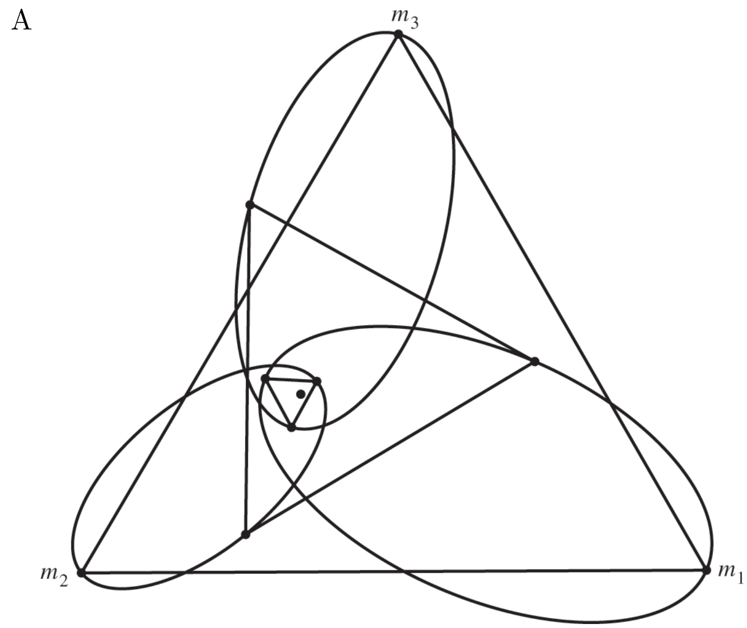

Figure 6: Lagrange's triangular solutions for the three-body problem: The masses m_i form equilateral triangles of different sizes and orientations at all times.

and choose the x-axis to coincide with with direction of $\vec{s}$. Then

$$\vec{r}_1 = -\alpha s \vec{e}_x\,, \quad \vec{r}_2 = \beta s \vec{e}_x\,, \tag{43}$$

and the position vector of the mass m is

$$\vec{r} \equiv \vec{r}_3 = x\vec{e}_x + y\vec{e}_y\,. \tag{44}$$

This immediately tells us that the distances of m from the masses M_1 and M_2 are given by

$$s_1 = \sqrt{(x+\alpha s)^2 + y^2}\,, \quad s_2 = \sqrt{(x-\beta s)^2 + y^2}\,, \tag{45}$$

respectively. The three masses span the xy plane of the coordinate system, and we have $\vec{l} = l\vec{e}_z$, as in the two-body Kepler problem.

Assuming that the orbit is nearly circular[1] , the angular velocity with which the coordinate system rotates can be determined using Kepler's Third Law: Noting that the semi-major axis of the ellipse The period T of the revolution is

$$T = 2\pi\sqrt{\frac{s^3}{G(M_1+M_2)}} \tag{46}$$

(cf. worksheet #8), hence the angular velocity is

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{G(M_1+M_2)}{s^3}}\,. \tag{47}$$

For an ellipse, this is only the *average* angular velocity, because ϕ must change to compensate the changing distance between the mass and the center of the gravitational potential in order to keep $\vec{l}$ constant. For a circular trajectory, $\vec{\omega} \parallel \vec{l}$, i.e., $\vec{\omega} = \omega\vec{e}_z = \text{const.}$.

[1]This is an excellent approximation for many important applications in the Solar system, the planet-moon systems etc.

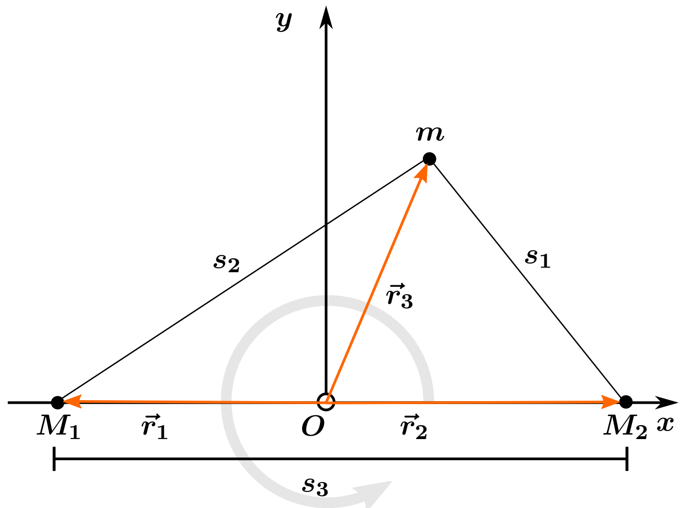

Figure 7: The synodic coordinate system for the restricted three-body problem.

3.2.2 Lagrangian in a Rotating Frame

Let us now construct the Lagrangian and the Lagrange equations in the synodic coordinate system, as parameterized by the coordinates $\vec{r}$ and velocities $\dot{\vec{r}}$. For comparison, we will also need an inertial frame, in which the position and velocity of $m$are given by $\vec{r}', \dot{\vec{r}}'$.

