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PHY422/820: Classical Mechanics

FS 2021

Worksheet #12 (Nov 15 – Nov 19)

November 15, 2021

1 Preparation

- Lemos, Sections 4.8-4.9, 5.1, 5.3-5.7
- Goldstein, Sections 6.1-6.5

2 Spinning Tops

2.1 The Symmetric Heavy Top

2.1.1 Defining the Lagrangian

Let us now consider a symmetric top with $A = B \neq C$ with a fixed point that is moving under the influence of gravity. Its dynamics is best described using a Lagrangian whose degrees of freedom are the Euler angles (ϕ, θ, ψ). Since we have no translation, the kinetic term is purely rotational,

$$T = \frac{1}{2}\vec{\omega} \cdot \boldsymbol{I} \cdot \vec{\omega}\,. \tag{1}$$

Since T is expressed in terms of tensors it is form-invariant under rotations, and we can evaluate it in the principal axis frame, where we have

$$T = \frac{1}{2}A(\omega_{x'}^2 + \omega_{y'}^2) + \frac{1}{2}C\omega_{z'}^2\,. \tag{2}$$

Using the expression for the body-fixed components of ω in terms of the Euler angles

$$\vec{\omega} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi,\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi,\ \dot{\phi}\cos\theta + \dot{\psi})^T\,, \tag{3}$$

we have

$$T = \frac{1}{2}A\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2\,. \tag{4}$$

Denoting the distance of the top's center-of-mass from the fixed point by l, the potential term can be written as

$$V = Mgl\cos\theta\,, \tag{5}$$

where we have used that θ is the angle between the space-fixed axis z and the body-fixed axis z' (see worksheet #10). Thus, the Lagrangian for the top reads

$$L = \frac{1}{2}A\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta. \tag{6}$$

2.1.2 Conservation Laws and General Solution

Inspecting the Lagrangian (6), we immediately notice that ϕ and ψ are cyclic variables, so we obtain

$$\frac{\partial L}{\partial \dot{\phi}} = \left(A\sin^2\theta + C\cos^2\theta\right)\dot{\phi} + C\dot{\psi}\cos\theta \equiv p_\phi = \text{const.}\,, \tag{7}$$

$$\frac{\partial L}{\partial \dot{\psi}} = C\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = C\omega_{z'} \equiv p_\psi = \text{const.}\,. \tag{8}$$

We see that p_ψ is the z'-component of the angular momentum in the body-fixed frame, while p_ϕ is the z-component of the angular momentum in the space-fixed frame. The latter is conserved because gravity does not exert any torque in the space-fixed x or y direction. Moreover, the Lagrangian does not explicitly depend on the time, hence the total energy of the top is conserved:

$$E = T + V = \frac{1}{2}A\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta\,. \tag{9}$$

We can rewrite it in terms of the conserved momenta as

$$E = \frac{1}{2}A\dot{\theta}^2 + \frac{(p_\phi - p_\psi\cos\theta)^2}{2A\sin^2\theta} + \frac{p_\psi^2}{2C} + Mgl\cos\theta\,. \tag{10}$$

Since the third term on the right-hand side is constant, it is convenient to introduce

$$E' = E - \frac{p_\psi^2}{2C} = \frac{1}{2}A\dot{\theta}^2 + \frac{(p_\phi - p_\psi\cos\theta)^2}{2A\sin^2\theta} + Mgl\cos\theta \tag{11}$$

which now only depends on θ and $\dot{\theta}$. This allows us to follow the same strategy we used for one-dimensional potentials to determine the general solution for $\theta(t)$: We separate the variables and integrate to obtain

$$\int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{\frac{2}{A}\left(E' - V_{\text{eff}}(\theta')\right)}} = t - t_0\,, \tag{12}$$

where we have introduced the effective potential

$$V_{\text{eff}}(\theta) \equiv \frac{(p_\phi - p_\psi\cos\theta)^2}{2A\sin^2\theta} + Mgl\cos\theta\,. \tag{13}$$

Once we have $\theta(t)$, we can plug it into Eqs. (7) and (8) and integrate them to obtain $\phi(t)$ and $\psi(t)$. The integral in Eq. (12) cannot be solved analytically in general, but we can discuss the dynamics of the top qualitatively using the effective potential, as in our treatment of central-force problems.

2.1.3 Discussion of the Effective Potential

First, we will determine the general shape of the effective potential:

- A physical solution for $\theta(t)$ must be confined to the interval $[0, \pi]$ at all times. At the borders of the interval, the denominator of the first term in the effective potential (13) will vanish, and $V_{\text{eff}}(\theta) \to +\infty$.

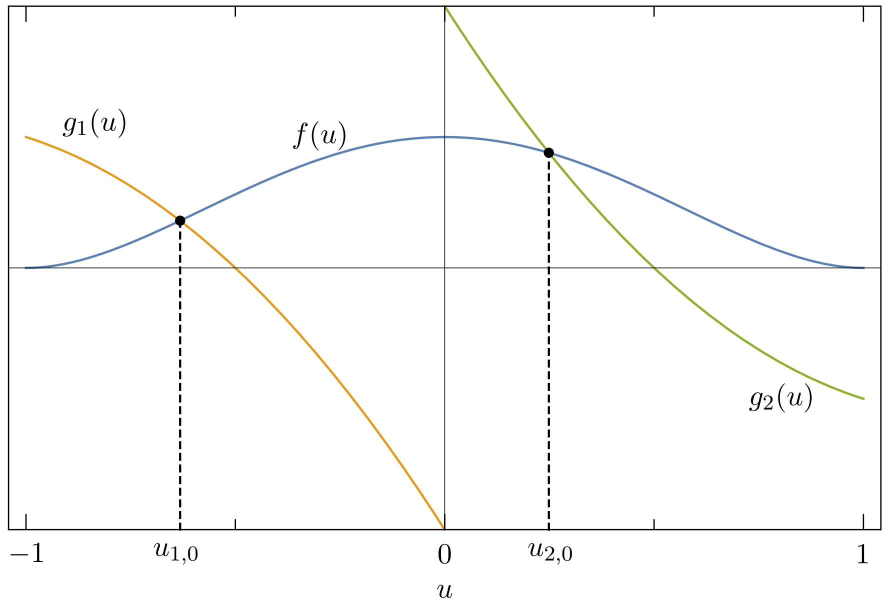


Figure 1: Graphical determination of the extrema of $V_{\text{eff}}(\theta)$ (cf. Eq. (18)).

