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# PHY422/820: Classical Mechanics

FS 2021

Worksheet #10 (Nov 1 – Nov 5)

November 12, 2021

## 1 Preparation

- Lemos, Chapter 3 and Sections 4.1-4.8
- Goldstein, Chapter 4 (skip 4.5) and Sections 5.1-5.6

## 2 Rigid Body Kinematics

### 2.1 The Concept of the Rigid Body

#### 2.1.1 Definition

We can define a rigid body as a system of $N$ pointlike masses $m_n$ that are moving subject to constraints of the form

$$r_{ij} = |\vec{r}_i(t) - \vec{r}_j(t)| = \text{const.}\,, \qquad i,j = 1,\ldots,n\,, \tag{1}$$

such that the body does not break apart or deform over time. Of course, this is an idealization, just like the concept of the pointlike mass.

Microscopically, a system consisting of $N$ point masses would have up to $3N$ degrees of freedom, e.g., the three components of each particle's position vector. For any realistic body, this would be an immense number: Consider for example a wooden chair, which weighs a few kilograms and mostly consists of organic molecules, i.e., carbon atoms. A kilogram of carbon has on the order of $10^{26}$ atoms, so we would naively have $10^{78}$ degrees of freedom. Clearly, there must be an equally immense amount of constraints to reduce this number to the actual number of degrees of freedom a of a realistic rigid body without additional external constraints, which is $f = 6$.

We can understand why there should be six degrees of freedom if we construct the rigid body from the ground up: A mass $m_1$ has three degrees of freedom. If we add a second mass $m_2$ we add two degrees of freedom instead of three because the distance between $m_1$ and $m_2$ is fixed by the constraint. Adding a third mass $m_3$ we only gain one degree of freedom because the distances $r_{12}, r_{13}$ are now fixed. This brings our total to $f = 6$. The position of any additional mass would be completely fixed by its distances with any three of the existing masses, so that $f$ cannot increase any further.

In practice, the motion of a rigid body is most conveniently described by using the three coordinates of one of its points, usually its center of mass, and three angles that describe its

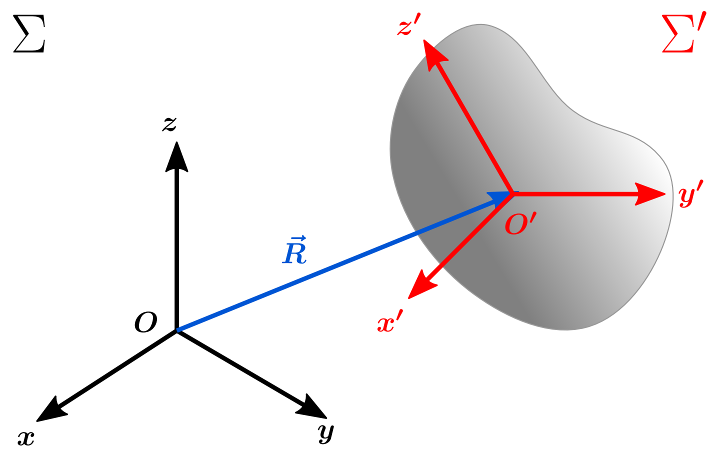


Figure 1: Laboratory ($\Sigma$) and body-fixed frames ($\Sigma'$). The center of mass of the rigid body is usually chosen as the origin of $\Sigma'$. In the laboratory frame, it's position is given by the vector $\vec{R}$.

orientation with respect to that point. The former allow us to describe the rigid body's translational motion, while the latter account for its rotational motion. Since we are familiar with the description of translation from our treatment of pointlike masses, we will focus on the new aspects that rotations add to the dynamics of mechanical systems.

#### 2.1.2 Laboratory and Body-Fixed Frames

In Figure 1, we introduce the two coordinate systems that will play a central role in our discussion of rigid-body kinematics and dynamics:

- The **laboratory frame** $\Sigma$ with coordinates $(x, y, z)$ and unit vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$, and
- the **body-fixed frame** $\Sigma'$ with coordinates $(x', y', z')$ and unit vectors $\{\vec{e}'_1, \vec{e}'_2, \vec{e}'_3\} = \{\vec{e}'_x, \vec{e}'_y, \vec{e}'_z\}$. In principle, we could place the origin of $\Sigma'$ in any point of the rigid body, but the most convenient choice for most of our applications is the body's center of mass. Note that its position in the laboratory frame is given by the vector $\vec{R}(t)$, which therefore describes the offset of the two systems at any given time. It can be used to treat the translation of the rigid body.

Let us focus on the rigid body's rotational degrees of freedom, and assume that the origins of $\Sigma$ and $\Sigma'$ coincide. A general vector $\vec{a}$ can be expressed in terms of the basis vectors of the two systems as

$$\vec{a} = \sum_{i=1}^{3} a_i \vec{e}_i = \sum_{i=1}^{3} a'_i \vec{e}'_i \,. \tag{2}$$

Projecting on either set of basis vectors by taking inner products, we can easily relate the components in the two coordinate frames:

$$a'_i = \vec{e}'_i \cdot \vec{a} = \sum_j a_j \, \vec{e}'_i \cdot \vec{e}_j \equiv \sum_j R_{ij} a_j \,. \tag{3}$$

The scalar products of the basis vectors define the entries of the matrix $\boldsymbol{R}$. Since

$$R_{ij} = \vec{e}'_i \cdot \vec{e}_j = |\vec{e}'_i| \, |\vec{e}_j| \, \cos\angle(\vec{e}'_i, \vec{e}_j) = \cos\angle(\vec{e}'_i, \vec{e}_j) \,, \tag{4}$$

they are also known as *directional cosines.*

As discussed above, the rotation between the two coordinate systems can be described by three degrees of freedom, so there must be some redundancies between the nine entries of the matrix $\boldsymbol{R}$. Noting that the length of $\vec{a}$ must not depend on the choice of coordinates, we find that

$$\begin{aligned} \vec{a}\cdot\vec{a} = \sum_i a'_i a'_i = \sum_i \left(\sum_j R_{ij} a_j\right)\left(\sum_k R_{ik} a_k\right) = \sum_{jk}\left(\sum_i R_{ij} R_{ik}\right) a_j a_k \\ \stackrel{!}{=} \sum_j a_j a_j \,, \end{aligned} \tag{5}$$

which implies

$$\sum_i R_{ij} R_{ik} = \sum_i R^T_{ji} R_{ik} = \delta_{jk}\,, \tag{6}$$

or in matrix form

$$\boldsymbol{R}^T \boldsymbol{R} = \mathbb{1}\,, \tag{7}$$

where $\mathbb{1}$ is the $3\times 3$ identity matrix.

Multiplying Eq. (3) from the left with $\boldsymbol{R}^T$, we find

$$\sum_j R^T_{ij} a'_j = \sum_{jk} R^T_{ij} R_{jk} a_k = \sum_k \delta_{ik} a_k = a_i\,, \tag{8}$$

i.e., the $a_i$ are obtained from the $a'_i$ by applying the transpose of $\boldsymbol{R}$. Considering the length of $\vec{a}$ in terms of the $a_i$ and following the same strategy as above, we can derive the relation

$$\boldsymbol{R}\boldsymbol{R}^T = \mathbb{1}\,, \tag{9}$$

or in summary,

$$\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{R}^T\boldsymbol{R} = \mathbb{1}\,. \tag{10}$$

We note that these relations imply that

$$\boldsymbol{R}^T = \boldsymbol{R}^{-1} \tag{11}$$

and

$$\det \boldsymbol{R}^T \boldsymbol{R} = \det \boldsymbol{R}^T \det \boldsymbol{R} = (\det \boldsymbol{R})^2 \stackrel{!}{=} \det \mathbb{1} = 1\,, \tag{12}$$

hence

$$\det \boldsymbol{R} = \pm 1\,. \tag{13}$$

This means that $\boldsymbol{R}$ is an **orthogonal matrix**. These matrices form a representation of $O(3)$, the **group of orthogonal transformations** of $\mathbb{R}^3$ which preserve the scalar product (or inner product) of any two vectors (see homework problem H18). The elements of this group with $\det \boldsymbol{R} = +1$ describe rotations and form a subgroup of $O(3)$, the **special orthogonal group SO(3)**. In contrast, the elements $\det \boldsymbol{R} = -1$ involve a reflection $-\mathbb{1}$ followed (or preceded) by a rotation. They do not form a subgroup since a product of an even number of such transformations would be a proper rotation again, due to $(-\mathbb{1}^{2n} = \mathbb{1}$ (also see H18).

**Box 2.1: Lie Groups and Infinitesimal Generators**

A group like $SO(3)$ whose elements are parameterized by continuous variables and which therefore has the geometric structure of a smooth manifold is called a *Lie*[a] *group*. Following our discussion of manifolds from early in the course, we note that the finite range of the intervals for the Euler angles implies that the $SO(3)$ manifold is compact. We also saw that compact groups whose boundaries are "glued together", e.g., because the parameter intervals repeat periodically as in the case of angles, corresponds to products of spheres in $n-$dimensional spaces. $SO(3)$, in particular, is isomorphic to a three-dimensional hypersphere in a four-dimensional Euclidean space.

Now that we know that $SO(3)$ has the geometric structure of a manifold, we can also construct a tangent space in the usual way, by considering the partial derivatives of a general element $\boldsymbol{R}(t)$ with respect to the angles $\phi, \theta, \psi$. If we do so, we find that the "tangent vectors" are the antisymmetric matrices

$$\boldsymbol{L}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{L}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{L}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{I2.1-1}$$

which are called the (infinitesimal) **generators** of the group (cf. homework problem H19). The "tangent space" spanned by the $\boldsymbol{L}_i$ is called the *Lie algebra* of the group, and is denoted by $so(3)$ (notice the lower-case notation).

[a] Pronounced "Lee" group, named after Norwegian mathematician Sophus Lie.

**Active and Passive Rotations**

In principle, there are two equally valid ways of interpreting Eqs. (3) and (8), which are illustrated in Fig. 2: In the **passive** view, the vector $\vec{a}$ is invariant, and the $a_i$ and $a'_i$ are merely the vector's components with respect to the basis vectors of the coordinate systems $\Sigma$ and $\Sigma'$ that are connected by the rotation $\boldsymbol{R}$. In the **active** view, $\vec{a}'$ is a new vector in the *same* coordinate system, e.g., in $\Sigma$.

From the figure, it is clear that the components of $\vec{a}$ in $\Sigma'$ in the passive case match the components of the actively rotated vector $\vec{a}'$. Note, however, that the active and passive rotations are in *opposite* directions!

### 2.1.3 The Theorems of Euler and Chasles

The formal proof that the motion of a rigid body can be split into translation and rotation is provided by the theorem's of Euler and Chasles. Let us first consider a rigid body that is fixed in space. Then the following theorem applies:

**Euler's Theorem of Rigid Body Motion**

In three-dimensional space, the most general displacement of a rigid body with one fixed point is a rotation about some axis through this fixed point.