The potential is the same in both frames since it only depends on the distances of the masses, which are invariant under a rotation. For the velocity, we have the relation

$$\dot{\vec{r}}' = \dot{\vec{r}} + \vec{\omega} \times \vec{r}\,, \tag{48}$$

hence the kinetic energy is given by

$$T = \frac{1}{2} m \dot{\vec{r}}'^2 = \frac{1}{2} m \left(\dot{\vec{r}} + \vec{\omega} \times \vec{r} \right)^2 = \frac{1}{2} m \dot{\vec{r}}^2 + m \dot{\vec{r}} \cdot (\vec{\omega} \times \vec{r}) + \frac{1}{2} m \left(\vec{\omega} \times \vec{r} \right)^2 \,. \tag{49}$$

Thus, the Lagrangian in the noninertial frame is given by

$$L = \frac{1}{2} m \dot{\vec{r}}^2 + m \dot{\vec{r}} \cdot (\vec{\omega} \times \vec{r}) + \frac{1}{2} m \left(\vec{\omega} \times \vec{r} \right)^2 - V(\vec{r}) \equiv \frac{1}{2} m \dot{\vec{r}}^2 - U(\vec{r}, \dot{\vec{r}})\,, \tag{50}$$

where we have introduced the velocity-dependent potential

$$U(\vec{r}, \dot{\vec{r}}) = V(\vec{r}) - m \dot{\vec{r}} \cdot (\vec{\omega} \times \vec{r}) - \frac{1}{2} \left(\vec{\omega} \times \vec{r} \right)^2 \,. \tag{51}$$

The associated generalized force is

$$\begin{aligned}
\vec{Q} &= \frac{d}{dt} \frac{\partial U}{\partial \dot{\vec{r}}} - \frac{\partial U}{\partial \vec{r}} \\
&= -m \frac{d}{dt} \left(\vec{\omega} \times \vec{r} \right) + m \frac{\partial}{\partial \vec{r}} \left(\dot{\vec{r}} \cdot (\vec{\omega} \times \vec{r}) + \frac{1}{2} \left(\vec{\omega} \times \vec{r} \right)^2 \right) - \frac{\partial V}{\partial \vec{r}} \\
&= -m \dot{\vec{\omega}} \times \vec{r} - m \vec{\omega} \times \dot{\vec{r}} + m \frac{\partial}{\partial \vec{r}} \left(\left(\dot{\vec{r}} \times \vec{\omega} \right) \cdot \vec{r} + \frac{1}{2} \left(\omega^2 r^2 - (\vec{\omega} \cdot \vec{r})^2 \right) \right) - \frac{\partial V}{\partial \vec{r}}
\end{aligned}$$

$$
\begin{aligned}
&= -m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times \dot{\vec{r}} + m\dot{\vec{r}} \times \vec{\omega} + \frac{1}{2}m\left(2\omega^2\vec{r} - 2(\vec{\omega}\cdot\vec{r})\omega\right) - \frac{\partial V}{\partial \vec{r}} \\
&= -m\dot{\vec{\omega}} \times \vec{r} - 2m\vec{\omega} \times \dot{\vec{r}} + m\vec{\omega} \times (\vec{r} \times \vec{\omega}) - \frac{\partial V}{\partial \vec{r}} \\
&= -m\dot{\vec{\omega}} \times \vec{r} - 2m\vec{\omega} \times \dot{\vec{r}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - \frac{\partial V}{\partial \vec{r}} \,.
\end{aligned}
\tag{52}
$$

We can identify the following contributions, which are the usual **pseudoforces** or **fictitious forces** associated with a rotating frame:

1. the **Euler force**
$$\vec{F}_E \equiv -m\dot{\vec{\omega}} \times \vec{r}\,, \tag{53}$$
which is caused by any variation of $\vec{\omega}$,

2. the **Coriolis force**
$$\vec{F}_C \equiv -2m\vec{\omega} \times \dot{\vec{r}}\,, \tag{54}$$

3. and the **centrifugal force**
$$\vec{F}_{cf} \equiv -m\vec{\omega} \times (\vec{\omega} \times \vec{r})\ . \tag{55}$$

For the circular restricted three-body problem under consideration, $\omega = \text{const.}$ and $\vec{\omega} \perp \vec{r}$, hence $\vec{F}_E = 0$ and the centrifugal term simplifies to

$$\vec{F}_{cf} = -m\left(\omega^2\vec{r} - \underbrace{(\vec{\omega}\cdot\vec{r})}_{=0}\vec{\omega}\right) = -m\omega^2\vec{r}. \tag{56}$$

The Lagrange equations for the Lagrangian (50) read

$$m\ddot{\vec{r}} + m\dot{\vec{\omega}} \times \vec{r} + 2m\vec{\omega} \times \dot{\vec{r}} + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{\partial V}{\partial \vec{r}} = 0\,, \tag{57}$$

and with the aforementioned simplifications,

$$m\ddot{\vec{r}} + 2m\vec{\omega} \times \dot{\vec{r}} + m\omega^2\vec{r} - \frac{\partial V}{\partial \vec{r}} = 0\,. \tag{58}$$