- The derivative of $V_{\text{eff}}(\theta)$ is

$$\begin{aligned} V'_{\text{eff}}(\theta) &= \frac{2\,(p_\phi - p_\psi\cos\theta)\,p_\psi\sin\theta\cdot 2A\sin^2\theta - (p_\phi - p_\psi\cos\theta)^2\cdot 4A\sin\theta\cos\theta}{4A^2\sin^4\theta} - Mgl\sin\theta \\ &= \frac{(p_\phi - p_\psi\cos\theta)\,p_\psi\cdot\sin^2\theta - (p_\phi - p_\psi\cos\theta)^2\cdot\cos\theta}{A\sin^3\theta} - Mgl\sin\theta \\ &= \frac{(p_\phi - p_\psi\cos\theta)\,(p_\psi\sin^2\theta - p_\phi\cos\theta + p_\psi\cos^2\theta)}{A\sin^3\theta} - Mgl\sin\theta \\ &= \frac{(p_\phi - p_\psi\cos\theta)\,(p_\psi - p_\phi\cos\theta)}{A\sin^3\theta} - Mgl\sin\theta\,, \end{aligned} \tag{14}$$

which allows us to find the extrema as solutions of

$$V'_{\text{eff}}(\theta_0) = \frac{(p_\phi - p_\psi\cos\theta_0)\,(p_\psi - p_\phi\cos\theta_0)}{A\sin^3\theta_0} - Mgl\sin\theta_0 = 0\,. \tag{15}$$

Substituting $u = \cos\theta$, we have

$$\begin{aligned} \frac{dV_{\text{eff}}}{du} &= \frac{dV_{\text{eff}}}{d\theta}\frac{d\theta}{du} \\ &= \left(\frac{(p_\phi - p_\psi\cos\theta)\,(p_\psi - p_\phi\cos\theta)}{A\sin^3\theta} - Mgl\sin\theta\right)\left(-\frac{1}{\sin\theta}\right) \\ &= -\frac{(p_\phi - p_\psi\cos\theta)\,(p_\psi - p_\phi\cos\theta)}{A\sin^4\theta} + Mgl \\ &= -\frac{(p_\phi - p_\psi u)\,(p_\psi - p_\phi u)}{A(1-u^2)^2} + Mgl \\ &\overset{!}{=} 0\,. \end{aligned} \tag{16}$$

We can find the solutions by introducing functions

$$f(u) \equiv MglA(1-u^2)^2\,, \quad g(u) = (p_\phi - p_\psi u)\,(p_\psi - p_\phi u)\,, \tag{17}$$

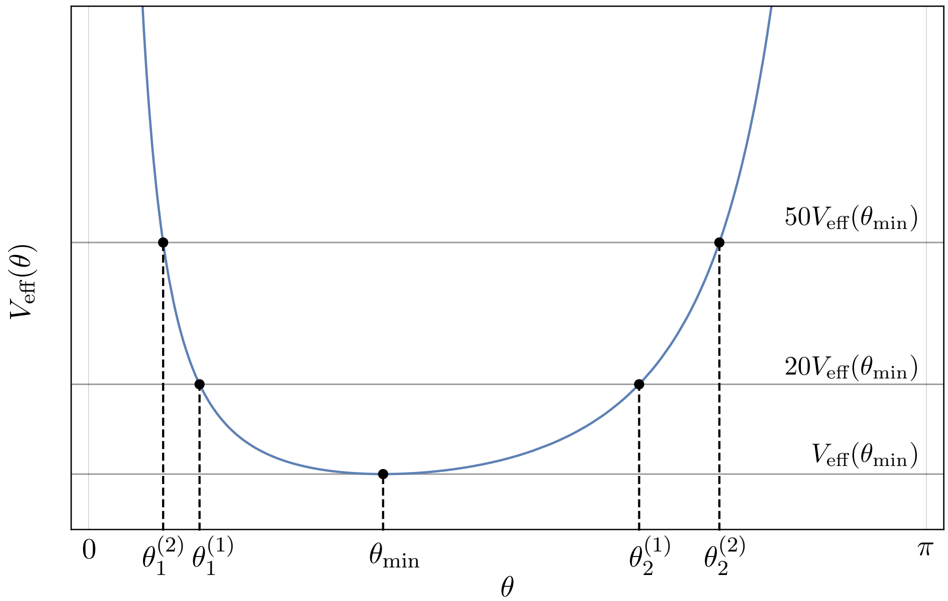


Figure 2: The effective potential for the heavy top.

and determining their intersection

$$f(u) \stackrel{!}{=} g(u). \tag{18}$$

Typical examples are shown in Fig. 1: For physical parameters, the functions intersect in a single point u_0. Since $V_{\text{eff}}(\theta) \to \infty$ at the borders of the integral, this extremum must be a *global minimum*.

From the preceding discussion, we can infer that the effective potential has the shape shown in Fig. 2, and this allows us to classify the main types of motion for the symmetric heavy top.

2.1.4 General Features of the Motion

The motion of the symmetric heavy top will be a superposition of the following effects:

- a **precession** of the symmetry axis around the space-fixed z axis with instantaneous angular velocity $\dot{\phi}$
- a **nutation**[1], that is, a periodic change of the tilt angle $\theta(t)$ between the top's symmetry axis and the space-fixed z axis,
- and a precession of body-fixed angular momentum around the symmetry axis (chosen to be the body-fixed z' axis) with angular velocity $\dot{\psi}$, just as in the case of the free symmetric top.

Combining Eqs. (7) and (8), we can determine the angular velocity $\dot{\phi}$:

$$p_\phi - p_\psi \cos\theta = (A\sin^2\theta + C\cos^2\theta)\dot{\phi} + C\dot{\psi}\cos\theta - C\dot{\psi}\cos\theta - C\dot{\phi}\cos^2\theta\,, \tag{19}$$

hence

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{A\sin^2\theta} = p_\psi \frac{\frac{p_\phi}{p_\psi} - \cos\theta}{A\sin^2\theta}\,. \tag{20}$$

We distinguish the following cases:

[1] From Latin *nutare*, "to nod, to sway."

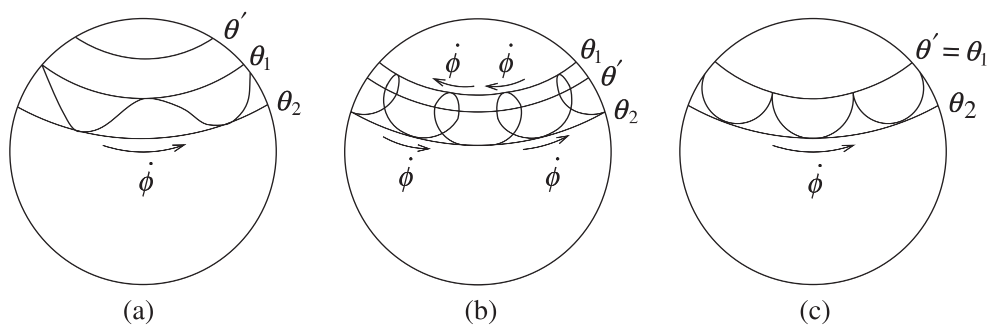


Figure 3: Types of motion for the symmetric heavy top.

- $\boldsymbol{E' > V_{\text{eff}}(\theta_0), |p_\phi| > |p_\psi|}$: The angular velocity $\dot{\phi}$ has the same sign at all times, and the top will precess in the same direction. The symmetry axis will trace out a curve on the unit sphere, as shown in Fig. 3(a). While precessing, the axis will nutate between the turning points θ_1 and θ_2 that are determined from $E' = V_{\text{eff}}(\theta_1) = V_{\text{eff}}(\theta_2)$, as usual.

