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PHY422/820: Classical Mechanics

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Worksheet #11 (Nov 8 – Nov 12)

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1 Preparation

- Lemos, Chapter 4
- Goldstein, Chapter 5

2 Rigid Body Dynamics

2.1 The Euler Equations

In the (inertial) laboratory frame Σ, the rotational dynamics of a rigid body with one fixed point O is described by the "rotational Second Law"

$$\left(\frac{d\vec{L}}{dt}\right)_\Sigma = \vec{N}\,, \tag{1}$$

where $\vec{N}$ is the torque about O due to external forces. Using the relation

$$\left(\frac{d}{dt}\right)_\Sigma = \left(\frac{d}{dt}\right)_{\Sigma'} + \vec{\omega}\times \tag{2}$$

between time derivatives in the laboratory and body-fixed frames (see worksheet #10), we can write

$$\left(\frac{d\vec{L}}{dt}\right)_{\Sigma'} + \vec{\omega}\times\vec{L} = \vec{N}\,. \tag{3}$$

Noting that $\vec{L} = \boldsymbol{I}\vec{\omega}$ and that the moment of inertia tensor $\boldsymbol{I}$ is time-independent in the body-fixed frame[1], we can write

$$\boldsymbol{I}\dot{\vec{\omega}} + \vec{\omega}\times(\boldsymbol{I}\vec{\omega}) = \vec{N}\,, \tag{4}$$

where we have used that the time derivative of $\vec{\omega}$ is the same in Σ and Σ':

$$\left(\frac{d\vec{\omega}}{dt}\right)_\Sigma = \left(\frac{d\vec{\omega}}{dt}\right)_{\Sigma'} + \vec{\omega}\times\vec{\omega} = \left(\frac{d\vec{\omega}}{dt}\right)_{\Sigma'}\,. \tag{5}$$

[1] In the laboratory frame, we have

$$\boldsymbol{I}(t) = \boldsymbol{R}(t)\boldsymbol{I}_0\boldsymbol{R}^T(t)$$

because the axes of the rigid body are rotating in space.

Choosing the principal-axis frame as our Σ', we can evaluate the cross product in Eq. (4),

$$\vec{\omega} \times \boldsymbol{I}\vec{\omega} = \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix} \times \begin{pmatrix} A\omega_{x'} \\ B\omega_{y'} \\ C\omega_{z'} \end{pmatrix} = \begin{pmatrix} (C-B)\omega_{y'}\omega_{z'} \\ (A-C)\omega_{z'}\omega_{x'} \\ (B-A)\omega_{x'}\omega_{y'} \end{pmatrix} , \tag{6}$$

and we eventually obtain the **Euler equations**:

$$A\dot{\omega}_{x'} + (C-B)\omega_{y'}\omega_{z'} = N_{x'} , \tag{7}$$
$$B\dot{\omega}_{y'} + (A-C)\omega_{z'}\omega_{x'} = N_{y'} , \tag{8}$$
$$C\dot{\omega}_{z'} + (B-A)\omega_{x'}\omega_{y'} = N_{z'} . \tag{9}$$

2.2 The Free Top

As a first application we consider a free top with $\vec{N} = 0$. (Note that this does *not* mean that no forces are acting on the top!) Then the Euler equations read

$$A\dot{\omega}_{x'} + (C-B)\omega_{y'}\omega_{z'} = 0 , \tag{10}$$
$$B\dot{\omega}_{y'} + (A-C)\omega_{z'}\omega_{x'} = 0 , \tag{11}$$
$$C\dot{\omega}_{z'} + (B-A)\omega_{x'}\omega_{y'} = 0 . \tag{12}$$

Multiplying each equation by the appropriate $\omega_{i'}$ and adding them, we obtain

$$A\dot{\omega}_{x'}\omega_{x'} + B\dot{\omega}_{y'}\omega_{y'} + C\dot{\omega}_{z'}\omega_{z'} = \frac{1}{2}\frac{d}{dt}\left(A\omega_{x'}^2 + B\omega_{y'}^2 + C\omega_{z'}^2\right) = 0 . \tag{13}$$

Thus, the rotational kinetic energy — which is the total energy of the torque-free top with a fixed point — is conserved:

$$E = T_{\text{rot}} = \frac{1}{2}\left(A\omega_{x'}^2 + B\omega_{y'}^2 + C\omega_{z'}^2\right) = \text{const.} \tag{14}$$

Multiplying the Euler equations by the components of $\vec{L}$ instead, addition yields

$$A^2\dot{\omega}_{x'}\omega_{x'} + B^2\dot{\omega}_{y'}\omega_{y'} + C^2\dot{\omega}_{z'}\omega_{z'} = \frac{1}{2}\frac{d}{dt}\left(A^2\omega_{x'}^2 + B^2\omega_{y'}^2 + C^2\omega_{z'}^2\right) = 0 . \tag{15}$$

We notice that the expression in parentheses is nothing but $\vec{L}^2$, hence

$$\frac{d\vec{L}^2}{dt} = 0 . \tag{16}$$

Thus, the *length* of $\vec{L}$ is conserved.