3.2.3 Jacobi Integral and Roche Potential

Since the Lagrangian (50) does not depend explicitly on time, we know that the Jacobi integral will be conserved:

$$
\begin{aligned}
h(\vec{r},\dot{\vec{r}}) &= \frac{\partial L}{\partial \dot{\vec{r}}} \cdot \dot{\vec{r}} - L \\
&= m\left(\dot{\vec{r}} + \vec{\omega} \times \vec{r}\right) \cdot \dot{\vec{r}} - \frac{1}{2}m\dot{\vec{r}}^2 - m\dot{\vec{r}} \cdot (\vec{\omega} \times \vec{r}) - \frac{1}{2}m\,(\vec{\omega} \times \vec{r})^2 + V(\vec{r}) \\
&= \frac{1}{2}m\dot{\vec{r}}^2 - \frac{1}{2}m\left(\omega^2 r^2 - \underbrace{(\vec{\omega}\cdot\vec{r})}_{=0}{}^2\right) + V(\vec{r}) = \text{const.}
\end{aligned}
\tag{59}
$$

Dividing by m and introducing the **Roche potential**

$$\Phi(\vec{r}) = \frac{V(\vec{r})}{m} - \frac{1}{2}\omega^2 r^2\,, \tag{60}$$

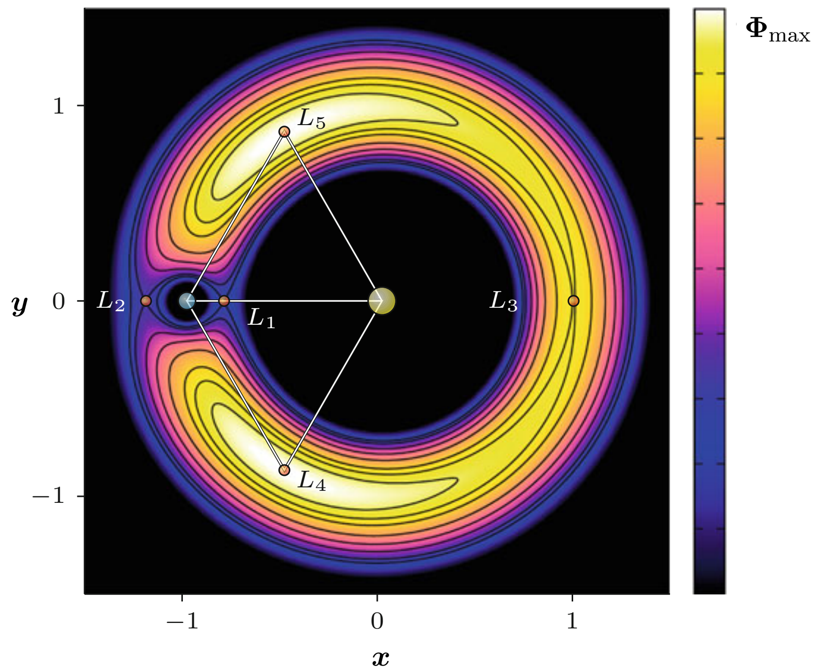

Figure 8: Roche potential and Lagrange points L_1–L_5 for $M_2/M_1 = 40$. Equipotential lines are shown in black.

we can rewrite the conservation law as

$$h = \frac{1}{2}\dot{\vec{r}}^2 + \Phi(\vec{r}) = \text{const.} \tag{61}$$

This is, in fact, the *original* Jacobi integral that gave all others its name. It was derived (with an opposite sign) in Jacobi's original work on the restricted three-body problem.

Using Eq. (47), we can rewrite the Roche potential as

$$\begin{aligned} \Phi(\vec{r}) &= -\frac{GM_1}{s_1} - \frac{GM_2}{s_2} - \frac{M_1 + M_2}{2s^3} r^2 \\ &= -\frac{GM_1}{\sqrt{(x + \alpha s)^2 + y^2}} - \frac{GM_2}{\sqrt{(x - \beta s)^2 + y^2}} - \frac{G(M_1 + M_2)}{2} \frac{x^2 + y^2}{s^3}\,. \end{aligned} \tag{62}$$

A contour plot for $M_2/M_1 = 40$ is shown in Fig. 8. Equipotential lines are shown in black. At large distances, the third term dominates and the potential is approximately rotationally symmetric. In the vicinity of the primary masses M_1 and M_2, the potential is dominated by the first and second terms in Eq. (62), respectively. The equipotential lines are initially disjointed and nearly circular, but as Φ grows, they become distorted into teardrop shapes that eventually touch in the point L_1, which will be discussed below. The regions enclosed by the critical equipotential line that goes through L_1 are the so-called **Roche lobes**: A mass orbiting within either of them will be gravitationally bound to the corresponding primary mass M_1 or M_2. Since $M_2 \gg M_1$, its Roche lobe is significantly bigger than that of M_1.