- $\boldsymbol{E' > V_{\text{eff}}(\theta_0), |p_\phi| < |p_\psi|}$: This case leads to a richer variety of phenomena. Introducing the auxiliary angle $\theta' = \arccos \frac{p_\phi}{p_\psi}$, we can have the three behaviors shown in Figs. 3(a-c). If the sign of $\dot{\phi}$ stays the same during the entire motion, the angle θ' lies outside of the physically allowed range $\theta(t) \in [\theta_1, \theta_2]$ for a top with a fixed energy. If $\theta_1 < \theta' \leq \theta_2$, $\dot{\phi}$ will change sign during a nutation, which causes the corkscrew-shaped trajectory shown in Fig. 3(b).

  Last but not least, $\theta' = \theta_1$ yields the trajectory shown in Fig. 3(c). This case would occur if a top with a fixed energy that is spinning around its symmetry axis is released: The initial conditions are

$$\theta = \theta_1\,, \quad \dot{\theta} = 0\,, \quad \dot{\phi} = 0\,, \quad \dot{\psi} = \omega_{z'}\,, \tag{21}$$

  which implies

$$p_\phi = C\omega_{z'} \cos\theta_1\,, \tag{22}$$
$$p_\psi = C\omega_{z'}\,, \tag{23}$$
$$E = \frac{1}{2}C\omega_{z'}^2 + Mgl\cos\theta_1 \tag{24}$$

  and

$$\cos\theta' = \frac{p_\phi}{p_\psi} = \cos\theta_1\,. \tag{25}$$

- $\boldsymbol{E' = V_{\text{eff}}(\theta_0)}$: The tilt angle stays constant $\theta(t) = \theta_0$, and the axis precesses without nutating, with a constant precession frequency:

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta_0}{A\sin^2\theta_0} = \text{const.} \tag{26}$$

  Let us now determine the initial conditions under which this can be achieved. Introducing

$$\sigma = p_\phi - p_\psi\cos\theta_0 \tag{27}$$

  we also have

$$p_\psi - p_\phi\cos\theta_0 = p_\psi\sin^2\theta_0 - \sigma\cos\theta_0 \tag{28}$$

and this allows us to rewrite Eq. (15) as a quadratic equation in σ:

$$\sigma^2 \cos\theta_0 - \sigma p_\psi \sin^2\theta_0 + MglA\sin^4\theta_0 = 0\,. \tag{29}$$

The general solutions are

$$\begin{aligned}\sigma_\pm &= \frac{p_\psi \sin^2\theta_0 \pm \sqrt{p_\psi^2 \sin^4\theta_0 - 4MglA\sin^4\theta_0\cos\theta_0}}{2\cos\theta_0} \\ &= \frac{C\omega_{z'}\sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MglA\cos\theta_0}{C^2\omega_{z'}^2}}\right)\,,\end{aligned} \tag{30}$$

where we have used $p_\psi = C\omega_{z'}$ in the final step. We see that real solutions can only exist if

$$\omega_{z'} \geq \omega_0 = \frac{2}{C}\sqrt{MglA\cos\theta_0}\,, \tag{31}$$

which requires $\theta_0 \leq \frac{\pi}{2}$. Clearly, the existence of two solutions implies that there will be two separate precession frequencies associated with σ_+ and σ_-.

In the limit $\omega_{z'} \gg \omega_0$, we have

$$\sigma_+ = \frac{C\omega_{z'}\sin^2\theta_0}{2\cos\theta_0}\left(2 + O\left(\frac{\omega_0^2}{\omega_{z'}^2}\right)\right) = \frac{C\omega_{z'}\sin^2\theta_0}{\cos\theta_0} + O\left(\frac{\omega_0^2}{\omega_{z'}^2}\right) \tag{32}$$

and

$$\sigma_- = \frac{C\omega_{z'}\sin^2\theta_0}{2\cos\theta_0}\left(1 - \left(1 - \frac{1}{2}\frac{\omega_0^2}{\omega_{z'}^2}\right)\right) = \frac{C\omega_{z'}\sin^2\theta_0}{4\cos\theta_0}\frac{\omega_0^2}{\omega_{z'}^2}\,, \tag{33}$$

where we have expanded the square-root terms for small $(\omega_0/\omega_{z'})^2$. Note that σ_- is suppressed by one order in the expansion compared to σ_+. Plugging these results back into the expression for the precession frequency, we find a fast precession frequency,

$$\dot{\phi}_+ = \frac{\sigma_+}{A\sin^2\theta_0} = \frac{C\omega_{z'}}{A\cos\theta_0} + O\left(\frac{\omega_0^2}{\omega_{z'}^2}\right)\,, \tag{34}$$

and a slow precession frequency

$$\dot{\phi}_- = \frac{\sigma_-}{A\sin^2\theta_0} = \frac{C\omega_{z'}}{4A\cos\theta_0}\frac{\omega_0^2}{\omega_{z'}^2} = \frac{C\omega_{z'}}{4A\cos\theta_0}\frac{4}{C^2\omega_{z'}^2}MglA\cos\theta_0 = \frac{Mgl}{C\omega_{z'}}\,. \tag{35}$$

2.1.5 The Vertical Top

One of the most frequently encountered examples is that of a symmetric top that is set to spin in an upright position, with initial conditions $\theta(0) = 0$ and $\dot{\psi} = \omega_{z'}$. From Eqs. (7) and (8), we see that

$$p_\phi = p_\psi = C\omega_{z'}\,, \tag{36}$$

and the effective potential becomes

$$\begin{aligned}V_{\text{eff}}(\theta) &= \frac{C^2\omega_{z'}^2}{2A}\frac{(1-\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta \\ &= \frac{C^2\omega_{z'}^2}{2A}\frac{4\sin^4\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}} + Mgl\cos\theta\end{aligned}$$

$$= \frac{C^2\omega_{z'}^2}{2A}\tan^2\frac{\theta}{2} + Mgl\cos\theta\,. \tag{37}$$

Now we consider what happens for small nutations with respect to the vertical axis. Expanding the effective potential for small angles, we have

$$V_{\text{eff}}(\theta) = \frac{C^2\omega_{z'}^2}{2A}\left(\frac{\theta}{2}\right)^2 + Mgl\left(1 - \frac{1}{2}\theta^2\right) = \frac{1}{2}\left(\frac{C^2\omega_{z'}^2}{4A} - Mgl\right)\theta^2 + Mgl\,. \tag{38}$$

Thus, the effective potential is a parabola that is offset by the constant Mgl in energy. For

$$\frac{C^2\omega_{z'}^2}{4A} - Mgl > 0\,, \tag{39}$$

the tops' motion with. This implies that its initial angular velocity must exceeds a critical value (cf. Eq. (31))

$$\omega_{z'} > \omega_c = \frac{2}{C}\sqrt{MglA}\,. \tag{40}$$

This explains the typical behavior of the upright top: As it is set in motion with a sufficiently large $\omega_{z'}$, it will be spinning with barely a wobble. Eventually, the rotation will slow down due to frictional energy losses, and the top will start to wobble, before eventually falling over when $\omega_{z'}$ drops below the critical value.