To prove it, we consider a rigid body that moves from its initial to its final configuration via a rotation around an axis through the fixed point. By construction, the axis itself would be invariant

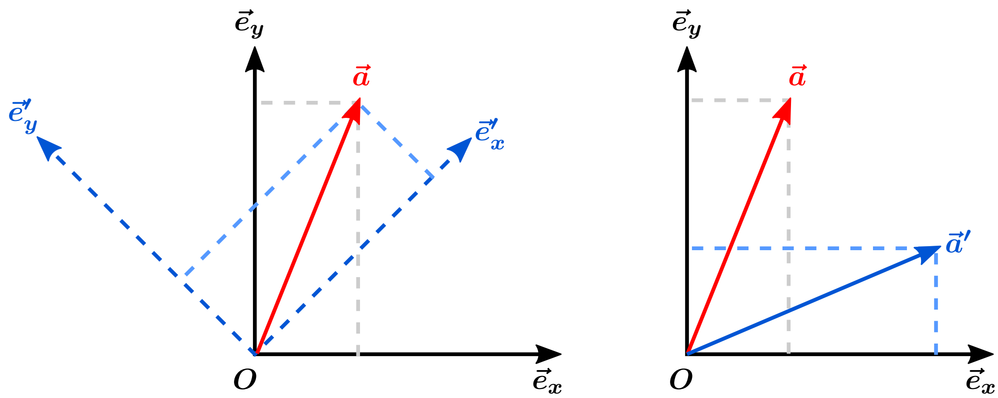


Figure 2: Passive (*left*) and active (*right*) view of rotations.

under a rotation $\boldsymbol{R}$, so we must find a vector $\vec{n}$ that satisfies

$$\boldsymbol{R}\vec{n} = \vec{n}\,, \tag{14}$$

i.e., an eigenvector of the rotational matrix $\boldsymbol{R}$. This equation is equivalent to

$$(\boldsymbol{R} - \mathbb{1})\vec{n} = 0\,, \tag{15}$$

which only allows a non-trivial solution ($\vec{n} \neq 0$) if $\det(\boldsymbol{R} - \mathbb{1})$ vanishes. Now note that

$$\boldsymbol{R} - \mathbb{1} = \boldsymbol{R} - \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{R}(\mathbb{1} - \boldsymbol{R}^T) = -\boldsymbol{R}(\boldsymbol{R}^T - \mathbb{1}) = -\mathbb{1}\boldsymbol{R}(\boldsymbol{R} - \mathbb{1})^T\,. \tag{16}$$

This implies

$$\det(\boldsymbol{R} - \mathbb{1}) = \det(-\mathbb{1})\det\boldsymbol{R}\det(\boldsymbol{R} - \mathbb{1})^T = (-1)^3 \cdot 1 \cdot \det(\boldsymbol{R} - \mathbb{1}) = -\det(\boldsymbol{R} - \mathbb{1})\,, \tag{17}$$

which can only be true if

$$\det(\boldsymbol{R} - \mathbb{1}) = 0\,, \tag{18}$$

completing the proof.

It is worth noting that the final steps of the proof explicitly relied on the dimension of space through $\det(-\mathbb{1}) = -1$. In an even-dimensional space, $\det(-\mathbb{1}) = (-1)^n = 1$, the argument from the proof fails, and Euler's theorem *does not hold*! For example, in a two-dimensional space no vector is left unaltered by a rotation. Indeed, for a rotation in the plane the axis of rotation is *perpendicular* to the plane, hence any vector along the rotation axis would actually not lie in the plane and would therefore not be part of the vector space!

If we don't restrict the body to a fixed point in space, we clearly add a translational component to the motion as the origin of the body-fixed frame moves away from the laboratory-fixed coordinate system. Euler's theorem was generalized to this situation by Chasles in 1830, who formulated the following theorem of his own:

**Chasles' Theorem**

The most general displacement of a rigid body in a three-dimensional space is a translation along a line that is either followed or preceded by a rotation around an axis that is parallel to that line.

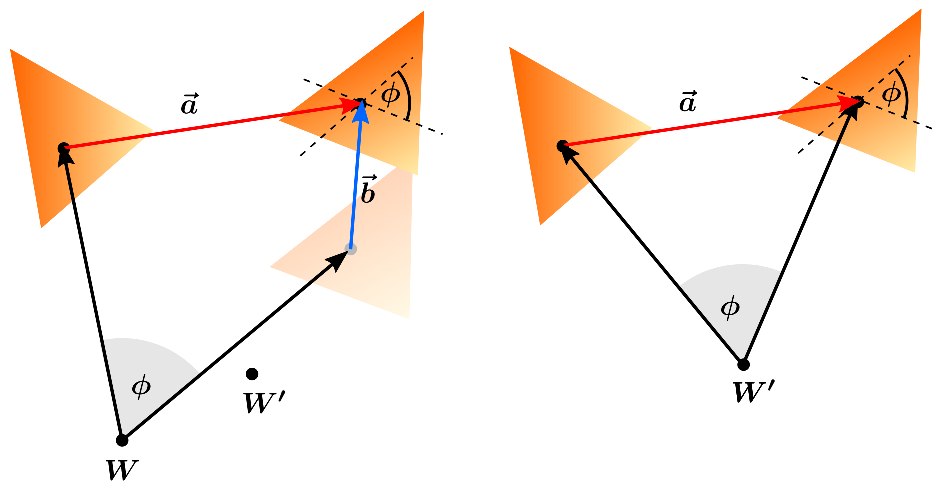


Figure 3: Illustration of Chasles' theorem: A translation perpendicular to the rotational axis followed by a rotation can be turned into a pure rotation around an axis through a point $W'$ determined by Eqs. (22) and xx.

It is intuitively clear that we can add three translational degrees of freedom to the rigid body if we no longer hold a point of the rigid body fixed as we did in Euler's theorem. If the rigid body undergoes a rotation $\boldsymbol{R}$ and subsequent translation by a vector $\vec{a}$, the positions $\vec{r}$ and $\vec{r}'$ in the laboratory and body-fixed frames are related by the transformation

$$\vec{r} = \boldsymbol{R}^T \vec{r}' + \vec{a}\,. \tag{19}$$

It is easy to see that the vector $\vec{a}$ is the new position of the body-fixed frame's origin in the laboratory frame.

Now let us consider the same rotation $\boldsymbol{R}$ around an axis through a point other than the origin. Denoting the position of this point in the laboratory frame by $\vec{w}$, rotating around an axis through $\vec{w}$ followed by a translation by a vector $\vec{b}$ can be written as

$$\vec{r} - \vec{w} = \boldsymbol{R}^T(\vec{r}' - \vec{w}) + \vec{b}\,, \tag{20}$$

or

$$\vec{r} = \boldsymbol{R}^T \vec{r}' - (\boldsymbol{R}^T - \mathbb{1})\vec{w} + \vec{b}\,. \tag{21}$$

This will be the same displacement as (19) if

$$\vec{b} - (\boldsymbol{R}^T - \mathbb{1})\vec{w} = \vec{a} \qquad \Rightarrow \qquad \vec{b} - \vec{a} = (\boldsymbol{R}^T - \mathbb{1})\vec{w}\,. \tag{22}$$

Thus, $\vec{b}$ will be uniquely determined from this equation if $\vec{w}$ is given. Conversely, however, we note that $\vec{w}$ would *not* be expressible in terms of $\vec{a}, \vec{b}$ and $\boldsymbol{R}$: We could attempt to solve Eq. (22) for $\vec{w}$, but in the proof of Euler's theorem we crucuially relied on the fact that $\boldsymbol{R}^T - \mathbb{1}$ is not invertible.

Let us now introduce the unit vector $\vec{n}$ in the direction of the rotational axis, which is an eigenvector of $\boldsymbol{R}$ (and $\boldsymbol{R}^T$), as demonstrated above:

$$\boldsymbol{R}\vec{n} = \boldsymbol{R}^T \vec{n} = \vec{n}\,. \tag{23}$$

Taking the scalar product of $\vec{n}$ with Eq. (22), we obtain

$$\vec{n} \cdot (\vec{b} - \vec{a}) = \vec{n} \cdot (\boldsymbol{R}^T \vec{w}) - \vec{n} \cdot \vec{w}$$

$$
\begin{aligned}
&= \sum_{ij} n_i R^T_{ij} w_j - \vec{n} \cdot \vec{w} = \sum_{ij} w_j R_{ji} n_i - \vec{n} \cdot \vec{w} \\
&= (\boldsymbol{R}\vec{n}) \cdot \vec{w} - \vec{n} \cdot \vec{w} = \vec{n} \cdot \vec{w} - \vec{n} \cdot \vec{w} = 0 \,,
\end{aligned} \tag{24}
$$

i.e., $\vec{b} - \vec{a}$ is perpendicular to the rotational axis. Thus, the only difference between performing the rotation $\boldsymbol{R}$ around an axis through the origin of the body-fixed frame or around a parallel axis through the point $\vec{w}$ is a translation in the perpendicular plane. We will be reminded of this finding when we discuss the **parallel-axis theorem**.

[Finish.]

## 2.2 Parametrization of Rotations

### 2.2.1 Matrix Parameterization and Euler Angles

We can characterize a general rotation in three dimension by three degrees of freedom: the usual azimuthal and polar angles $\phi$ and $\theta$ of a spherical coordinate system, which characterize the direction of the axis of rotation, and a third angle $\psi$ that specifies *how far* we rotate the coordinate system. From this argument, we can infer the ranges of the angles: $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, as usual in spherical coordinates, and $\psi \in [0, 2\pi]$.

To construct a general rotation, we consider the basic $SO(3)$ matrices which describe rotations around the axes of a Cartesian coordinate system, spanned by the right-handed triad of unit vectors $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$:

$$
\boldsymbol{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} , \; \boldsymbol{R}_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} , \; \boldsymbol{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{25}
$$

A general rotation can be described by a product of three such matrices as long as we do not rotate around the same axis twice, because then we could replace the consecutive rotations by a single rotation with the combined angle, and we would not used all three degrees of freedom. In physics, the most frequently used convention is to perform consecutive rotations around the $z$, $x$, and $z$ axes by the angles $\phi, \theta$ and $\psi$, which are referred to as **Euler angles** in this context. Importantly, we are rotating the basis vectors, which requires that we consider $\boldsymbol{R}_i$ with angles $-\alpha$, and implies a *passive* view of the rotation (see Sec. 2.1.2).