3.2.4 Lagrange Points

The Collinear Points L_1, L_2, L_3

Let us now try and determine whether the equation of motion (62) admits any equilibrium solutions, which would indicate points at which our mass m would remain at rest relative to the primary masses M_1 and M_2. Rewriting Eq. (58) in terms of the Roche potential, we have

$$\ddot{\vec{r}} + 2\vec{\omega} \times \dot{\vec{r}} + \frac{\partial \Phi}{\partial \vec{r}} = 0\,. \tag{63}$$

Assuming static equilibrium (in the synodic frame), we have $\ddot{\vec{r}} = \dot{\vec{r}} = 0$, and the equation of motion turns into

$$\frac{\partial \Phi}{\partial \vec{r}} = 0\,. \tag{64}$$

The solutions are the extrema of the Roche potential. Evaluating the gradient, we have

$$0 = \frac{\partial \Phi}{\partial x} = \frac{GM_1(x+\alpha s)}{((x+\alpha s)^2 + y^2)^{3/2}} + \frac{GM_2(x-\beta s)}{((x-\beta s)^2 + y^2)^{3/2}} - \omega^2 x\,, \tag{65}$$

$$\begin{aligned} 0 = \frac{\partial \Phi}{\partial y} &= \frac{GM_1 y}{((x+\alpha s)^2 + y^2)^{3/2}} + \frac{GM_2 y}{((x-\beta s)^2 + y^2)^{3/2}} - \omega^2 y \\ &= y\left(\frac{GM_1}{((x+\alpha s)^2 + y^2)^{3/2}} + \frac{GM_2}{((x-\beta s)^2 + y^2)^{3/2}} - \omega^2\right) \end{aligned} \tag{66}$$

From the second equation, we immediately obtain a class of solutions with $y = 0$. This means that these extrema are collinear, and that they result from Euler's collinear solutions (cf. Sec. 3.1). Thus, it is no surprise that we will obtain a fifth-order polynomial when we plug in $y = 0$ into Eq. (65). This equation cannot be solved analytically, but we can visualize the solution by graphing a cross section of $\Phi(x, y)$ for $y = 0$. This is shown in Fig. 9 for the same parameters that were used in Fig. 8.

We see that the extrema are *local maxima* of $\Phi(x, 0)$ and *saddle points* of the full Roche potential $\Phi(x, y)$. Adopting the usual naming conventions, the extrema are referred to as the **Lagrange points** L_1, L_2, L_3. L_1 lies between the primary masses, where the Roche lobes touch, L_2 lies behind on the far side of the mass M_1 as viewed from M_2, and similarly, L_3 lies on the far side of M_2.

While these Lagrange points are *unstable equilibria*, they play an important role in spaceflight because the gravitational forces nearly cancel out in these points. Thus, a space craft that is stationed at one of the Lagrange points can keep its position by regularly performing trajectory adjustments at a small fuel expenditure. Similarly, Lagrange points are optimal locations for performing course adjustment on long-distance missions, and the trajectories of space craft are designed to incorporate Lagrange points of different three-body systems within our greater Solar system: e.g., the Earth-Moon L_1 point was used in the Apollo program's trajectories, and the James Webb telescope, the "successor" to the Hubble telescope, will join probes like Planck and WMAP at the Sun-Earth L_2 point, where the Earth shields them from the Solar wind.

The Triangular Points L_4 and L_5

Since we found Lagrange points stemming from Euler's collinear solutions to the three-body problem, it stands to reason that we can find additional Lagrange points associated with the triangular

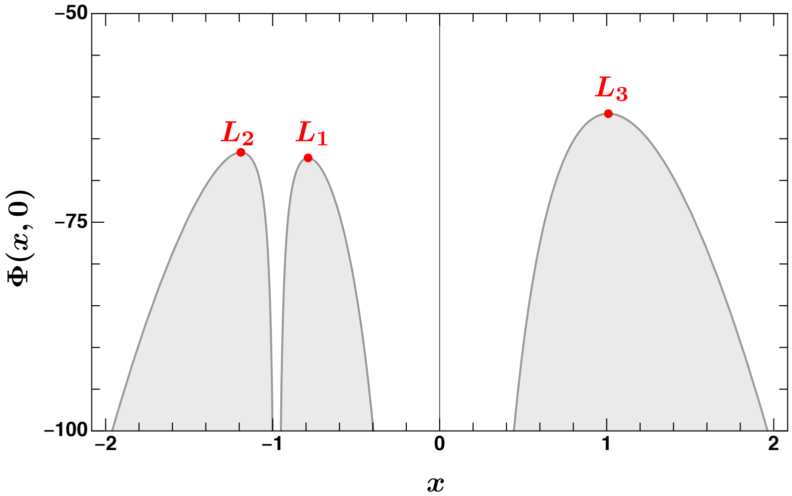

Figure 9: Cross section of the Roche potential $\Phi(x, 0)$. The Lagrange points L_1, L_2 and L_3 are highlighted in red.