3 Small Oscillations

Harmonic oscillators are ubiquitous in classical mechanics (and beyond). In fact, their importance extends beyond free wave phenomena or systems that explicitly contain springs: As we have seen throughout this course, oscillatory motion is also so prevalent in physics because it occurs when a system is displaced out of a stable equilibrium. Consider the potential (or effective potential) as a function of the generalized coordinates and perform a Taylor expansion around an equilibrium point:

$$V(\vec{q}_0 + \Delta\vec{q}) = V(\vec{q}_0) + \sum_{i=1}^{n}\left.\frac{\partial V}{\partial q_i}\right|_{\vec{q}_0}\Delta q_i + \frac{1}{2}\sum_{ij}\left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\vec{q}_0}\Delta q_i \Delta q_j + O(\Delta q^3)\,. \tag{41}$$

In an equilibrium configuration, the (generalized) forces on the system vanish, i.e.,

$$Q_i = -\frac{\partial V}{\partial q_i} = 0\,, \tag{42}$$

and the potential is approximately quadratic in the vicinity. If the matrix of second derivatives (i.e., the Hessian) is positive definite at $\vec{q}_0$, the potential will be that of a harmonic oscillator — and that is precisely the condition under which an equilibrium is stable, as we saw in our discussions of orbits or the motion of rigid bodies!

In this next segment of the course, we will develop the tools to describe small oscillations around equilibria for general mechanical systems with multiple degrees of freedom.

3.1 Coupled Oscillators

3.1.1 Example: Two Coupled Pendula

We start the discussion of coupled oscillators by recalling the procedure for solving the equations of motion of such systems. As an example, we consider two equal pendula of length l and mass m

that perform small oscillations while being coupled by an ideal spring with spring constant k and natural length d (see Fig. 4). Letting the y axis point upward, the coordinates of the masses are given by

$$x_1 = l\sin\theta_1\,, \qquad \dot{x}_1 = l\dot{\theta}_1\cos\theta_1\,, \tag{43}$$
$$y_1 = l(1-\cos\theta_1)\,, \qquad \dot{y}_1 = l\dot{\theta}_1\sin\theta_1\,, \tag{44}$$
$$x_2 = d + l\sin\theta_2 \qquad \dot{x}_2 = l\dot{\theta}_2\cos\theta_2\,, \tag{45}$$
$$y_2 = l(1-\cos\theta_2) \qquad \dot{y}_2 = l\dot{\theta}_2\sin\theta_2\,, \tag{46}$$

so the kinetic energy reads

$$T = \frac{1}{2}m\left(\dot{x}_1^2+\dot{y}_1^2+\dot{x}_2^2+\dot{y}_2^2\right) = \frac{1}{2}ml^2\left(\dot{\theta}_1^2+\dot{\theta}_2^2\right)\,. \tag{47}$$

For the potential term, we first consider the spring potential:

$$\begin{aligned} V_{\text{sp}} &= \frac{k}{2}\left(\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}-d\right)^2 \\ &= \frac{k}{2}\left((x_2-x_1)\sqrt{1+\left(\frac{y_2-y_1}{x_2-x_1}\right)^2}-d\right)\,, \end{aligned} \tag{48}$$

where we have used that $x_2 - x_1 > 0$ for small-angle oscillations. For small angles, we also have

$$x_2 - x_1 = d + l\theta_2 - l\theta_1 + O(\theta^3)\,, \tag{49}$$
$$y_2 - y_1 = l\frac{\theta_2^2}{2} - l\frac{\theta_1^2}{2} + O(\theta^4) = \frac{l}{2}\left(\theta_2^2-\theta_1^2\right) + O(\theta^4)\,. \tag{50}$$

We can expand the ratio in the potential,

$$\begin{aligned} \frac{y_2-y_1}{x_2-x_1} &= \frac{\frac{l}{2}\left(\theta_2^2-\theta_1^2\right)+O(\theta^4)}{d+l(\theta_2-\theta_1)+O(\theta^3)} = \frac{l}{2d}\left(\theta_2^2-\theta_1^2+O(\theta^4)\right)\frac{1}{1+\frac{l}{d}(\theta_2-\theta_1)+O(\theta^3))} \\ &= \frac{l}{2d}\left(\theta_2^2-\theta_1^2+O(\theta^4)\right)\left(1-\frac{l}{d}(\theta_2-\theta_1)+O(\theta^2)\right) \\ &= \frac{l}{2d}\left(\theta_2^2-\theta_1^2\right)+O(\theta^3)\,, \end{aligned} \tag{51}$$

so

$$\sqrt{1+\left(\frac{y_2-y_1}{x_2-x_1}\right)^2} = 1+\frac{1}{2}\left(\frac{y_2-y_1}{x_2-x_1}\right)^2+\ldots = 1+O(\theta^4) \tag{52}$$

Now the spring potential reads

$$V_{\text{sp}} = \frac{k}{2}\left(x_2-x_1-d+O(\theta^4)\right)^2 = \frac{k}{2}\left(l\theta_2+d-l\theta_1-d+O(\theta^3)\right) \tag{53}$$

and the total potential becomes

$$\begin{aligned} V &= \frac{k}{2}l^2\left(\theta_2-\theta_1+O(\theta^3)\right)^2+mgl\left(\frac{\theta_1^2}{2}+\frac{\theta_2^2}{2}\right)+O(\theta^3) \\ &= \frac{k}{2}l^2\left(\theta_2-\theta_1\right)^2+mgl\left(\frac{\theta_1^2}{2}+\frac{\theta_2^2}{2}\right)+O(\theta^3)\,. \end{aligned} \tag{54}$$

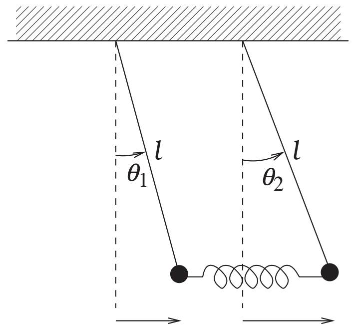


Figure 4: Coordinates for two pendula that are coupled by a spring.