The conventional sequence of rotations that transforms between two coordinate systems in the *passive point of view* is illustrated in Fig. 2.1.2:

- We start by rotating the initial coordinate system around the $z$ axis by an angle $\phi$ using the rotation matrix

  $$
  \boldsymbol{D} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{26}
  $$

  This rotation leaves the $z$ axis invariant, while the $x$ and $y$ axes are tilted by the angle $\phi$ compared to their previous directions. We denote the intermediate axes resulting from this rotation by $(\xi, \eta, \zeta)$.

- In the intermediate coordinate system, we rotate around the $\xi$ axis (also known as the line of nodes) by an angle $\theta$, which tilts the entire coordinate system against the $\zeta$ axis. The

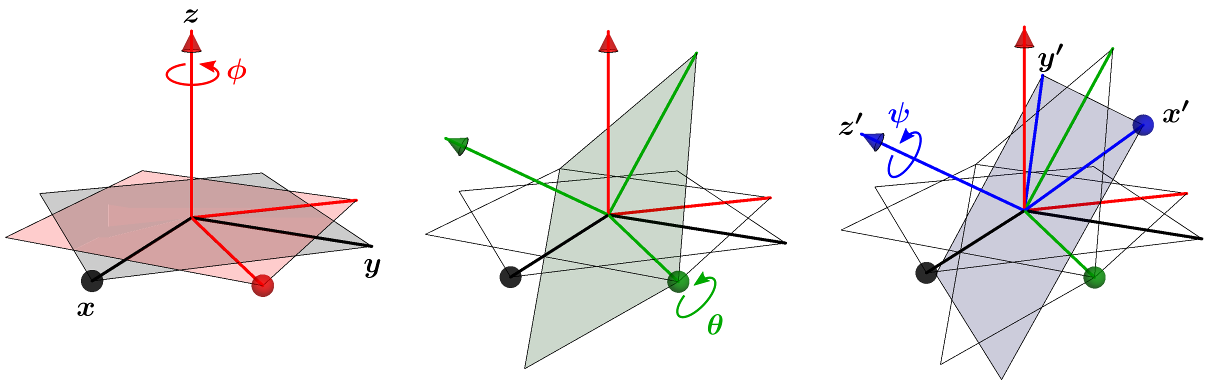


Figure 4: Transformation between the laboratory (black) and body-fixed (blue) coordinate systems by a sequence of rotations with the Euler angles $\phi, \theta, \psi$.

rotation matrix is given by

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} , \tag{27}$$

but note that this representation refers to the basis vectors of the *intermediate* coordinate system, not to the initial one. The new intermediate axes are $(\xi', \eta', \zeta')$.

- Finally, we rotate around the $\zeta'$ axis by the angle $\psi$ to obtain the final coordinate system, using

$$\boldsymbol{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{28}$$

which is represented in the unit vectors associated with $(\xi', \eta', \zeta')$. We note that the $\zeta'$ axis is invariant under the rotation and identical to the body-fixed $z'$ axis.

The complete rotation between the two coordinate system can be expressed as a single rotation given by

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{BCD} \\ &= \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & \cos\theta\sin\psi\cos\phi + \cos\psi\sin\phi & \sin\theta\sin\psi \\ -\cos\theta\cos\psi\sin\phi - \sin\psi\cos\phi & \cos\theta\cos\psi\cos\phi - \sin\psi\sin\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} . \end{aligned} \tag{29}$$

It is straightforward to verify that $\boldsymbol{A}$ is an $SO(3)$ matrix.

### 2.2.2 Vectorial Parameterization

We can also characterize a general rotation by specifying the direction of the rotation axis $\vec{n}$ (e.g., through the angular variables of a spherical coordinate system) and a third angle $\phi$ that specifies *how far* we rotate the coordinate system.

Let $\vec{n}$ be a unit vector along the rotational axis. We can write the initial and rotated vectors as

$$\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp} = \vec{n}(\vec{n}\cdot\vec{r}) + \vec{r}_{\perp} \tag{30}$$

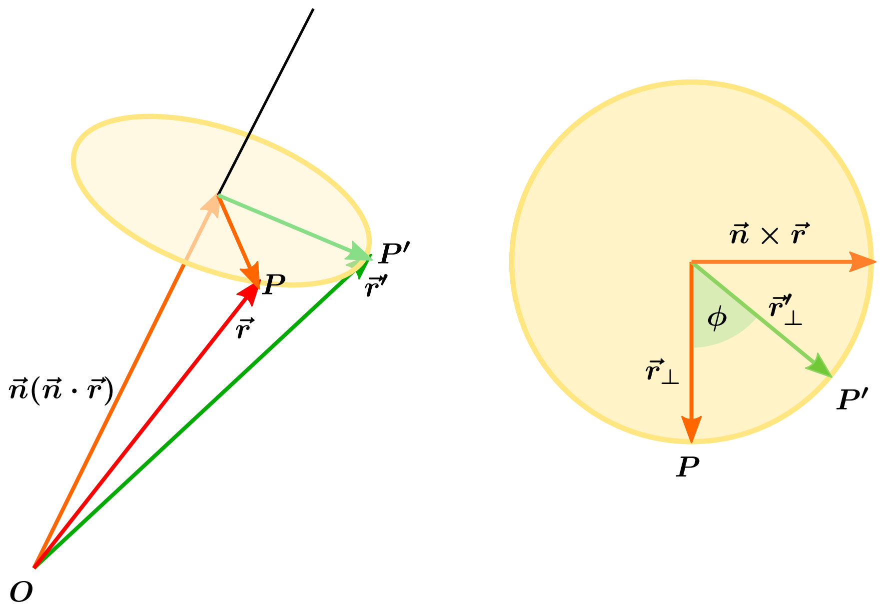


Figure 5: Vectorial description of a rotation by an angle $\phi$ around an axis indicated by the unit vector $\vec{n}$.

$$\vec{r}' = \vec{r}'_{\parallel} + \vec{r}'_{\perp} = \vec{n}(\vec{n}\cdot\vec{r}) + \vec{r}'_{\perp}\,, \tag{31}$$

where we have used that the projection of $\vec{r}$ on the axis $\vec{n}$ remains unchanged under rotations. Next, we consider the top view of the rotational cone to determine $\vec{r}'_{\perp}$. From Fig. 5, it is clear that $\vec{r}_{\perp}$ and

$$\vec{n}\times\vec{r} = \vec{n}\times\vec{r}_{\perp} \tag{32}$$

are a pair of equally long orthogonal vectors in the top plane of the cone. This allows us to write

$$\begin{aligned}\vec{r}'_{\perp} &= \vec{r}_{\perp}\cos\phi + (\vec{n}\times\vec{r})\sin\phi \\ &= (\vec{r} - \vec{n}(\vec{n}\cdot\vec{r}))\cos\phi + (\vec{n}\times\vec{r})\sin\phi\end{aligned} \tag{33}$$

and we obtain

$$\boxed{\vec{r}' = \vec{r}\cos\phi + \vec{n}(\vec{n}\cdot\vec{r})(1-\cos\phi) + (\vec{n}\times\vec{r})\sin\phi} \tag{34}$$

(cf. discussion of Noether's theorem for spatial rotations on worksheet #4).

### 2.3 The Angular Velocity

Since we know that the basis vectors of the laboratory and body-fixed frames are related by Eqs. (3) and (8), we can consider the implications for the kinematic variables. A given point $\vec{r}(t)$ can be expanded in either frame as

$$\vec{r}(t) = \sum_i r_i(t)\vec{e}_i = \sum_i r'_i\vec{e}'_i(t)\,, \tag{35}$$

where Eq. (8) implies that the coefficients are connected by the transpose of $\boldsymbol{R}(t)$:

$$r_i(t) = \sum_j r'_j R_{ji}(t) = \sum_j R^T_{ij}(t) r'_j \,. \tag{36}$$

Next, we consider the velocity in the laboratory frame:

$$\frac{d\vec{r}}{dt} = \sum_i r'_i \frac{d\vec{e}'_i}{dt} = \sum_{ij} r'_i \dot{R}_{ij} \vec{e}_j = \sum_{ijk} r'_i \dot{R}_{ij} R^T_{jk} \vec{e}'_k \equiv \sum_{ik} r'_i \Omega_{ik} \vec{e}'_k \,, \tag{37}$$

where we have introduced the **angular velocity tensor $\boldsymbol{\Omega}$** with elements

$$\boxed{\Omega_{ij} = \sum_k \frac{dR_{ik}}{dt} R^T_{kj} \,. \tag{38}}$$

We can easily see that $\boldsymbol{\Omega}$ is antiysmmetric:

$$\frac{d}{dt}\left(\boldsymbol{R}\boldsymbol{R}^T\right) = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{R}}^T \stackrel{!}{=} \frac{d}{dt}\mathbb{1} = 0 \,, \tag{39}$$

hence

$$\boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^T = -R\dot{\boldsymbol{R}}^T = -\left(\dot{\boldsymbol{R}}\boldsymbol{R}^T\right)^T = -\boldsymbol{\Omega}^T \,. \tag{40}$$

In group exercise G23, we have seen there is a unique mapping between antisymmetric matrices and vectors. It implicitly defines a vector $\vec{\omega}$ associated with the matrix $\Omega$ through the relationship

$$\Omega_{ij} = -\sum_k \epsilon_{ijk} \omega'_k \,. \tag{41}$$

We can solve for the components by multiplying with $\epsilon_{ijk}$ and summing:

$$\begin{aligned} \sum_{ij} \epsilon_{kij} \Omega_{ij} &= -\sum_{ijl} \epsilon_{ijk}\epsilon_{ijl}\omega'_l = -\sum_{jl}\left(\delta_{jj}\delta_{kl} - \delta_{jk}\delta_{jl}\right)\omega'_l = -\sum_l (3\delta_{kl} - \delta_{kl})\omega'_l \\ &= -2\omega'_l \,, \end{aligned} \tag{42}$$

i.e.,

$$\boxed{\omega'_i = \frac{1}{2}\sum_{jk} \epsilon_{ijk} \Omega_{jk} \,. \tag{43}}$$

Plugging Eq. (43) back into (37), we can identify the components of $(\vec{\omega} \times \vec{r})$ in the body-fixed frame and obtain

$$\boxed{\frac{d\vec{r}}{dt} = -\sum_{ijk} r'_i \epsilon_{ijk} \omega'_j \vec{e}'_k = -\sum_k (\vec{r} \times \vec{\omega})'_k \vec{e}'_k = \vec{\omega} \times \vec{r} \,. \tag{44}}$$

The vector $\vec{\omega}$ is the **instantaneous angular velocity** of the rigid body.

**Representations of $\vec{\omega}$**

In applications, it will often be necessary to express $\vec{\omega}$ in terms of the angular velocities associated with the Euler angles, $\dot{\phi}, \dot{\theta}, \dot{\psi}$. Formally, we can write

$$\vec{\omega} = \dot{\phi}\vec{e}_\phi + \dot{\theta}\vec{e}_\theta + \dot{\psi}\vec{e}_\psi \, , \tag{45}$$

but since some of the Euler angles are rotations around intermediate axes in a sequence of transformations, as discussed in Sec. 2.2.1, finding the unit vectors indicating the intermediate rotational axes is not entirely straightforward.