Finally, let us consider under which conditions the direction of $\vec{L}$ is conserved as well. This requires

$$A\dot{\omega}_{x'} = 0 , \tag{17}$$
$$B\dot{\omega}_{y'} = 0 , \tag{18}$$
$$C\dot{\omega}_{z'} = 0 , \tag{19}$$

so we must have

$$(C-B)\omega_{y'}\omega_{z'} = 0 , \tag{20}$$

$$(A - C)\omega_{z'}\omega_{x'} = 0\,, \tag{21}$$
$$(B - A)\omega_{x'}\omega_{y'} = 0\,. \tag{22}$$

Let us consider the possible solutions:

- A trivial solution to these equations is obtained for a rigid body with degenerate moments of inertia, $A = B = C$. Then $\vec{\omega}$ is fixed and $\vec{L}$ is parallel for any possible $\vec{\omega}$.

- If two moments of inertia are identical, e.g., $A = B$, then one of the equations will be trivially satisfied. This implies that $\vec{e}_{z'}$, the principal axis associated with C, is the symmetry axis of the rigid body. If the rotational axis is parallel to $\vec{e}_{z'}$, $\vec{L} = C\vec{\omega} =$ const. is a solution. Alternatively, $\vec{\omega}$ can be a constant rotational axis in the $x'y'$-plane ($\omega_{z'} = 0$), but then $\vec{L}$ will *not* be parallel to $\vec{\omega}$ in general.

- If all three moments of inertia are distinct, the only possible solutions are rotations around the principal axes, e.g., $\omega_{x'} = \omega_{y'} = 0\,, \omega_{z'} \neq 0$ and $\vec{L} = C\vec{\omega} =$ const.

2.3 Stability of Rotation and the Intermediate-Axis Theorem

3 Group Exercises

Problem G27 – Rotating Cuboid

Consider a homogenous rotating cuboid with side lengths a, b, c and mass M.

1. Compute the principal moments of inertia with respect to the cuboid's center of mass.

   HINT: The diagonalization of the moment-of-inertia tensor can be avoided through an appropriate choice of coordinate system.

2. Determine the cuboid's equations of motion in the body-fixed frame, the **Euler equations for the rigid body**, by starting from

$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{N}\,, \quad \vec{L} = \boldsymbol{I}\vec{\omega} = (A\omega_{x'}, B\omega_{y'}, C\omega_{z'})^T\,, \tag{23}$$

   where all vectors are expressed in the **body-fixed frame**, and A, B and C denote the principal moments of inertia.

3. Consider the force-free rotation of the cuboid around a principal axis, e.g., $\vec{\omega}_0 = (\omega_0, 0, 0)^T =$ const. Under which conditions is a rotation around this axis stable?

   HINT: Assume a small perturbation of the rotational axis,

$$\vec{\omega} = \vec{\omega}_0 + \vec{\epsilon} = \vec{\omega}_0 + (\epsilon_{x'}, \epsilon_{y'}, \epsilon_{z'})^T\,, \tag{24}$$

   and determine the conditions under which the amplitude of the perturbation $\vec{\epsilon}$ remains small. Omit terms of order $O(\epsilon^2)$ and higher.

Problem G28 – Rotating Platelet

[cf. problem G27] Consider a thin rectangular platelet of mass m with side lengths a, b and a homogeneous mass distribution. Choose a coordinate system whose origin is the platelet's center of mass.

1. Express the platelet's mass density $\rho(x, y, z)$ using δ and step functions.

2. Determine the moment of inertia tensor $\boldsymbol{I}$ in the chosen center-of-mass frame, and determine the principal axes.

   HINT: You can use your results from problem G27, or compute $\boldsymbol{I}$ explicitly for practice.

3. Derive the Euler equations for the platelet in the body-fixed frame.

4. Compute the torque $\vec{N}$ that is required to make the platelet rotate with a *constant* angular velocity around its *diagonal*. What happens if the platelet is quadratic, i.e., $a = b$?