Putting everything together, we obtain the Lagrangian

$$L = \frac{1}{2} m l^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 \right) - \frac{k}{2} l^2 \left(\theta_2 - \theta_1 \right)^2 - mgl \left(\frac{\theta_1^2}{2} + \frac{\theta_2^2}{2} \right) . \tag{55}$$

Next, we derive the Lagrange equations. The partial derivatives are

$$\frac{\partial L}{\partial \dot{\theta}_j} = m l^2 \dot{\theta}_j \, , \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_j} = m l^2 \ddot{\theta}_j \, , \tag{56}$$

$$\frac{\partial L}{\partial \theta_1} = k l^2 \left(\theta_2 - \theta_1 \right) - mgl\theta_1 \, , \tag{57}$$

$$\frac{\partial L}{\partial \theta_2} = -k l^2 \left(\theta_2 - \theta_1 \right) - mgl\theta_2 \, , \tag{58}$$

and the equations of motion read

$$m l^2 \ddot{\theta}_1 - k l^2 \left(\theta_2 - \theta_1 \right) + mgl\theta_1 = 0 \, , \tag{59}$$

$$m l^2 \ddot{\theta}_2 + k l^2 \left(\theta_2 - \theta_1 \right) + mgl\theta_2 = 0 \, . \tag{60}$$

The usual procedure for solving this system of equation is to make the ansatz

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} \theta_{1,0} \\ \theta_{2,0} \end{pmatrix} e^{i\omega t} \, . \tag{61}$$

Note that the same frequency ω is used for both components of the solution. We have

$$\ddot{\theta}_j(t) = -\omega^2 \theta_j(t) \, , \tag{62}$$

and we can use this to write the system of equations of motion in matrix form:

$$\begin{pmatrix} -ml^2\omega^2 + mgl + kl^2 & -kl^2 \\ -kl^2 & -ml^2\omega^2 + mgl + kl^2 \end{pmatrix} \begin{pmatrix} \theta_{1,0} \\ \theta_{2,0} \end{pmatrix} = 0 \, . \tag{63}$$

This is an eigenvalue problem whose solutions are the **characteristic frequencies** ω^2 and associated **characteristic vectors** that define the **normal modes** of the system of coupled oscillators.

The characteristic polynomial of the matrix is

$$\det \begin{pmatrix} -ml^2\omega^2 + mgl + kl^2 & -kl^2 \\ -kl^2 & -ml^2\omega^2 + mgl + kl^2 \end{pmatrix}$$

$$= (-ml^2\omega^2 + mgl + kl^2)^2 - (-kl^2)^2 = 0\,. \tag{64}$$

Moving the final term to the right and taking the square root on both sides, we end up with the equation

$$-ml^2\omega^2 + mgl + kl^2 = \pm kl^2\,. \tag{65}$$

Its solutions are

$$\omega_+^2 = \frac{g}{l}\,, \quad \omega_-^2 = \frac{g}{l} + \frac{2k}{m}\,. \tag{66}$$

We can determine the characteristic vectors by plugging the frequencies back into Eq. (63). For ω_+^2, we have

$$\begin{aligned} 0 &= \begin{pmatrix} -ml^2\frac{g}{l} + mgl + kl^2 & -kl^2 \\ -kl^2 & -ml^2\frac{g}{l} + mgl + kl^2 \end{pmatrix} \begin{pmatrix} \rho_1^{(+)} \\ \rho_2^{(+)} \end{pmatrix} \\ &= \begin{pmatrix} kl^2 & -kl^2 \\ -kl^2 & kl^2 \end{pmatrix} \begin{pmatrix} \rho_1^{(+)} \\ \rho_2^{(+)} \end{pmatrix}\,, \end{aligned} \tag{67}$$

which means that the components of the characteristic vector need to satisfy

$$\rho_1^{(+)} = \rho_2^{(+)}\,. \tag{68}$$

Thus, a normalized solution is

$$\vec{\rho}^{\,(+)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,. \tag{69}$$

For ω_+^2, we find

$$\begin{aligned} 0 &= \begin{pmatrix} -ml^2\left(\frac{g}{l} + 2\frac{k}{m}\right) + mgl + kl^2 & -kl^2 \\ -kl^2 & -ml^2\left(\frac{g}{l} + 2\frac{k}{m}\right) + mgl + kl^2 \end{pmatrix} \begin{pmatrix} \rho_1^{(-)} \\ \rho_2^{(-)} \end{pmatrix} \\ &= \begin{pmatrix} -kl^2 & -kl^2 \\ -kl^2 & -kl^2 \end{pmatrix} \begin{pmatrix} \rho_1^{(-)} \\ \rho_2^{(-)} \end{pmatrix}\,, \end{aligned} \tag{70}$$

which implies

$$\rho_1^{(-)} = -\rho_2^{(-)}\,, \tag{71}$$

so a solution for the second characteristic vector is

$$\vec{\rho}^{\,(-)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\,. \tag{72}$$

Figure 5 visualizes the two normal modes we have found. For $\vec{\rho}^{\,(+)}$, the two pendula are in sync, while the masses swing in opposite directions for $\vec{\rho}^{\,(-)}$, as indicated by the signs of the characteristic vectors.

3.1.2 Normal Coordinates and Normal Modes

In the example discussed in the previous section, we determined the normal modes of a system of coupled oscillators *after* deriving the equations of motion. The characteristic vectors associated with these modes constitute a complete basis for the description of the system — in fact, if we

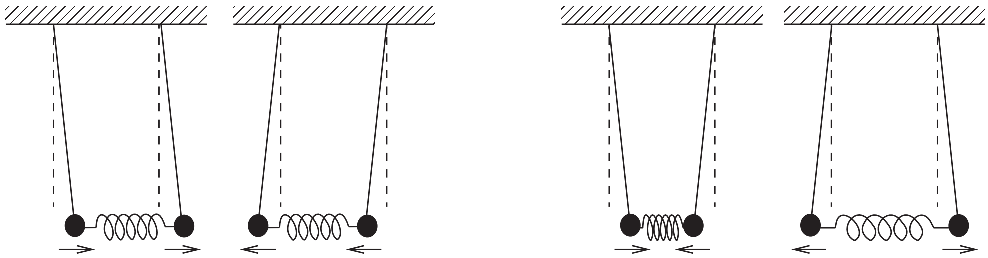


Figure 5: Normal modes for two pendula that are coupled by a spring.

make a change of basis from our initial choice of coordinates and basis vectors to the normal mode basis, the equations of motion are *decoupled* and can be solved independently.

The central premise of the Lagrangian formalism, however, is to formulate the problem in the most efficient choice of generalized coordinates from the very beginning (unless we need to deal with some constraints explicitly). If we can identify the matrix appearing in the equations of motion of a system of oscillators already in the Lagrangian, we can diagonalize it and express L directly in terms of normal coordinates with respect to the characteristic vectors. The Lagrange equations for the normal coordinates are then particularly easy to derive, because they correspond to simple, uncoupled oscillators.