The rotation by the angle $\phi$ is around the $z$ axis of the laboratory frame, so

$$\vec{e}_\phi = \vec{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \, . \tag{46}$$

The rotation by $\theta$ is around the $\xi$ axis of the first intermediate coordinate system. Referring to Sec. 2.2.1, we see that the unit vector can be obtained by rotating $\vec{e}_\xi$ back to the laboratory frame using $\boldsymbol{D}^T$:

$$\vec{e}_\theta = \boldsymbol{D}^T \vec{e}_\xi = \boldsymbol{D}^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \tag{47}$$

(also compare with Fig. 4). In the same way, the $\zeta'$ axis is obtained by pulling back the unit vector $\vec{e}_{\zeta'} = \vec{e}_{z'}$ to the laboratory frame:

$$\vec{e}_\psi = (\boldsymbol{C}\boldsymbol{D})^T \vec{e}_{\zeta'} = \boldsymbol{D}^T \boldsymbol{C}^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{pmatrix} \, . \tag{48}$$

Using the representation of the axes in the laboratory frame, we obtain

$$\boxed{\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}_\Sigma = \begin{pmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \dot{\phi} + \dot{\psi}\cos\theta \end{pmatrix}_\Sigma \, .} \tag{49}$$

The components of $\vec{\omega}$ in the body-fixed frame can be obtained readily by applying the rotation matrix (29):

$$\boxed{\vec{\omega} = \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix}_{\Sigma'} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\psi} + \dot{\phi}\cos\theta \end{pmatrix}_{\Sigma'} \, .} \tag{50}$$

### Non-Existence of an Antiderivative Vector for $\omega$

In this section, we have introduced $\vec{\omega}$ as a particular representation[1] of the angular velocity tensor $\Omega$, which illustrates that it is a more complicated object than a typical linear velocity vector $\dot{\vec{r}}$. The latter can be written as a derivative of $\vec{r}$, but the same is not possible for $\vec{\omega}$, as we will now prove.

Let us assume that there were an $\vec{\Omega}(\phi, \theta, \psi)$ whose time derivative is $\vec{\omega}$. The components of such a vector in either the laboratory or body-fixed frame would have to satisfy

$$\frac{d\Omega_i}{dt} = \frac{\partial\Omega_i}{\partial\phi}\dot{\phi} + \frac{\partial\Omega_i}{\partial\theta}\dot{\theta} + \frac{\partial\Omega_i}{\partial\psi}\dot{\psi} \stackrel{!}{=} \omega_i \,. \tag{51}$$

Comparing this equation with the representation of $\vec{\omega}$ in the laboratory frame, Eq. (49), we would identify

$$\frac{\partial\Omega_x}{\partial\phi} = 0\,, \qquad \frac{\partial\Omega_x}{\partial\theta} = \cos\phi\,, \qquad \frac{\partial\Omega_x}{\partial\psi} = \sin\theta\sin\phi\,, \tag{52}$$

$$\frac{\partial\Omega_y}{\partial\phi} = 0\,, \qquad \frac{\partial\Omega_y}{\partial\theta} = \sin\phi\,, \qquad \frac{\partial\Omega_y}{\partial\psi} = -\sin\theta\cos\phi\,, \tag{53}$$

$$\frac{\partial\Omega_z}{\partial\phi} = 1\,, \qquad \frac{\partial\Omega_z}{\partial\theta} = 0\,, \qquad \frac{\partial\Omega_z}{\partial\psi} = \cos\theta\,. \tag{54}$$

But if we now consider the mixed second partial derivatives, we find that

$$\frac{\partial^2\Omega_x}{\partial\theta\partial\phi} = 0\,, \quad \frac{\partial^2\Omega_x}{\partial\phi\partial\theta} = -\sin\phi\,, \quad \frac{\partial^2\Omega_y}{\partial\theta\partial\phi} = 0\,, \quad \frac{\partial^2\Omega_y}{\partial\phi\partial\theta} = \cos\phi\,, \tag{55}$$

and we see that

$$\frac{\partial^2\Omega_i}{\partial\theta\partial\phi} \neq \frac{\partial^2\Omega_i}{\partial\phi\partial\theta}\,, \quad \text{etc.}. \tag{56}$$

Moreover, we cannot even satisfy the equations at specific fixed values of the Euler angles, since

$$\frac{\partial^2\Omega_x}{\partial\theta\partial\phi} = 0 \stackrel{!}{=} -\sin\phi = \frac{\partial^2\Omega_x}{\partial\phi\partial\theta} \quad \Rightarrow \phi = n\pi, \quad n \in \mathbb{Z}\,, \tag{57}$$

but we would simultaneously need

$$\frac{\partial^2\Omega_y}{\partial\theta\partial\phi} = 0 \stackrel{!}{=} \sin\phi = \frac{\partial^2\Omega_y}{\partial\phi\partial\theta} \quad \Rightarrow \phi = \left(n + \frac{1}{2}\right)\pi, \quad n \in \mathbb{Z}\,. \tag{58}$$

Thus, we conclude that $\vec{\omega}$ cannot be the time derivative of a vector.

## 2.4 Dynamics in Non-Inertial Frames

Most of the discussion in the previous sections focused on transformations between a spatially fixed and a rotating coordinate system. The relations we found are not specific to the rigid body but apply for any mass point. More importantly, they relate the mass point's kinematics and dynamics in the non-inertial rotating frame to those in the inertial laboratory frame.

[1] For those who are interested, $\vec{\omega}$ is the *dual* representation of $\mathbf{\Omega}$ that is generated by the application of the so-called **Hodge star operator** $\star$ that plays a central role in differential geometry. In the special case of $\mathbb{R}^3$, it is generating the mapping between skew-symmetric matrices and vectors (problem G23), which is implemented through the calculation we used to derive Eq. (43). The Hodge star operator has interesting applications in physics: For example, it can be used to formulate Maxwell's equations on general manifolds (e.g., curved spacetime), and to make their geometric structures manifest.

In the previous section, we saw that the displacement of a purely rotating body can be expressed as

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} . \tag{59}$$

This expression would also describe the motion of an object that is at rest in the rotating frame from the vantage point of an observer in the laboratory frame, *provided both frames have the same origin.* If we generalize this expression to allow translations in the body-fixed frame, we obtain

$$\boxed{\left(\frac{d\vec{r}}{dt}\right)_{\Sigma} = \left(\frac{d\vec{r}}{dt}\right)_{\Sigma'} + \vec{\omega} \times \vec{r} .} \tag{60}$$

In our discussion of the synondic frame for the restricted three-body problem on worksheet #9, we saw that this expression was all we needed to set up a Lagrangian and derive the forces and equations of motion. In particular, we obtained a velocity-dependent potential that generated the pseudoforces we would observe from within the rotating coordinate system.

In the following, we want to derive the same result in a slightly different way. Equation (60) suggests that we can relate the operations of taking a time derivative in the laboratory and body-fixed frames via

$$\boxed{\left(\frac{d}{dt}\right)_{\Sigma} = \left(\frac{d}{dt}\right)_{\Sigma'} + \vec{\omega} \times \ .} \tag{61}$$

This is a relationship between two **linear differential operators** that act from the left on any vector of the three-dimensional space $\mathbb{R}^3$. Equation (61) immediately implies that

$$\left(\frac{d\vec{\omega}}{dt}\right)_{\Sigma} = \left(\frac{d\vec{\omega}}{dt}\right)_{\Sigma'} + \vec{\omega} \times \vec{\omega} = \left(\frac{d\vec{\omega}}{dt}\right)_{\Sigma'} , \tag{62}$$

i.e., the angular acceleration in both coordinate frames is going to be the same.

We can also use Eq. (61) to derive the relation between the accelerations of a mass in $\Sigma$ and $\Sigma'$:

$$\begin{aligned} \left(\frac{d^2\vec{r}}{dt^2}\right)_{\Sigma} &= \left(\left(\frac{d}{dt}\right)_{\Sigma'} + \vec{\omega}\times\right)\left(\left(\frac{d\vec{r}}{dt}\right)_{\Sigma'} + \vec{\omega} \times \vec{r}\right) \\ &= \left(\frac{d^2\vec{r}}{dt^2}\right)_{\Sigma'} + \left(\frac{d\vec{\omega}}{dt}\right)_{\Sigma'} \times \vec{r} + 2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{\Sigma'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) . \end{aligned} \tag{63}$$

Multiplying with $m$, we see that the equation of motion in the inertial laboratory frame,

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{\Sigma} = \vec{F} , \tag{64}$$

takes the form

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{\Sigma'} = \vec{F} + \vec{F}_E + \vec{F}_C + \vec{F}_{cf} , \tag{65}$$

with the same **pseudoforces** or **fictitious forces** that we previously discussed on worksheet #9:

1. the **Euler force**
$$\vec{F}_E \equiv -m \left(\frac{d\vec{\omega}}{dt}\right)_{\Sigma'} \times \vec{r} = \left(\frac{d\vec{\omega}}{dt}\right)_{\Sigma} \times \vec{r}, \tag{66}$$
which is caused by any variation of $\vec{\omega}$,
2. the **Coriolis force**
$$\vec{F}_C \equiv -2m\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{\Sigma'} , \tag{67}$$
3. and the **centrifugal force**
$$\vec{F}_{cf} \equiv -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \ . \tag{68}$$

**Frames with Different Origins**

Next, we want to consider the case where the origins of the two frames are in motion with respect to each other. Without loss of generality, we can assume that $\Sigma$ is at rest and $\Sigma'$ is moving and potentially accelerating. Let $\vec{R}$ denote the position of the origin of $\Sigma'$ as viewed from $\Sigma$ (cf. Fig. 1). If $\vec{r}$ is the position of an object in $\Sigma'$, its position in $\Sigma$ will be given by $\vec{R}+\vec{r}$. In general, it is clear that
$$m \left(\frac{d^2\vec{r}}{dt^2}\right)_{\Sigma} = \vec{F} - m \left(\frac{d^2\vec{R}}{dt^2}\right)_{\Sigma} , \tag{69}$$
and plugging in Eq. (63), we have
$$m \left(\frac{d^2\vec{r}}{dt^2}\right)_{\Sigma'} = \vec{F} - m \left(\frac{d^2\vec{R}}{dt^2}\right)_{\Sigma} - m \left(\frac{d\vec{\omega}}{dt}\right)_{\Sigma'} \times \vec{r} - 2m\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{\Sigma'} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \ . \tag{70}$$

### Application: Dynamics on the Rotating Earth

As an application of our expressions for the dynamics in a non-inertial reference frame, we consider the impact of the Earth's rotation on an object that is in free fall. To this end, we introduce a Cartesian coordinate system at some latitude $\lambda$ whose $z$ axis is perpendicular to the surface of the Earth, while the $x$ and $y$ axes points towards the equator and eastward along the meridian, respectively, (see Fig. 6).