solution (39)–(41). To do so, we consider Eq. (41) for our mass m in the synodic frame. First, we note that

$$M = m + M_1 + M_2 \approx M_1 + M_2 \tag{67}$$

hence

$$\vec{r} = \frac{M_1 - M_2}{2(M_1 + M_2)}\vec{s} - \frac{\sqrt{3}}{2}\vec{n} \times \vec{s}\,. \tag{68}$$

Using

$$\frac{M_1 - M_2}{M_1 + M_2} = \beta - \alpha = \beta - (1 - \beta) = 2\beta - 1 \tag{69}$$

as well as $\vec{s} = s\vec{e}_x$ and $\vec{n} = \vec{e}_z$, we obtain

$$\vec{r} = \frac{2\beta - 1}{2}s\vec{e}_x - \frac{\sqrt{3}}{2}s\vec{e}_z \times \vec{e}_x = \frac{2\beta - 1}{2}s\vec{e}_x - \frac{\sqrt{3}}{2}s\vec{e}_y\,. \tag{70}$$

Since the synodic frame is symmetric under a reflection across the x axis (i.e., $y \to -y$), we get a second solution with the opposite sign in the y component. Thus, we have found two additional Lagrange points

$$L_4 : \left(\frac{(2\beta - 1)}{2}s, -\frac{\sqrt{3}}{2}s, 0\right)\,, \qquad L_5 : \left(\frac{(2\beta - 1)}{2}s, \frac{\sqrt{3}}{2}s, 0\right)\,. \tag{71}$$

Let us plug them into Eqs. (65)–(66) to check if they are extrema of the Roche potential. First, we note that for both points

$$x + \alpha s = \beta s - \frac{s}{2} + \alpha s = \underbrace{(\alpha + \beta)}_{=1} s - \frac{s}{2} = \frac{s}{2}\,, \tag{72}$$

$$x - \beta s = \beta s - \frac{s}{2} - \beta s = -\frac{s}{2}\,, \tag{73}$$

and therefore

$$(x + \alpha s)^2 + y^2 = \frac{1}{4}s^2 + \frac{3}{4}s^2 = s^2\,, \tag{74}$$

$$(x-\beta s)^2+y^2=s^2\,, \tag{75}$$

which is as expected for an equilateral triangle. Using these values, we find that the gradient indeed vanishes for L_4:

$$\begin{aligned}
\left.\frac{\partial\Phi}{\partial x}\right|_{L_4} &= \frac{GM_1 s}{2s^3}-\frac{GM_2 s}{2s^3}-\omega^2\frac{(2\beta-1)}{2}s \\
&= \frac{GM_1 s}{2s^3}-\frac{GM_2 s}{2s^3}-\frac{G(M_1+M_2)}{s^3}\frac{(\beta-\alpha)}{2}s \\
&= \frac{G(M_1+M_2)}{2s^2}-\frac{G(M_1+M_2)}{2s^2}\frac{M_1-M_2}{M_1+M_2} \\
&= 0\,,
\end{aligned} \tag{76}$$

$$\begin{aligned}
\left.\frac{\partial\Phi}{\partial y}\right|_{L_4} &= -\frac{GM_1\sqrt{3}s}{2s^3}-\frac{GM_2\sqrt{3}s}{2s^3}+\omega^2\frac{\sqrt{3}}{2}s \\
&= -\frac{GM_1\sqrt{3}s}{2s^3}-\frac{GM_2\sqrt{3}s}{2s^3}+\frac{G(M_1+M_2)}{s^3}\frac{\sqrt{3}}{2}s \\
&= 0\,.
\end{aligned} \tag{77}$$

Since L_5 results from L_4 by a sign change of the y coordinate, Eqs. (65) and (66) imply

$$\left.\frac{\partial\Phi}{\partial x}\right|_{L_5}=\left.\frac{\partial\Phi}{\partial x}\right|_{L_4}=0\,, \tag{78}$$

$$\left.\frac{\partial\Phi}{\partial y}\right|_{L_5}=-\left.\frac{\partial\Phi}{\partial y}\right|_{L_4}=0\,, \tag{79}$$

and L_5 is also an extremum.

While it is clear from Fig. 8 that L_4 and L_5 are maxima of the Roche potential (at least for the chosen masses), it is worthwhile to analyze the stability of these points in more detail. To do so, we need to compute the second derivatives of Φ in order to construct the Hessian (cf. worksheet #4). We have

$$\begin{aligned}
\frac{\partial^2\Phi}{\partial x^2} &= \frac{GM_1((x+\alpha s)^2+y^2)^{3/2}-3GM_1(x+\alpha s)^2((x+\alpha s)^2+y^2)^{1/2}}{((x+\alpha s)^2+y^2)^3} \\
&\quad+\frac{GM_2((x-\beta s)^2+y^2)^{3/2}-3GM_2(x-\beta s)^2((x-\beta s)^2+y^2)^{1/2}}{((x-\beta s)^2+y^2)^3}-\omega^2\,,
\end{aligned} \tag{80}$$