[...] Consider a general Lagrangian of the form

$$L = \frac{1}{2}\sum_{jk} M_{jk}\dot{q}_j\dot{q}_k - V(\vec{q})\,, \tag{73}$$

where $\vec{q} = (q_1, \ldots, q_n)$ and $\mathbf{M}$ is the mass tensor in generalized coordinates (cf. worksheet #5):

$$M_{jk} = \sum_{i=1}^{A} m_i \frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_j}{\partial q_k} = M_{kj}\,. \tag{74}$$

Let us consider displacements η_j around an equilibrium position of the system:

$$q_j = q_{0j} + \eta_j\,, \tag{75}$$

$$\dot{q}_j = \dot{\eta}_j\,. \tag{76}$$

The potential can be expanded as

$$\begin{aligned} V(\vec{q}) &= V(\vec{q}_0) + \sum_j \underbrace{\left.\frac{\partial V}{\partial q_j}\right|_{\vec{q}_0}}_{=0} \eta_j + \frac{1}{2}\sum_{jk}\left.\frac{\partial^2 V}{\partial q_j \partial q_k}\right|_{\vec{q}_0}\eta_j\eta_j + O(\eta^3) \\ &\equiv V(\vec{q}_0) + \frac{1}{2}\sum_{jk} V_{jk}\eta_j\eta_k + O(\eta^3)\,, \end{aligned} \tag{77}$$

where we have used that the gradient of the potential vanishes in equilibrium, and introduced the symmetric matrix

$$V_{jk} = \frac{\partial^2 V}{\partial q_j \partial q_k} = V_{kj}\,, \tag{78}$$

which is nothing but the Hessian of the system evaluated at $\vec{q}_0$. Dropping cubic and higher terms in η, a general potential therefore looks like a system of coupled oscillators near an equilibrium configuration. For a system of coupled harmonic oscillators, the higher-order terms vanish and the expression for the potential is exact.

Introducing

$$T_{jk} \equiv M_{jk}|_{\vec{q}_0} = T_{kj} \,, \tag{79}$$

and dropping the constant $V(\vec{q}_0)$ we can write the Lagrangian as a **quadratic form**,

$$L = \frac{1}{2}\dot{\vec{\eta}} \cdot \boldsymbol{T} \cdot \dot{\vec{\eta}} - \frac{1}{2}\vec{\eta} \cdot \boldsymbol{V} \cdot \vec{\eta} = \frac{1}{2}\sum_{jk} \dot{\eta}_j T_{jk} \dot{\eta}_k - \frac{1}{2}\sum_{jk} \eta_j V_{jk} \eta_k \,. \tag{80}$$

Equations of Motion and Generalized Eigenvalue Problem

The partial derivatives are

$$\begin{aligned} \frac{\partial L}{\partial \eta_j} &= -\frac{1}{2}\sum_{kl} \frac{\partial}{\partial \eta_j}\left(\eta_k V_{kl} \eta_l\right) = -\frac{1}{2}\sum_{kl}\left(\frac{\partial \eta_k}{\partial \eta_j} V_{kl}\eta_l + \eta_k V_{kl}\frac{\partial \eta_l}{\partial \eta_j}\right) \\ &= -\frac{1}{2}\sum_{kl}\left(\delta_{jk} V_{kl} \eta_l + \eta_k V_{kl} \delta_{jl}\right) \\ &= -\sum_k V_{jk}\eta_k \,, \end{aligned} \tag{81}$$

and

$$\frac{\partial L}{\partial \dot{\eta}_j} = \sum_k T_{jk}\dot{\eta}_k \,, \tag{82}$$

where we have used the properties of the Kronecker delta and combined terms by using the freedom to rename summation indices. In terms of the matrices $\boldsymbol{T}$ and $\boldsymbol{V}$, the Lagrange equations therefore read

$$\sum_k \left(T_{jk}\ddot{\eta}_k + V_{jk}\eta_k\right) = 0 \,, \quad j = 1, \ldots, n \,. \tag{83}$$

To identify the normal modes, we make the usual switch to complex coordinates

$$\eta_j \longrightarrow z_j = z_{0j} e^{i\omega t} \,, \quad z_j \in \mathbb{C} \,, \tag{84}$$

and plug them into the equations of motion to obtain

$$\sum_k \left(-\omega^2 T_{jk} z_{0k} + V_{jk} z_{0k}\right) = 0 \,, \tag{85}$$

or in matrix form

$$\left(\boldsymbol{V} - \omega^2 \boldsymbol{T}\right)\vec{z}_0 = 0 \,. \tag{86}$$

This is, in fact, a **generalized eigenvalue problem** since $\boldsymbol{T}$ appears in place of the identity matrix $\mathbb{1}$. It contains the information about the **curvature of the configuration manifold** iself in the vicinity of the equilibrium configuration we expand around. If the configuration manifold is flat, e.g., because we use a Cartesian basis to describe the sytem, $\boldsymbol{T}$ will be simply a diagonal matrix with the masses of the oscillators on the diagonal. If $\boldsymbol{T}$ is invertible, we can in principle convert Eq. (86) into a regular eigenvalue problem by multiplying with $\boldsymbol{T}^{-1}$ from the left,

$$\left(\boldsymbol{T}^{-1}\boldsymbol{V} - \omega^2\mathbb{1}\right)\vec{z}_0 = 0\,. \tag{87}$$

Solving Eq. (86), we obtain the characteristic frequencies ω_s^2 as the eigenvalues, and the characteristic vectors $\vec{\rho}^{(s)}$. Since $\boldsymbol{T}$ and $\boldsymbol{V}$ are real and symmetric, the ω_s^2 are guaranteed to be real as well. The characteristic vectors will be real as well, since their entries are just linear combinations of the original coordinates we used to describe the oscillators. A general oscillation of the system can then be expanded in the basis of normal modes instead of our initial choice of basis vectors (denoted here by $\{\vec{e}^{(1)}, \ldots, \vec{e}^{(n)}\}$):

$$\vec{\eta}(t) = \sum_{j=1}^{n}\eta_j(t)\vec{e}^{(j)} = \sum_{s=1}^{n}\zeta_s(t)\vec{\rho}^{(s)}\,, \tag{88}$$

where ζ_s are the **normal coordinates** of the problem.

Determination of the Normal Coordinates

Let us now assume that we have a Lagrangian given by Eq. (80), and introduce normal coordinates that are related to our initial displacement coordinates by a matrix $\boldsymbol{A}$ that is to be determined:

$$\vec{\eta} = \boldsymbol{A}\vec{\zeta}\,, \quad \eta_k, \zeta_k \in \mathbb{R}. \tag{89}$$

Plugging this relation into the Lagrangian, we have

$$L = \frac{1}{2}\dot{\vec{\zeta}}\cdot\boldsymbol{A}^T\boldsymbol{T}\boldsymbol{A}\cdot\dot{\vec{\zeta}} - \frac{1}{2}\vec{\zeta}\cdot\boldsymbol{A}^T\boldsymbol{V}\boldsymbol{A}\cdot\vec{\zeta}\,. \tag{90}$$

If the normal coordinates are to be decoupled, $\boldsymbol{A}^T\boldsymbol{T}\boldsymbol{A}$ and $\boldsymbol{A}^T\boldsymbol{V}\boldsymbol{A}$ must each be diagonal because they depend on $\dot{\vec{\zeta}}$ and $\vec{\zeta}$, respectively, so there can be no cancellations in off-diagonal matrix elements between the two terms.

To proceed, we first define the **inner product induced by $\boldsymbol{T}$**:

$$\left(\vec{a},\ \vec{b}\right) \equiv \vec{a}\cdot\boldsymbol{T}\cdot\vec{b} = \sum_{kl} a_k T_{kl} b_l \tag{91}$$

It generalizes the standard scalar product from a flat space to the geometry of configuration manifold or, more precisely, the tangent vector spaces to the configuration manifold at the equilibrium configuration. For $T_{kl} = \delta_{kl}$, we get back the regular scalar product.