Since the origin is not moving in the body-fixed frame, we must have

$$\left(\frac{d\vec{R}}{dt}\right)_{\Sigma'} = 0\,, \quad \left(\frac{d^2\vec{R}}{dt^2}\right)_{\Sigma'} = 0\,, \tag{71}$$

which implies

$$\left(\frac{d^2\vec{R}}{dt^2}\right)_{\Sigma} = \left(\frac{d\vec{\omega}}{dt}\right)_{\Sigma'} \times \vec{R} + \vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right)\,. \tag{72}$$

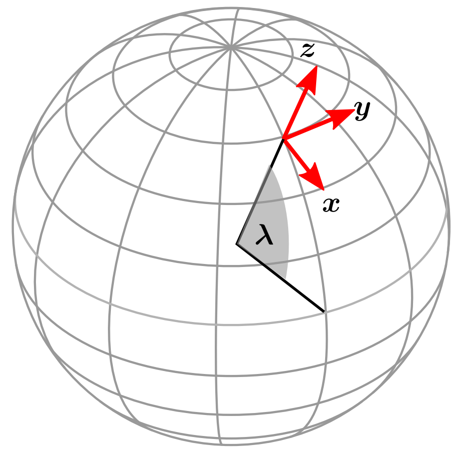


Figure 6: Dynamics on the rotating Earth

Noting that the angular velocity of the Earth is essentially constant, we obtain

$$m\left(\frac{d^2\vec{R}}{dt^2}\right)_{\Sigma} = m\vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right)\,. \tag{73}$$

Plugging this into Eq. (70) and using $\vec{F} = m\vec{g}$, the equation of motion becomes

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{\Sigma'} = m\vec{g}_{\text{eff}} - 2m\omega \times \left(\frac{d\vec{r}}{dt}\right)_{\Sigma'} - \vec{\omega} \times (\vec{\omega} \times \vec{r})\,, \tag{74}$$

where we have introduced the **effective gravitational acceleration**

$$\vec{g}_{\text{eff}} = \vec{g} - \vec{\omega} \times \left(\vec{\omega} \times \vec{R}\right)\,. \tag{75}$$

In the coordinate system $\Sigma'$,

$$\vec{g} = -g\vec{e}_z\,, \quad \vec{\omega} = -\omega\cos\lambda\vec{e}_x + \omega\sin\lambda\vec{e}_z\,, \tag{76}$$

and we also have

$$\vec{R} = R\vec{e}_z\,, \tag{77}$$

where $R$ is Earth's radius. Evaluating $\vec{g}_{\text{eff}}$, we find that

$$\vec{g}_{\text{eff}} = -\left(g - R\omega^2\cos^2\lambda\right)\vec{e}_z + R\omega^2\sin\lambda\cos\lambda\vec{e}_x\,, \tag{78}$$

hence the centrifugal effects due to Earth's rotation not only reduce the size of the $\vec{g}$ but also cause a deflection of a falling mass in north-south direction! Since

$$R\omega^2 = 6.371 \cdot 10^6\,\text{m} \cdot \frac{2\pi}{(24 \cdot 3600\,\text{s})^2} = 3.36 \cdot 10^{-2}\,\frac{\text{m}}{\text{s}^2} \tag{79}$$

the effect is small compared to $g = 9.8\,\mathrm{m/s}^2$, but it may not be negligible in precision measurements. The final term in Eq. (74) is at least an additional two to three orders of magnitude smaller for motion near the earth's surface since $|\vec{r}| \ll R$.

The Coriolis term in Eq. (74), on the other hand, can have a non-negligible effect. For a plane that is moving at the speed of sound, i.e. $v \approx 1000\,\mathrm{km/h}$, the Coriolis acceleration is on the order of $0.005\,g$. It is responsible for the formation of vortices in the large-scale dynamics of the Earth's oceans and atmosphere, most notably atmospheric cyclones. On a smaller scale, the Coriolis effect causes deflections in projectile motion, or the precession of Foucault's pendulum.

# 3 Rigid Body Dynamics

## 3.1 The Kinetic Energy of the Rigid Body

[...] Let $\vec{O}$ be the position of the origin of the body-fixed frame $\Sigma'$ in the laboratory frame $\Sigma$, and $\vec{r}_i'$ the positions of a mass element $\Delta m_i$. Then

$$\frac{d\vec{r}_i}{dt} = \frac{d}{dt}\left(\vec{O} + \vec{r}_i'\right) + \vec{\omega} \times \vec{r}_i'\,, = \frac{d\vec{O}}{dt} + \vec{\omega} \times \vec{r}_i'\,, \tag{80}$$

where we have used that the mass segment must be at a fixed distance from the origin due to the rigid body constraints. Plugging this into the kinetic energy, we obtain

$$\begin{aligned} T &= \frac{1}{2}\sum_i \Delta m_i \dot{\vec{r}}_i^2 \\ &= \frac{1}{2}\sum_i \Delta m_i \left(\dot{\vec{O}} + \vec{\omega} \times \vec{r}_i'\right)^2 \\ &= \frac{1}{2}\sum_i \Delta m_i \left(\dot{\vec{O}}^2 + 2\dot{\vec{O}} \cdot (\vec{\omega} \times \vec{r}_i') + (\vec{\omega} \times \vec{r}_i')^2\right) \\ &\equiv T_1 + T_2 + T_3\,. \end{aligned} \tag{81}$$

Since $\vec{R}$ does not depend on the index $i$, hence we can carry out the sum and obtain

$$T_1 = \frac{1}{2}\sum_i \Delta m_i \dot{\vec{O}}^2 = \frac{1}{2} M \dot{\vec{O}}^2\,, \tag{82}$$

where $M$ is the total mass of the rigid body. Thus, $T_1$ is the kinetic energy of the **translation** of the origin of the body-fixed frame — and therefore the rigid body itself — as seen by an observer in the laboratory frame $\Sigma$. Next, we have

$$\begin{aligned} T_2 &= \sum_i \Delta m_i \dot{\vec{O}} \cdot (\vec{\omega} \times \vec{r}_i') = M\dot{\vec{O}} \cdot \left(\vec{\omega} \times \left(\sum_i \frac{\Delta m_i \vec{r}_i'}{M}\right)\right) \\ &= M\dot{\vec{O}} \cdot \left(\vec{\omega} \times \vec{R}'\right)\,, \end{aligned} \tag{83}$$

where we have introduced the coordinate of the rigid body's center of mass *in* $\Sigma'$. Finally, we have the **kinetic energy of the rigid body's rotation** around the origin of the body-fixed frame:

$$T_3 = \frac{1}{2}\sum_i \Delta m_i \left(\vec{\omega} \times \vec{r}_i'\right)^2 = \frac{1}{2}\sum_i \Delta m_i \underbrace{(\vec{\omega} \times \vec{r}_i')}_{\vec{a}} \cdot \underbrace{(\vec{\omega} \times \vec{r}_i')}_{\vec{b}\times\vec{c}}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_i \Delta m_i \underbrace{\left((\vec{\omega}\times\vec{r}_i')\times\vec{\omega}\right)}_{\vec{a}\times\vec{b}}\cdot\underbrace{\vec{r}_i'}_{\vec{c}} = \frac{1}{2}\sum_i \Delta m_i \left(\vec{\omega}^2\,\vec{r}_i' - \left(\vec{\omega}\cdot\vec{r}_i'\right)\vec{\omega}\right)\cdot\vec{r}_i' \\
&= \frac{1}{2}\sum_i \Delta m_i \left(\vec{\omega}^2\,\vec{r}_i'^2 - \left(\vec{\omega}\cdot\vec{r}_i'\right)^2\right) \\
&\equiv \frac{1}{2}\vec{\omega}\cdot\boldsymbol{I}\cdot\vec{\omega}\,, \qquad (84)
\end{aligned}
$$

where we have introduce the **moment of inertia tensor**

$$
\boxed{I_{ab} = \sum_i \Delta m_i \left(\vec{r}_i^{\,2}\delta_{ab} - r_{ia}r_{ib}\right)\,. \qquad (85)}
$$

We will discuss $\boldsymbol{I}$ further in Secs. 3.3–3.5. Let us stress here already that **the moment-of-inertial tensor depends on the choice of origin and the orientation of the coordinate system** that we use to evaluate Eq. (85) (see Secs. 3.4 and 3.5).

**Center-of-Mass System**

The kinetic energy has a particularly simple form if we use the center of mass as the origin of our body-fixed frame. Then its position in $\Sigma'$ is $\vec{R}' = 0$ and we see that the mixed term in the kinetic energy vanishes:

$$
T_2 = M\dot{\vec{O}}\cdot\left(\vec{\omega}\times\vec{R}'\right) = M\dot{\vec{R}}\cdot\left(\vec{\omega}\times\vec{R}'\right) = 0\,. \qquad (86)
$$

We are left with

$$
\boxed{T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\vec{\omega}\cdot\boldsymbol{I}\cdot\vec{\omega}\,, \qquad (87)}
$$

which implies that we describe the motion of the rigid body as a translation of its center of mass accompanied by a rotation around that point (cf. Sec. 2.1.3).

## 3.2 The Angular Momentum of the Rigid Body

We can treat the angular momentum of the rigid body analogously to the kinetic energy. Starting from its form in the inertial system and using Eq. (80), we find

$$
\begin{aligned}
\vec{L} &= \sum_i \Delta m_i \left(\vec{r}_i\times\dot{\vec{r}}_i\right) \\
&= \sum_i \Delta m_i \left(\left(\vec{O}+\vec{r}_i'\right)\times\left(\dot{\vec{O}}+\vec{\omega}\times\vec{r}_i'\right)\right) \\
&= M\vec{O}\times\dot{\vec{O}} + M\left(\vec{R}'\times\dot{\vec{O}} + \vec{O}\times\left(\vec{\omega}\times\vec{R}'\right)\right) + \sum_i \Delta m_i \left(\vec{r}_i'\times\left(\vec{\omega}\times\vec{r}_i'\right)\right) \\
&= \vec{L}_1 + \vec{L}_2 + \vec{L}_3\,. \qquad (88)
\end{aligned}
$$

Here, $\vec{L}_1$ is the angular momentum of the body-fixed frame's origin in the laboratory frame, the mixed terms $\vec{L}_2$ are due to any deviation between the rigid body's center of mass and $O$, and $\vec{L}_3$ is the angular momentum for rotations around $O$. As in the case of the kinetic energy, we can make the mixed term $\vec{L}_2$ vanish by attaching the body-fixed frame to the center of mass, so that $\vec{O} = \vec{R}$ and $\vec{R}' = 0$. Also noting that

$$\sum_i \Delta m_i \left(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)\right) = \sum_i \Delta m_i \left(\vec{\omega}\left(\vec{r}_i' \cdot \vec{r}_i'\right) - \left(\vec{\omega} \cdot \vec{r}_i'\right) \vec{r}_i'\right) \tag{89}$$

or in components

$$\begin{aligned} L_{3a} = &= \sum_i \Delta m_i \left(\vec{r}_i'^2 \omega_a - \left(\sum_b \omega_b r_{ib}'\right) r_{ia}'\right) \\ &= \sum_b \left(\sum_i \Delta m_i \left(\vec{r}_i'^2 \delta_{ab} - r_{ia}' r_{ib}'\right)\right) \omega_b \\ &= \sum_b I_{ab} \omega_b \,, \end{aligned} \tag{90}$$

we obtain

$$\boxed{\vec{L} = M \vec{R} \times \dot{\vec{R}} + \boldsymbol{I} \cdot \vec{\omega} \equiv \vec{L}_{\rm cm} + \vec{L}' \,.} \tag{91}$$

Thus, we see that the **total angular momentum** of the rigid body is a contribution due to the motion of its center of mass around the origin of the laboratory frame, plus the angular momentum associated with the rigid body's rotation around the center of mass. The latter is sometimes referred to as the **relative** or **intrinsic angular momentum**.