$$\frac{\partial^2\Phi}{\partial y\,\partial x}=-\frac{3GM_1(x+\alpha s)y}{((x+\alpha s)^2+y^2)^{5/2}}-\frac{3GM_2(x-\beta s)y}{((x-\beta s)^2+y^2)^{5/2}}\,, \tag{81}$$

$$\begin{aligned}
\frac{\partial^2\Phi}{\partial y^2} &= \frac{GM_1}{((x+\alpha s)^2+y^2)^{3/2}}+\frac{GM_2}{((x-\beta s)^2+y^2)^{3/2}}-\omega^2 \\
&\quad-\frac{3GM_1y^2}{((x+\alpha s)^2+y^2)^{5/2}}-\frac{3GM_2y^2}{((x-\beta s)^2+y^2)^{5/2}}\,.
\end{aligned} \tag{82}$$

Since L_4 and L_5 are related by symmetry, we only need to consider one of the points, so let us choose L_4. Using Eqs. (72)–(75), we obtain

$$\left.\frac{\partial^2\Phi}{\partial x^2}\right|_{L_4}=\frac{GM_1s^3-\frac{3}{4}GM_1s^3}{s^6}+\frac{GM_2s^3-\frac{3}{4}GM_2s^3}{s^6}-\omega^2=\frac{1}{4}\frac{G(M_1+M_2)}{s^3}-\omega^2=-\frac{3}{4}\omega^2 \tag{83}$$

$$\left.\frac{\partial^2\Phi}{\partial y\,\partial x}\right|_{L_4} = -\frac{3\sqrt{3}GM_1s^2}{4s^5} + \frac{3\sqrt{3}GM_2s^2}{4s^5} = -\frac{3\sqrt{3}}{4}\omega^2\,(2\beta-1)\ , \tag{84}$$

$$\left.\frac{\partial^2\Phi}{\partial y^2}\right|_{L_4} = \frac{GM_1}{s^3} + \frac{GM_2}{s^3} - \omega^2 - \frac{9GM_1s}{4s^3} - \frac{9GM_2}{4s^3} = \omega^2 - \omega^2 - \frac{9}{4}\omega^2 = -\frac{9}{4}\omega^2\,. \tag{85}$$

Thus, the Hessian of the Roche potential is given by

$$\mathcal{H}[\Phi] = -\frac{3}{4}\omega^2\begin{pmatrix} 1 & \sqrt{3}(2\beta-1) \\ \sqrt{3}(2\beta-1) & 3 \end{pmatrix}\,. \tag{86}$$

Its eigenvalues are

$$\mathcal{H}_\pm = -\frac{3}{2}\left(1 \pm \sqrt{1-3\beta+3\beta^2}\right)\omega^2\,, \tag{87}$$

which are both negative for $0 \leq \beta \leq 1$. Comparing with the definition (42), we see that this range precisely corresponds to the physically allowed mass ratios. Thus, the Hessian is negative definite, and L_4 is a maximum of the Roche potential, as expected. For L_5, the off-diagonal matrix elements of $\mathcal{H}[\Phi]$ change signs but the eigenvalues are unchanged, hence L_5 is a maximum, too.

So far, so unsurprising. Let us perform a stability analysis of the Roche potential in the vicinity of L_4. To this end, we introduce the relative displacements

$$\xi \equiv x - x_0\,, \quad \eta \equiv y - y_0\,, \tag{88}$$

where (x_0, y_0) are the coordinates of the Lagrange point. We expand the Roche potential through second order:

$$\begin{aligned}
\Phi(x,y) &= \Phi(x_0,y_0) + \left.\frac{\partial\Phi}{\partial x}\right|_{(x_0,y_0)}(x-x_0) + \left.\frac{\partial\Phi}{\partial y}\right|_{(x_0,y_0)}(y-y_0) \\
&\quad + \frac{1}{2}\left.\frac{\partial^2\Phi}{\partial x^2}\right|_{(x_0,y_0)}(x-x_0)^2 + \frac{1}{2}\left.\frac{\partial^2\Phi}{\partial y^2}\right|_{(x_0,y_0)}(y-y_0)^2 \\
&\quad + \left.\frac{\partial^2\Phi}{\partial x\partial y}\right|_{(x_0,y_0)}(x-x_0)(y-y_0) + \mathcal{O}\left((x-x_0)^3, (y-y_0)^3\right) \\
&= \Phi(x_0,y_0) + \frac{1}{2}\mathcal{H}_{\xi\xi}\,\xi^2 + \frac{1}{2}\mathcal{H}_{\eta\eta}\,\eta^2 + \mathcal{H}_{\xi\eta}\,\xi\eta + \mathcal{O}\left(\xi^3,\eta^3\right)\ ,
\end{aligned} \tag{89}$$

where we have used that the gradient of Φ vanishes at the Lagrange point, and introduced

$$\mathcal{H}_{\xi\xi} \equiv \left.\frac{\partial^2\Phi}{\partial x^2}\right|_{(x_0,y_0)}\ , \quad \mathcal{H}_{\eta\eta} \equiv \left.\frac{\partial^2\Phi}{\partial x^2}\right|_{(x_0,y_0)}\ , \quad \mathcal{H}_{\xi\eta} \equiv \left.\frac{\partial^2\Phi}{\partial x\partial y}\right|_{(x_0,y_0)}\ . \tag{90}$$