The characteristic vectors $\vec{\rho}^{(s)}$ are orthonormal with respect to the new inner product:

$$\left(\vec{\rho}^{(r)},\ \vec{\rho}^{(s)}\right) = \sum_{kl}\rho_k^{(r)}T_{kl}\rho_l^{(s)} = \delta_{rs}\,. \tag{92}$$

Let us now define the **modal matrix $\boldsymbol{A}$** in terms of the characteristic vectors

$$\boldsymbol{A} \equiv \begin{pmatrix} \rho_1^{(1)} & \dots & \rho_1^{(n)} \\ \vdots & \ddots & \vdots \\ \rho_n^{(1)} & \dots & \rho_n^{(n)} \end{pmatrix} , \tag{93}$$

or componentwise

$$A_{ks} = \rho_k^{(s)} . \tag{94}$$

Then

$$\sum_{kl} \rho_k^{(r)} T_{kl} \rho_l^{(s)} = \sum_{kl} A_{kr} T_{kl} A_{ls} = \sum_{kl} A_{rk}^T T_{kl} A_{ls} \overset{!}{=} \delta_{rs} , \tag{95}$$

which means that $\boldsymbol{T}$ will be represented by the identity matrix in the basis of characteristic vectors:

$$\boldsymbol{A}^T \boldsymbol{T} \boldsymbol{A} = \mathbb{1} . \tag{96}$$

For the potential term, the eigenvalue problem implies

$$\sum_l V_{kl} \rho_l^{(s)} = \omega_s^2 \sum_l T_{kl} \rho_l^{(s)} . \tag{97}$$

Defining the diagonal matrix

$$\boldsymbol{W} = \text{diag}\left(\omega_1^2, \dots, \omega_n^2\right) , \tag{98}$$

we can rewrite this as

$$\sum_{kl} V_{kl} A_{ls} = \sum_l T_{kl} A_{ls} = \sum_{lr} T_{kl} A_{lr} W_{rs} , \tag{99}$$

or in matrix form as

$$\boldsymbol{V} \boldsymbol{A} = \boldsymbol{T} \boldsymbol{A} \boldsymbol{W} . \tag{100}$$

Multiplying from the left by $\boldsymbol{A}^T$ and using the fact that $\boldsymbol{T}$ is the identity matrix in the basis of characteristic vectors, we have

$$\boldsymbol{A}^T \boldsymbol{V} \boldsymbol{A} = \underbrace{\boldsymbol{A}^T \boldsymbol{T} \boldsymbol{A}}_{=\mathbb{1}} \boldsymbol{W} = \boldsymbol{W} , \tag{101}$$

so the modal matrix also renders $\boldsymbol{V}$ diagonal.

Plugging our results for the transformed matrices back into Eq. (90), the Lagrangian becomes

$$L = \frac{1}{2} \dot{\vec{\zeta}} \cdot \boldsymbol{A}^T \boldsymbol{T} \boldsymbol{A} \cdot \dot{\vec{\zeta}} - \frac{1}{2} \vec{\zeta} \cdot \boldsymbol{A}^T \boldsymbol{V} \boldsymbol{A} \cdot \vec{\zeta} = \frac{1}{2} \dot{\vec{\zeta}} \cdot \dot{\vec{\zeta}} - \frac{1}{2} \vec{\zeta} \cdot \boldsymbol{W} \cdot \vec{\zeta} = \frac{1}{2} \sum_s \left(\dot{\zeta}_s^2 - \omega_s^2 \zeta_s^2 \right) , \tag{102}$$

and the Lagrange equations yield the decoupled equations of motion

$$\ddot{\zeta}_s + \omega_s^2 \zeta_s = 0 , \quad s = 1, \dots, n . \tag{103}$$

as desired. The normal coordinates can now be determined from original ansatz (89) by mutiplying it with $\boldsymbol{A}^T \boldsymbol{T}$ from the left-hand side and using $\boldsymbol{A}^T \boldsymbol{T} \boldsymbol{A} = \mathbb{1}$. We obtain

$$\vec{\zeta} = \boldsymbol{A}^T \boldsymbol{T} \vec{\eta} \tag{104}$$

using the coefficient matrix $\boldsymbol{T}$ and the modal matrix $\boldsymbol{A}$ obtained from the characteristic vectors of our generalized eigenvalue problem.

[Add remark about additional generalized forces / cf. homework.]

3.1.3 Example: A Linear Triatomic Molecule

We now demonstrate the introduction of normal coordinates in the Lagrangian for the example of a linear triatomic molecule, which we describe as a central mass M that is connected to two equal masses m via identical springs with spring constants k. The spring potential can be viewed as the harmonic approximation to the (classical) intermolecular interaction close to an equilibrium configuration, i.e.,

$$V(|x_i - x_j|) = V_0 + \frac{1}{2}\sum_{ij} \left.\frac{\partial^2 V}{\partial x_i \partial x_j}\right|_{|x_{i,0}-x_{j,0}|} \Delta x_i \Delta x_j + O(\Delta x^3)\,. \tag{105}$$

Referring to the coordinates introduced in Fig. 6, we have

$$V = \frac{k}{2}\left(x_2 - x_1 - l\right)^2 + \frac{k}{2}\left(x_3 - x_2 - l\right)^2\,, \tag{106}$$

where l is the natural length of the spring. If we expand the squares in V, we get

$$V = \frac{k}{2}\left(x_1^2 + 2x_2^2 + x_3^2 + 2l^2 - 2x_1x_2 + 2lx_1 - 2lx_2 - 2x_2x_3 - 2lx_2 + 2lx_3\right)\,, \tag{107}$$

which cannot be written as a quadratic form in the coordinates due to the presence of terms like lx_i. Thus, we first have to introduce coordinates that directly measure the displacement out of equilibrium:

$$\eta_1 = x_1 + l\,, \quad \eta_2 = x_2\,, \quad \eta_3 = x_3 - l \tag{108}$$

(other choices are possible as long as they allow us to write L as a quadratic form.) In terms of the new coordinates, the Lagrangian of the system can be written as

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2}M\dot{x}_2^2 - \frac{k}{2}\left(x_2 - x_1 - l\right)^2 - \frac{k}{2}\left(x_3 - x_2 - l\right)^2 \\ &= \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_3^2) + \frac{1}{2}M\dot{\eta}_2^2 - \frac{k}{2}\left(\eta_2 - \eta_1\right)^2 - \frac{k}{2}\left(\eta_3 - \eta_2\right)^2\,, \end{aligned} \tag{109}$$

or

$$L = \frac{1}{2}\dot{\vec{\eta}} \cdot \boldsymbol{T} \cdot \dot{\vec{\eta}} - \frac{1}{2}\vec{\eta} \cdot \boldsymbol{V} \cdot \vec{\eta} \tag{110}$$

with the matrices

$$\boldsymbol{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}\,, \qquad \boldsymbol{V} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\,. \tag{111}$$

Note that we have distributed terms of the form $2\eta_i\eta_j$ to the upper and lower off-diagonal parts of $\boldsymbol{V}$ so that the matrix is symmetric.