It is important to notice that $\vec{L}'$ is in general *not* parallel to $\vec{\omega}$, since the moment of inertia tensor is a matrix. This is illustrated in Fig. 7.

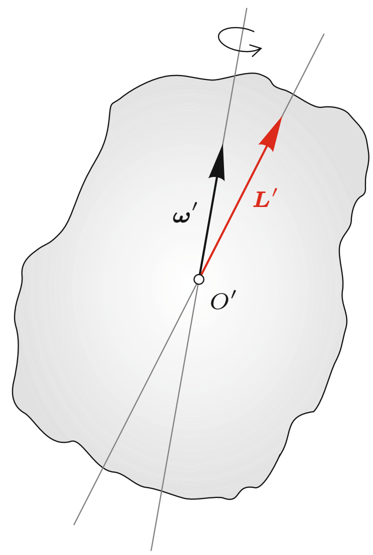


Figure 7: The intrinsic angular momentum $\vec{L}'$ and the angular velocity $\vec{\omega}$, which also indicates the instantaneous rotational axis, are not parallel in general.

We conclude our discussion by pointing out that the rotational kinetic energy of the rigid body in Eq. (87) can be expressed as

$$T_{\rm rot} = \frac{1}{2} \vec{\omega} \cdot \boldsymbol{I} \cdot \vec{\omega} = \frac{1}{2} \vec{L}' \cdot \vec{\omega} \,. \tag{92}$$

### 3.3 The Moment of Inertia Tensor

In the previous section, we have treated the rigid body as a sum of discrete mass elements $\Delta m_i$, but now we want to take the limit where we make these mass elements infinitesimally small while keeping the overall mass $M$ constant, of course. Then

$$\sum_i \Delta m_i \quad \rightarrow \quad \int dm = \int d^3 r \; \rho(\vec{r}) \,, \tag{93}$$

where $\rho(\vec{r})$ is the **mass density distribution**. Then we can define the **continuous moment of inertia tensor**

$$I_{ab} = \int d^3r\ \rho(\vec{r}) \left(\vec{r}^2 \delta_{ab} - r_a r_b\right) \ . \tag{94}$$

Here, the integral extends over all of space, hence we rely on $\rho(\vec{r})$ to restrict the integrand to the actual finite volume of the rigid body. **We stress again that $\boldsymbol{I}$ depends on the chosen origin and orientation of the coordinate system that we use to evaluate** (94)**.**

From Eq. (94), it is immediately apparent that $\boldsymbol{I}$ is **symmetric**, i.e.,

$$\boldsymbol{I}^T = \boldsymbol{I}\,, \quad I_{ab} = I_{ba}\,. \tag{95}$$

Thus, it can at most have six independent elements. Its diagonal entries are

$$\begin{aligned} I_{xx} &= \int d^3r\ \rho(\vec{r})\big(\vec{r}^2 \underbrace{\delta_{xx}}_{=1} - x^2\big) = \int d^3r\ \rho(\vec{r}) \left(x^2 + y^2 + z^2 - x^2\right) \\ &= \int d^3r\ \rho(\vec{r}) \left(y^2 + z^2\right) \ , \qquad (96) \\ I_{yy} &= \int d^3r\ \rho(\vec{r}) \left(x^2 + z^2\right) \ , \qquad (97) \\ I_{zz} &= \int d^3r\ \rho(\vec{r}) \left(x^2 + y^2\right) \ , \qquad (98) \end{aligned}$$

and we will see shortly that they are the **moments of inertia** for rotations around the axes of the chosen coordinate system. The off-diagonal elements have the structure

$$I_{xy} = \int d^3r\ \rho(\vec{r})\big(\vec{r}^2 \underbrace{\delta_{xy}}_{=0} - xy\big) = -\int d^3r\ \rho(\vec{r}) xy \tag{99}$$

etc. They are referred to as **products of inertia** and **deviation moments** in the literature.

### Moment of Inertia for Arbitrary Rotational Axis

The tensor $\boldsymbol{I}$ allows us to determine the rigid body's moment of inertia for rotations around an arbitrary rotational axis defined by $\vec{n}$. If the rigid body is rotating with an angular velocity $\omega$ around that axis, the rotational kinetic energy is

$$T_{\mathrm{rot}} = \frac{1}{2}\vec{\omega}\cdot\boldsymbol{I}\cdot\vec{\omega} = \frac{1}{2}\omega^2 \vec{n}\cdot\boldsymbol{I}\cdot\vec{n} \equiv \frac{1}{2} I \omega^2\,, \tag{100}$$

with

$$I = \vec{n}\cdot\boldsymbol{I}\cdot\vec{n} = \sum_{ab} I_{ab} n_a n_b\,. \tag{101}$$

As an example, let us consider a rotation around one of the axes of our coordinate system, e.g., $\vec{n} = \vec{e}_x = (1,0,0)^T$. In this case, we see that

$$I = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = I_{xx}\,. \tag{102}$$

If the rotation is around a general axis, the deviation moments contribute, e.g., $\vec{n} = \frac{1}{\sqrt{2}}(1,1,0)^T$:

$$
\begin{aligned}
I &= \frac{1}{2}\begin{pmatrix}1 & 1 & 0\end{pmatrix}\begin{pmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{xy} & I_{yy} & I_{yz}\\ I_{xz} & I_{yz} & I_{zz}\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1 & 0\end{pmatrix}\begin{pmatrix}I_{xx}+I_{xy}\\ I_{xy}+I_{yy}\\ I_{xz}+I_{yz}\end{pmatrix}\\
&= \frac{1}{2}\left(I_{xx} + 2I_{xy} + I_{yy}\right)\,. \qquad (103)
\end{aligned}
$$

### Example: Moment of Inertia Tensor of a Thin Hoop

As a concrete example, we want to compute the moment of inertia tensor of a homogenous hoop of mass $M$ and radius $R$. If the thickness of the hoop is small compared to its radius, we can describe it as an idealized mass distribution that is constructed using Dirac $\delta$-functions (see Box 3.1). Choosing our coordinate system such that the hoop lies in the $xy$ plane and that the origin coincides with the hoop's center, we can use cylindrical coordinates and make the ansatz

$$
\rho_M(\vec{r}) = \rho_M(\rho,\phi,z) = C\delta(\rho - R)\delta(z)\,, \qquad (104)
$$

where we indicated the mass density by $\rho_M$ to distinguish it from the radial coordinate $\rho$. We can determine the unknown constant $C$ by computing the volume integral over $\rho_M(\vec{r})$, which must equal the mass $M$ of the hoop:

$$
\begin{aligned}
M \overset{!}{=} \int d^3r\ \rho_M(\vec{r}) &= C\int_0^{2\pi} d\phi \int_0^{\infty} d\rho\ \rho \int_{-\infty}^{\infty} dz\ \delta(\rho - R)\delta(z)\\
&= C\int_0^{2\pi} d\phi \int_0^{\infty} d\rho\ \rho\delta(\rho - R) \int_{-\infty}^{\infty} dz\ \delta(z)\\
&= C\cdot 2\pi\cdot R\cdot 1\,. \qquad (105)
\end{aligned}
$$

Thus, we have

$$
\rho_M(\vec{r}) = \frac{M}{2\pi R}\delta(\rho - R)\delta(z)\,, \qquad (106)
$$

and we see that the constant corresponds to the **linear mass density** of the hoop, with units mass per length. As discussed in Box 3.1, the $\delta$ functions have the inverse dimension of their variables, so we see that the mass distribution has units mass per volume, as required.

Plugging $\rho_M(\vec{r})$ into Eq. (94), we first determine the diagonal entries:

$$
\begin{aligned}
I_{xx} &= \int d^3r\ \rho_M(\vec{r})\left(y^2+z^2\right) = \frac{M}{2\pi R}\int_0^{2\pi} d\phi \int_0^{\infty} d\rho \int_{-\infty}^{\infty} dz\ \rho\left(\rho^2\sin^2\phi + z^2\right)\delta(\rho - R)\delta(z)\\
&= \frac{M}{2\pi R}\int_0^{2\pi} d\phi\ R^3\sin^2\phi = \frac{MR^2}{2\pi}\int_0^{2\pi} d\phi\ \sin^2\phi = \frac{1}{2}MR^2\,, \qquad (107)\\
I_{yy} &= \int d^3r\ \rho_M(\vec{r})\left(x^2+z^2\right) = \frac{M}{2\pi R}\int_0^{2\pi} d\phi \int_0^{\infty} d\rho \int_{-\infty}^{\infty} dz\ \rho\left(\rho^2\cos^2\phi + z^2\right)\delta(\rho - R)\delta(z)\\
&= \frac{M}{2\pi R}\int_0^{2\pi} d\phi\ R^3\cos^2\phi = \frac{MR^2}{2\pi}\int_0^{2\pi} d\phi\ \cos^2\phi = \frac{1}{2}MR^2\,, \qquad (108)\\
I_{zz} &= \int d^3r\ \rho_M(\vec{r})\left(x^2+y^2\right) = \frac{M}{2\pi R}\int_0^{2\pi} d\phi \int_0^{\infty} d\rho \int_{-\infty}^{\infty} dz\ \rho^3\delta(\rho - R)\delta(z)
\end{aligned}
$$