In terms of the displacement coordinates, the equations of motion (58) become

$$\ddot{\xi} - 2\omega\dot{\eta} + \frac{\partial\Phi}{\partial\xi} = \ddot{\xi} - 2\omega\dot{\eta} + \mathcal{H}_{\xi\xi}\xi + \mathcal{H}_{\xi\eta}\eta = 0\,, \tag{91}$$

$$\ddot{\eta} + 2\omega\dot{\xi} + \frac{\partial\Phi}{\partial\eta} = \ddot{\eta} + 2\omega\dot{\xi} + \mathcal{H}_{\xi\eta}\xi + \mathcal{H}_{\eta\eta}\eta = 0\,. \tag{92}$$

Next, we determine the fundamental solutions of these coupled differential equations in the usual way, by making the ansatz

$$\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix}\xi_0 \\ \eta_0\end{pmatrix} e^{\lambda t}\,. \tag{93}$$

Writing the resulting equation in matrix form, we have

$$\begin{pmatrix} \lambda^2 + \mathcal{H}_{\xi\xi} & -2\omega\lambda + \mathcal{H}_{\xi\eta} \\ 2\omega\lambda + \mathcal{H}_{\xi\eta} & \lambda^2 + \mathcal{H}_{\eta\eta} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 \,. \tag{94}$$

Nontrivial solutions can only exist if the determinant of the coefficient matrix vanishes, i.e.,

$$\begin{aligned} &\left(\lambda^2 + \mathcal{H}_{\xi\xi}\right)\left(\lambda^2 + \mathcal{H}_{\eta\eta}\right) + 4\omega^2\lambda^2 - \mathcal{H}_{\xi\eta}^2 \\ &= \lambda^4 + \lambda^2\left(\mathcal{H}_{\xi\xi} + \mathcal{H}_{\eta\eta} + 4\omega^2\right) + \mathcal{H}_{\xi\xi}\mathcal{H}_{\eta\eta} - \mathcal{H}_{\xi\eta}^2 \\ &= 0 \,. \end{aligned} \tag{95}$$

The solutions are

$$\lambda_\pm^2 = -\frac{1}{2}\left(\mathcal{H}_{\xi\xi} + \mathcal{H}_{\eta\eta} + 4\omega^2\right) \pm \frac{1}{2}\sqrt{\left(\mathcal{H}_{\xi\xi} + \mathcal{H}_{\eta\eta} + 4\omega^2\right)^2 - 4\left(\mathcal{H}_{\xi\xi}\mathcal{H}_{\eta\eta} - \mathcal{H}_{\xi\eta}^2\right)} \,. \tag{96}$$

If $\lambda_\pm^2$ is real, then $\lambda_\pm$ are purely imaginary, and our fundamental solutions are oscillatory and motion near L_4 would be stable without any external stabilizing forces. This requires

$$\left(\mathcal{H}_{\xi\xi} + \mathcal{H}_{\eta\eta} + 4\omega^2\right)^2 \geq 4\left(\mathcal{H}_{\xi\xi}\mathcal{H}_{\eta\eta} - \mathcal{H}_{\xi\eta}^2\right) \,. \tag{97}$$

Plugging in the matrix elements of the Hessian, we have

$$\omega^4\left(-\frac{3}{4} - \frac{9}{4} + 4\right)^2 \geq 4\omega^4\left(\frac{27}{16} - \frac{27}{16}(2\beta - 1)^2\right) \tag{98}$$

and after some algebra, we obtain the stability condition

$$1 \geq \frac{27}{4}\left(1 - (2\beta - 1)^2\right) = 27\beta(1 - \beta) \,. \tag{99}$$

Its solutions are $\beta \leq \beta_-$ and $\beta \geq \beta_+$, where $\beta_\pm$ are obtained from

$$-27\beta^2 + 27\beta - 1 = 0 \,. \tag{100}$$

Solving, we obtain

$$\beta_\pm = \frac{-27 \pm \sqrt{27^2 - 4 \cdot 27}}{-2 \cdot 27} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{27}}\right) \,. \tag{101}$$

Since $\beta = M_1/(M_1 + M_2)$, β_+ must be applied in the case where $M_1 > M_2$, and β_- if $M_1 < M_2$. Focusing on this branch, which matches the parameters we have used in the figures above, we find

$$\beta = \frac{M_1}{M_1 + M_2} \leq \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right) \approx \frac{1}{25.96} \tag{102}$$

Thus, we find that motion around L_4 (and analogously, L_5) is **stable** if M_1 is at least approximately 26 times lighter than the combined mass $M_1 + M_2$. If we swap the masses M_1 and M_2, we will get the same result from the β_+ branch of Eq. (101).