### Box 3.1: The Dirac and Heaviside Distributions

For modeling mathematically idealized density (or charge) distributions, we often rely on the so-called **Dirac** $\delta(x)$ and **Heaviside step functions** $\Theta(x)$. They are not functions in the traditional mathematical sense, but **generalized functions** or **distributions**[a]. The $\delta(x)$, in particular, is meant to be used alongside regular functions within an integral, as we will see from its definition below. The same is true for many applications of the step function $\theta(x)$, although it is closer to a regular function.
The delta function, can be viewed as a generalization of the Kronecker symbol $\delta_{ij}$ to continuous variables. It is formally defined by its action in an integral:

$$\int_a^b dx\; f(x)\delta(x-x_0) = \begin{cases} f(x_0) & \text{for } x_0 \in [a,b]\,, \\ 0 & \text{else.} \end{cases} \tag{I3.1-1}$$

We can construct it as a limit of a series of functions, e.g., Gaussians,

$$g_a(x) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{x^2}{2a^2}}\,, \tag{I3.1-2}$$

so that

$$\lim_{a\to\infty} g_a(x) = \delta(x)\,. \tag{I3.1-3}$$

Plugging the $g_a(x)$ into an integral alongside a general function $f(x)$ and performing the limit on the integration result, we will then recover Eq. (I3.1-1).
The step function is defined as

$$\Theta(x) = \begin{cases} 1 & \text{if} \quad x > 0\,, \\ \frac{1}{2} & \text{if} \quad x = 0\,, \\ 0 & \text{if} \quad x < 0\,. \end{cases} \tag{I3.1-4}$$

We see that we can use it to impose finite limits on an integration that extends over the entire real axis: For example

$$\int_{-\infty}^{\infty} dx\; \Theta(x-a)\Theta(b-x) f(x) = \int_a^b dx\; f(x)\,. \tag{I3.1-5}$$

#### Use with Dimensionful Physical Quantities

In physical applications, the arguments of $\delta(x)$ and $\Theta(x)$ will usually be dimensionful quantities, and we need to check whether the two functions will have dimensions of their own as a consequence:

- From Eq. (I3.1-1), we can see that the units of $\delta(x)$ must be *the inverse* of the units of the measure $dx$ of the integral, so that the units on the left- and right-hand sides of the equation can be solely determined by $f(x)$. Thus, $\delta(x)$ is an inverse length, $\delta(\phi)$ an inverse angle, $\delta(t)$ an inverse time (i.e., a frequency), and so on.
- In contrast, the definition (I3.1-4) makes it clear that $\Theta(x)$ is dimensionless.

[a] Schwartz developed the formal theory of $\delta(x)$ and other distributions by defining them as **functionals** on a space of well-behaved (i.e., continuous and highly differentiable) test functions.

$$= \frac{MR^2}{2\pi} \int_0^{2\pi} d\phi \; = MR^2 \,, \tag{109}$$

where we have used that

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \; \sin^2\phi = \frac{1}{2\pi} \int_0^{2\pi} d\phi \; \cos^2\phi = \frac{1}{2} \,. \tag{110}$$

We could have anticipated that $I_{xx} = I_{yy}$ because of the symmetry of the hoop under rotations around the $z$ axis in our coordinate system, which is its **figure** or **symmetry axis**.

Now let us consider the deviation moments. Because of the $\delta(z)$ factor in $\rho_M(\vec{r})$, we immediately notice that

$$I_{xz} \sim \int_{-\infty}^{\infty} dz \; z\delta(z) = 0 \tag{111}$$

and $I_{yz} = 0$ as well. That leaves $I_{xy}$:

$$\begin{aligned} I_{xy} &= -\int d^3r \; \rho_M(\vec{r})xy = \frac{M}{2\pi R} \int_0^{2\pi} d\phi \int_0^{\infty} d\rho \int_{-\infty}^{\infty} dz \; \rho^3 \sin\phi \cos\phi \; \delta(\rho - R)\delta(z) \\ &= \frac{MR^2}{2\pi} \int_0^{2\pi} d\phi \; \frac{1}{2} \sin(2\phi) = 0 \,. \end{aligned} \tag{112}$$

Thus, the hoop's moment of inertia tensor with respect to its center of mass is given by

$$\boldsymbol{I} = \begin{pmatrix} \frac{1}{2}MR^2 & & \\ & \frac{1}{2}MR^2 & \\ & & MR^2 \end{pmatrix} = \frac{1}{2}MR^2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix} \,. \tag{113}$$

How would the moment of inertia tensor change if we let the hoop rotate around axes through a pivot point on its rim? For concreteness, we shift the hoop by $\vec{R}' = R\vec{e}_y$ in our coordinate sytem (see Fig. 8), so that points on the hoop are

$$x = R\cos\phi, \quad y = R(1 + \sin\phi), \quad z = 0 \,. \tag{114}$$

Then

$$\begin{aligned} I'_{xx} &= \frac{M}{2\pi R} \int_0^{2\pi} d\phi \; R^3 \,(1 + \sin\phi)^2 = \frac{MR^2}{2\pi} \int_0^{2\pi} d\phi \; \left(1 + 2\sin\phi + \sin^2\phi\right) \\ &= MR^2 \left(1 + 0 + \frac{1}{2}\right) = \frac{3}{2}MR^2 \,, \end{aligned} \tag{115}$$

$$I'_{yy} = \frac{M}{2\pi R} \int_0^{2\pi} d\phi \; R^3 \cos^2\phi = \frac{1}{2}MR^2 \,, \tag{116}$$

$$I'_{zz} = \frac{M}{2\pi R} \int_0^{2\pi} d\phi \; R^3 \left(\cos^2\phi + 1 + 2\sin\phi + \sin^2\phi\right) = \frac{MR^2}{2\pi} \int_0^{2\pi} d\phi \; (2 + 2\sin\phi) = 2MR^2 \,. \tag{117}$$

The deviation moments involving $z$ still vanish as in the previous calculation,

$$I'_{xz} = I'_{yz} = 0 \,, \tag{118}$$

and

$$I'_{xy} = -\frac{M}{2\pi R} \int_0^{2\pi} d\phi \; \cos\phi \,(1 - \sin\phi) = 0 \,, \tag{119}$$

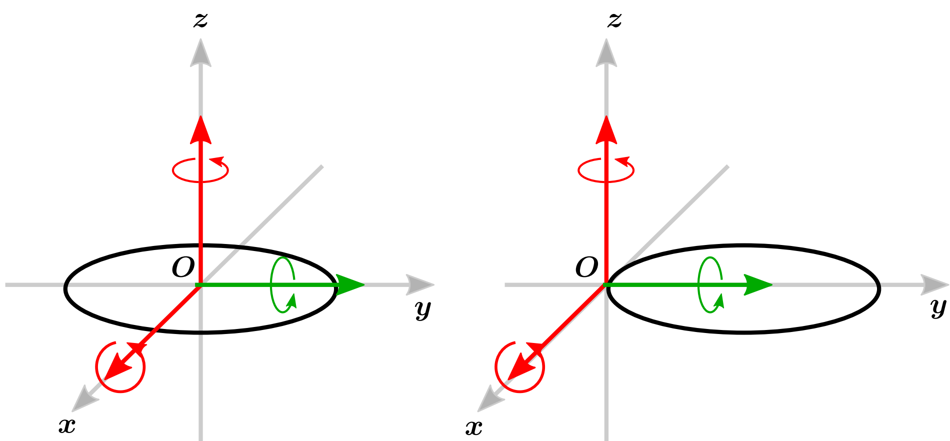


Figure 8: Rotations of a hoop around its center of mass or a pivot point on the rim: The moments of inertia around the $x$ and $z$ axes (indicated in red) change, while the moment of inertia for rotations around the $y$ axis (green) remain unchanged because the hoop was shifted in this direction.

because we are still integrating trigonometric functions over multiples of their full period. We see that

$$\boldsymbol{I}' = \frac{1}{2}MR^2 \begin{pmatrix} 3 & & \\ & 1 & \\ & & 4 \end{pmatrix} = \boldsymbol{I} + MR^2 \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \tag{120}$$

i.e. the moments of inertia along the $x$ and $z$ axes change while the moment the $y$ axis is the same as before, which we can trace back to the fact that we shifted the pivot point along the $y$ axis. We will come back to this result when we discuss the parallel-axis theorem in Sec. 3.5.

### 3.4 Principal Moments of Inertia and the Principal-Axis Frame

Since $\boldsymbol{I}$ is represented by a real symmetric matrix, we are always able to diagonalize it by solving the eigenvalue problem

$$\boldsymbol{I}\vec{v} = I\vec{v}\,. \tag{121}$$

Because of its symmetry, eigenvectors for non-degenerate eigenvalues are guaranteed to be orthogonal, so we the (normalized) eigenvectors can be used to construct an orthogonal transformation by arranging their column-vector representations into a $3 \times 3$ matrix:

$$\boldsymbol{R} \equiv \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix}\,. \tag{122}$$

Here, we have made use of the fact that eigenvectors are only defined up to a sign to ensure that $\det \boldsymbol{R} = 1$, so that the orthogonal transformation is a proper rotation. In its eigenbasis, $\boldsymbol{I}$ is diagonal,

$$\boldsymbol{R}\boldsymbol{I}\boldsymbol{R}^T = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \tag{123}$$

and its eigenvalues $A, B, C$ define the **principal moments of inertia**. The associated eigenvectors define the **principal axes** of the rigid body.

In the principal-axis frame, the rotational kinetic energy and the angular momentum have particularly easy forms:

$$T_{\text{rot}} = \frac{1}{2}\vec{\omega}\cdot\boldsymbol{I}\cdot\vec{\omega} = \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right) \tag{124}$$

and

$$\vec{L} = \boldsymbol{I}\vec{\omega} = \begin{pmatrix} A\omega_1 \\ B\omega_2 \\ C\omega_3 \end{pmatrix} . \tag{125}$$

### 3.5 The Parallel-Axis Theorem

In our earlier discussion, we saw that we can simplify the description of a rigid body's motion if we attach the body-fixed frame to its center of mass (cf. Secs. 3.1 and 3.2), and we can achieve further simplification by diagonalizing $\boldsymbol{I}$ (cf. Sec. 3.4). Thus, the optimal choice for the body-fixed frame is the principal-axis frame with the center of mass at the origin. In practice, however, constructing the moment-of-inertia tensor in the center-of-mass frame of a complicated mass distribution may not be so simple, and it could be more convenient to "break down" the rigid body into components and asssemble the total $\boldsymbol{I}$ from their respective tensors.

As an example, let us consider a dumbbell consisting of two spherical weights connected by a rod (cf. Fig. 9). The moment of inertia tensor for a sphere with respect to its center of mass is

$$\boldsymbol{I}_{\text{sphere}} = \frac{2}{5}MR^2\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \tag{126}$$

and that of an idealized thin rod of length $L$, aligned with the $x$ axis, is

$$\boldsymbol{I}_{\text{rod}} = \frac{1}{12}ML^2\begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} , \tag{127}$$

as we can verify by taking the limit of the tensor for a cylinder as $R \to 0$ (see problem G25). We note that it is in diagonal form, and that $I_{xx} = A = 0$ because the mass is idealized to lie exactly on the coordinate axis.

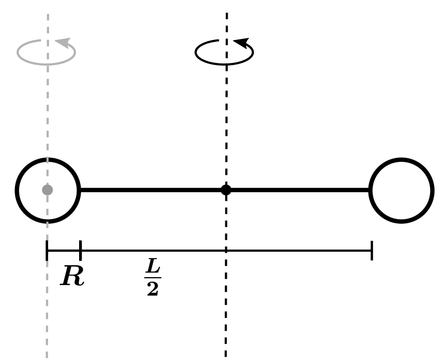


Figure 9: Illustration of the parallel-axis theorem for an idealized dumbbell consisting of two spheres of radius $R$ connected by a thin rod of length $L$.

From symmetry, it is clear that the center of mass of the dumbbell will coincide with the rod's center of mass, so the contribution from the rod to the overall tensor will be Eq. (127), but we cannot simply add $\boldsymbol{I}_{\text{sphere}}$ twice because the spheres are not rotating around an axis through their center of mass, but an external axis!

To see how the moment of inertia tensor changes if we rotate around a general pivot point, we choose that pivot as the origin of our new body-fixed coordinate system. This implies that the center of mass has the position $\vec{R} \neq 0$. Indicating the position vectors in the center-of-mass frame by primes for the present discussion, we have

$$\vec{r} = \vec{R} + \vec{r}\,' . \tag{128}$$

Plugging this into the definition of $\boldsymbol{I}$, we obtain

$$
\begin{aligned}
I_{ab} &= \int d^3r\ \rho(\vec{r})\left(\vec{r}^2\delta_{ab} - r_a r_b\right) \\
&= \int d^3r'\ \rho_{\rm cm}(\vec{r}')\left(\left(\vec{R}+\vec{r}'\right)^2\delta_{ab} - (R_a + r'_a)(R_b + r'_b)\right) \\
&= \int d^3r'\ \rho_{\rm cm}(\vec{r}')\left((\vec{R}^2 + 2\vec{r}'\cdot\vec{R} + \vec{r}'^2)\delta_{ab} - (R_aR_b + R_ar'_b + R_br'_a + r'_ar'_b)\right) \\
&= \int d^3r'\ \rho_{\rm cm}(\vec{r}')\left(\vec{R}^2\delta_{ab} - R_aR_b + \vec{r}'^2\delta_{ab} - r'_ar'_b + 2\vec{r}'\cdot\vec{R}\delta_{ab} - R_ar'_b - R_br'_a\right)\,,
\end{aligned}
\tag{129}
$$

where we substituted variables and introduced the density distribution as seen from the center-of-mass frame (which is usually easier to formulate):

$$
\rho_{\rm cm}(\vec{r}') = \rho(\vec{r}' - \vec{R})\,. \tag{130}
$$

The tensor structure in the first two terms of Eq. (129) does not depend on $\vec{r}'$, so we can perform the integration, which merely gives us the total mass:

$$
I^{(1)}_{ab} = \left(\vec{R}^2\delta_{ab} - R_aR_b\right)\int d^3r'\ \rho_{\rm cm}(\vec{r}') = M\left(\vec{R}^2\delta_{ab} - R_aR_b\right)\,. \tag{131}
$$

The next contribution is the moment of inertia tensor with respect to the center of mass:

$$
I^{(2)}_{ab} \equiv (I_{\rm cm})_{ab} = \int d^3r'\ \rho_{\rm cm}(\vec{r}')\left(\vec{r}'^2\delta_{ab} - r'_ar'_b\right)\,. \tag{132}
$$

That leaves the contribution

$$
I^{(3)}_{ab} = \int d^3r'\ \rho_{\rm cm}(\vec{r}')\left(2\vec{r}'\cdot\vec{R}\,\delta_{ab} - R_ar'_b - R_br'_a\right) = 2M\vec{R}'\cdot\vec{R}\,\delta_{ab} - MR_aR'_b - MR'_aR_b = 0\,, \tag{133}
$$

where we have used the definition of the center of mass of a continuous mass distribution

$$
\vec{R} \equiv \frac{1}{M}\int d^3r\ \rho(\vec{r})\vec{r}, \tag{134}
$$

and exploited that $\vec{R}' = 0$ in the center-of-mass system.

Our final result is the **parallel-axis theorem**:

**Parallel-Axis Theorem (Steiner, Huygens)**

For a rigid body of mass $M$, the moment-of-inertia tensor with respect to an arbitrary pivot point is the sum of the moment-of-inertia tensor with respect to the center of mass and a contribution describing the rotation of the total mass $M$, concentrated at the center of mass, around the pivot:

$$
I_{ab} = (I_{\rm cm})_{ab} + M\left(\vec{R}^2\delta_{ab} - R_aR_b\right)\,. \tag{135}
$$

**Example: Dumbbell**

Let us use the parallel-axis theorem to complete the determination of the dumbbell's moment of inertia. Choosing the center-of-mass of the rod as the pivot point and assuming that the rod is aligned with the $x$ axis of our coordinate system, we note that the sphere's centers of mass are offset by $\vec{R} = \pm(R + \frac{L}{2})\vec{e}_x$. For a single sphere, we have

$$\Delta \boldsymbol{I}_{\text{sphere}} = M \begin{pmatrix} 0 & 0 & 0 \\ 0 & \left(R + \frac{L}{2}\right)^2 & 0 \\ 0 & 0 & \left(R + \frac{L}{2}\right)^2 \end{pmatrix} , \tag{136}$$

so the tensor of the dumbbell is

$$\begin{aligned} \boldsymbol{I} &= \boldsymbol{I}_{\text{rod}} + 2\boldsymbol{I}_{\text{sphere}} + 2\Delta\boldsymbol{I}_{\text{sphere}} \\ &= M \begin{pmatrix} \frac{1}{12}L^2 + \frac{4}{5}R^2 & 0 & 0 \\ 0 & \frac{1}{12}L^2 + \frac{4}{5}R^2 + 2\left(R + \frac{L}{2}\right)^2 & 0 \\ 0 & 0 & \frac{1}{12}L^2 + \frac{4}{5}R^2 + 2\left(R + \frac{L}{2}\right)^2 \end{pmatrix} \\ &= M \begin{pmatrix} \frac{1}{12}L^2 + \frac{4}{5}R^2 & 0 & 0 \\ 0 & \frac{7}{12}L^2 + 2LR + \frac{14}{5}R^2 & 0 \\ 0 & 0 & \frac{7}{12}L^2 + 2LR + \frac{14}{5}R^2 \end{pmatrix} . \end{aligned} \tag{137}$$

**Example: Hoop**

Finally, let us revisit the rotation of the hoop around its rim from Sec. 3.3. In a coordinate system that is anchored in the pivot point on the hoop's rim, the center of mass of the hoop had the position $\vec{R} = R\vec{e}_y$, so the shift tensor $\Delta\boldsymbol{I}$ in Eq. (135) has the form

$$\Delta\boldsymbol{I} = M \begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2 - R^2 & 0 \\ 0 & 0 & R^2 \end{pmatrix} , \tag{138}$$

and we obtain

$$\boldsymbol{I} = \frac{1}{2}MR^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \frac{1}{2}MR^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{2}MR^2 \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} , \tag{139}$$

as before.

# 4 Group Exercises

### Problem G24 – Mass Densities

Which solids are described by the following mass densities?

$$\rho_1(\vec{r}) = \frac{M}{\pi R^2}\Theta\left(R-\rho\right)\delta\left(z\right)\,, \tag{140}$$

$$\rho_2(\vec{r}) = \frac{M}{\pi R^2 H}\Theta\left(R-\rho\right)\Theta\left(\frac{H}{2}-|z|\right)\,, \tag{141}$$

$$\rho_3(\vec{r}) = \frac{M}{\pi(a^2-b^2)H}\Theta\left(a-\rho\right)\Theta\left(\rho-b\right)\Theta\left(\frac{H}{2}-|z|\right)\,, \tag{142}$$

$$\rho_4(\vec{r}) = \frac{M}{4\pi R^2}\delta(r-R)\,. \tag{143}$$

Perform a volume integration over three-dimensional space in appropriate coordinates to show that you obtain the mass $M$ of the solid.

HINT:

$$\int_a^b f(x)\delta(x-x_0) = \begin{cases} f(x_0) & \text{if} \quad x_0 \in [a,b]\,, \\ 0 & \text{else,} \end{cases} \qquad \Theta(x) = \begin{cases} 1 & \text{if} \quad x > 0\,, \\ \frac{1}{2} & \text{if} \quad x = 0\,, \\ 0 & \text{if} \quad x < 0\,. \end{cases}$$

### Problem G25 – Moment of Inertia Tensor

A hollow cylinder of mass $M$ and radius $R$ can be described by the mass density

$$\rho(r,\varphi,z) = \frac{M}{2\pi R H}\Theta\left(\frac{H}{2}-|z|\right)\delta(r-R)\,, \tag{144}$$

where $r$ indicates the radial distance from the cylinder's symmetry axis, chosen to be the $z$ axis of our coordinate system.

1. Compute the moment of inertia tensor $\boldsymbol{I}$ of the cylinder, and determine its principal axes.

2. Use rotation matrices to determine $\boldsymbol{I}$ in coordinate systems that are rotated by *(i)* $\frac{2\pi}{3}$ around the original $z$ axis, and *(ii)* by $\frac{\pi}{4}$ around the original $y$ axis.

3. Verify by explicit calculation that the kinetic energy for a rotation around the axis $\vec{\omega} = \omega\vec{e}_z$ in the original coordinate system is identical to the kinetic energy we obtain for that motion in the two rotated coordinate systems.

HINT:

$$\boldsymbol{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}\,, \quad \boldsymbol{R}_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}\,, \quad \boldsymbol{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Problem G26 – Racing Solids

A solid sphere, a solid cylinder and a thin-walled, hollow cylinder (both of height $H$) with equal masses $M$ and radii $R$ are rolling down an inclined plane with inclination angle $\alpha$.

1. Compute the solids' moments of inertia with respect to rotations around principal axes through the center of mass that are relevant for the rolling motion studied here. **You do not have to compute the complete moment of inertia tensor.**

2. Describe the motion of the solids as a superposition of center-of-mass translation and rotation around the center of mass. Construct the Lagrangian, determine the equations of motion and state their general solutions.

3. Now describe the motion of the solids as a rotation around a principal axis through the point where the solid touches the inclined plane. Construct the Lagrangian for this case and show that you obtain the same equations of motion as in the previous part of the problem.

4. Which of the three solids will reach the bottom of the inclined plane in the shortest amount fo time after being released from rest at the top?