In the Solar system, the so-called Trojans or Sun-Jupiter Trojans are groups of several thousand asteroids that perform a libration, i.e., stable oscillatory motion around the Lagrange points L_4 and L_5 of the Sun-Jupiter system. They are orbiting the Sun on Jupiter's trajectory, at angular offsets of $\pm 60°$ from the planet itself. Since the first observations of Jupiter's trojans in the first decade of the 20th century, trojans have been discovered for Mars, Neptune, and even Earth.

4 Group Exercises

Problem G21 – Spacecraft Orbits

At the time $t = 0$, a spacecraft of mass $m \ll M_E$ is at a distance $2R_E$ from the Earth's center and traveling with a velocity $\vec{u}$ that is parallel to the tangent to the equator at a longitude $\lambda = 0$. Let us denote

$$u = |\vec{u}| = \alpha v_0 \,, \quad v_0 \equiv \sqrt{\frac{GM_E}{2R_E}} \,, \tag{103}$$

where $\alpha > 0$.

1. Show that the spacecraft will orbit the Earth on a circular trajectory if $\alpha = 1$.

2. At what velocity will the spacecraft escape from the Earth's gravitational well?

3. Show that the spacecraft will crash into the Earth at a longitude λ if

$$\alpha(\lambda) = \sqrt{\frac{1 - \cos\lambda}{2 - \cos\lambda}} \,. \tag{104}$$

   (Neglect drag effects due to the atmosphere.) Use your present results and the results of the previous parts to classify the trajectories.

Problem G22 – Lagrange's Solutions to the Three-Body Problem

Consider the gravitational three-body problem for the masses m_1, m_2, m_3,

1. Verify that the equations of motion in the center-of-mass frame, Eqs. (21)–(23), can be cast in the form (28).

2. Show that the decoupling condition $\vec{G} = 0$ for the equations of motion (28) is satisfied if the masses form an equilateral triangle.

3. Assuming that $\vec{s}_3(t)$ has been determined by solving eq. (28), show that the trajectories of the masses in the center-of-mass frame are given by

$$\vec{r}_1 = -\frac{2m_2 + m_3}{2M}\vec{s}_3 + \frac{\sqrt{3}}{2}\frac{m_3}{M}\vec{n} \times \vec{s}_3 \,, \tag{105}$$

$$\vec{r}_2 = \frac{2m_1 + m_3}{2M}\vec{s}_3 + \frac{\sqrt{3}}{2}\frac{m_3}{M}\vec{n} \times \vec{s}_3 \,, \tag{106}$$

$$\vec{r}_3 = \frac{m_1 - m_2}{2M}\vec{s}_3 - \frac{\sqrt{3}}{2}\frac{m_1 + m_2}{M}\vec{n} \times \vec{s}_3 \,, \tag{107}$$

   where $\vec{n}$ is the unit normal vector of the equilateral triangle.

   HINT: Express $\vec{n}$ in terms of the $\vec{s}_i$, and use that finite rotations of a vector $\vec{a}$ by an angle ϕ around the axis defined by $\vec{n}$ can be expressed as

$$\vec{a}' = (\vec{n} \cdot \vec{a})\vec{n} + (\vec{a} - \vec{n}(\vec{n} \cdot \vec{a}))\cos\phi + (\vec{a} \times \vec{n})\sin\phi \,. \tag{108}$$

Problem G23 – Vectors and Antisymmetric Matrices in Three Dimensions

In three dimensions, one can construct a unique mapping between vectors and antisymmetric (or skew-symmetric) matrices:

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^{3\times 3} : \quad \Phi(\vec{v}) = \Phi\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}, \tag{109}$$

or in components

$$[\Phi(\vec{v})]_{ij} = -\epsilon_{ijk} v_k \,, \quad \epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312 \quad \text{(cyclic permutations)}\,, \\ -1 & \text{if } ijk = 213, 132, 321 \quad \text{(anticyclic permutations)}\,, \\ 0 & \text{else}\,, \end{cases} \tag{110}$$

where ϵ is the usual Levi-Civita tensor.

1. Show that the mapping is *linear*, i.e.,

$$\Phi\left(\alpha\vec{u} + \beta\vec{v}\right) = \alpha\Phi(\vec{u}) + \beta\Phi(\vec{v})\,. \tag{111}$$

2. Show that the usual scalar product can be written as

$$\vec{u} \cdot \vec{v} = \frac{1}{2}\mathrm{tr}\,\left[\Phi(\vec{u})^T\Phi(\vec{v})\right]\,. \tag{112}$$

3. Show that the vector product can be obtained from

$$\vec{u} \times \vec{v} = \Phi(\vec{u})\vec{v} \tag{113}$$

or, alternatively,

$$\Phi(\vec{u} \times \vec{v}) = \Phi(\vec{u})\Phi(\vec{v}) - \Phi(\vec{v})\Phi(\vec{u}) = \big[\Phi(\vec{u}), \Phi(\vec{v})\big] \tag{114}$$

where we have introduced the commutator

$$\big[A, B\big] = AB - BA\,. \tag{115}$$