Now we solve the generalized eigenvalue problem for $\boldsymbol{V} - \omega^2\boldsymbol{T}$, starting with the determination of the characteristic polynomial:

$$\begin{aligned} 0 &= \det\begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{pmatrix} \\ &= \left(k - m\omega^2\right)\left[(2k - M\omega^2)(k - m\omega^2) - (-k)^2\right] - (-k)\left[(-k)(k - m\omega^2)\right] \\ &= \left(k - m\omega^2\right)\left[(2k - M\omega^2)(k - m\omega^2) - 2k^2\right] \\ &= \left(k - m\omega^2\right)\left[\underline{2k^2} - k(2m + M)\omega^2 + mM\omega^4 - \underline{2k^2}\right] \end{aligned}$$

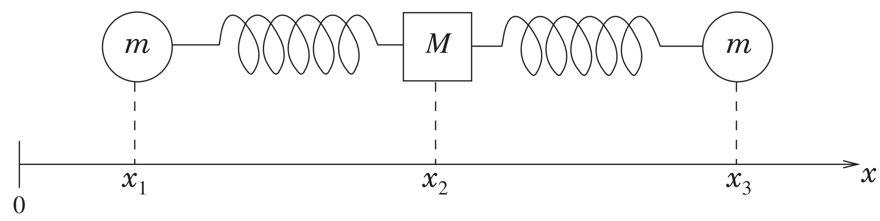


Figure 6: Coordinates for the linear triatomic molecule

$$= \omega^2 \left(k - m\omega^2\right) \left[-k(2m + M) + mM\omega^2\right] . \tag{112}$$

We can immediately read off the solutions

$$\omega_1^2 = 0 , \quad \omega_2^2 = \frac{k}{m} , \quad \omega_3^2 = \frac{k}{m} \left(1 + \frac{2m}{M}\right) . \tag{113}$$

Plugging these frequencies into the eigenvalue equation, we can determine the characteristic vectors. For $\omega_1^2 = 0$, we have

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = 0 , \tag{114}$$

and the first and third rows give us the equations

$$\rho_1 = \rho_2 , \quad \rho_2 = \rho_3 . \tag{115}$$

Thus, our first characteristic vector is

$$\vec{\rho}^{\,(1)} = C_1 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \tag{116}$$

with some normalization constant C_1.

Analogously, $\omega_2^2 = k/m$ yields

$$\begin{pmatrix} k - km\frac{k}{m} & -k & 0 \\ -k & 2k - M\frac{k}{m} & -k \\ 0 & -k & k - m\frac{k}{m} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} 0 & -k & 0 \\ -k & k(2 - \frac{M}{m}) & -k \\ 0 & -k & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = 0 , \tag{117}$$

and we obtain the equations

$$-k\rho_2 = 0 , \tag{118}$$

$$-k(\rho_1 + \rho_3) + k\left(2 - \frac{m}{M}\right) \rho_2 = 0 . \tag{119}$$

Thus, our second characteristic vector is

$$\vec{\rho}^{\,(2)} = C_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} . \tag{120}$$

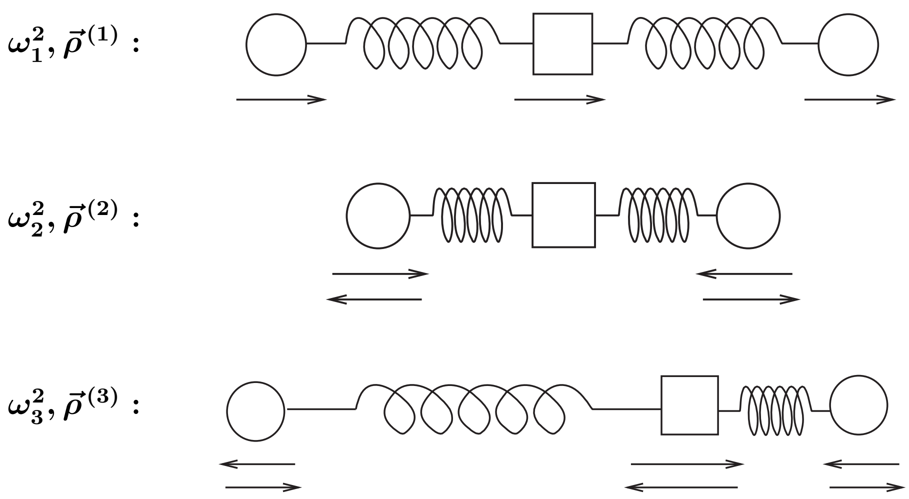


Figure 7: Normal modes of the linear triatomic molecule

Finally, we plug $\omega_3^2 = k/m\,(1+2m/M)$ into the eigenvalue equation and obtain

$$
\begin{aligned}
0 &= \begin{pmatrix} k - k\left(1+\frac{2m}{M}\right) & -k & 0 \\ -k & 2k - k\frac{M}{m}\left(1+\frac{2m}{M}\right) & -k \\ 0 & -k & k - k\left(1+\frac{2m}{M}\right) \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \\
&= \begin{pmatrix} -k\frac{2m}{M} & -k & 0 \\ -k & -k\frac{M}{m} & -k \\ 0 & -k & -k\frac{2m}{M} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \\
&.
\end{aligned}
\tag{121}
$$

Thus, the components of the characteristic vector must satisfy the equations

$$
-\frac{2m}{M}\rho_1 - \rho_2 = 0\,, \tag{122}
$$

$$
-k(\rho_1+\rho_3) - \frac{M}{m}\rho_2 = 0\,, \tag{123}
$$

$$
-\rho_2 - \frac{2m}{M}\rho_3 = 0\,, \tag{124}
$$

and we have

$$
\vec{\rho}^{\,(3)} = C_3 \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix} . \tag{125}
$$

The vectors $\rho^{(k)}$ are associated with different frequencies, so they are guaranteed to be orthonormal in the inner product induced by $\boldsymbol{T}$. The normalization condition yields

$$
\left(\vec{\rho}^{(s)},\ \vec{\rho}^{(s)}\right) = \vec{\rho}^{(s)} \cdot \boldsymbol{T} \cdot \vec{\rho}^{(s)} = m\left(\rho_1^{(s)2} + \rho_3^{(s)2}\right) + M\rho_2^{(s)2} = 1\,, \tag{126}
$$

which allows us to determine the normalization constants C_s:

$$C_1 = \frac{1}{\sqrt{2m(1+M/2m)}}\,, C_2 = \frac{1}{\sqrt{2m}}\,, C_3 = \frac{1}{\sqrt{2m(1+2m/M)}}\,. \tag{127}$$

Assembling the modal matrix $\boldsymbol{A}$ from the characteristic vectors, we have

$$\boldsymbol{A} = \begin{pmatrix}\vec{\rho}^{(1)} & \vec{\rho}^{(2)} & \vec{\rho}^{(3)}\end{pmatrix} = \frac{1}{\sqrt{2m}}\begin{pmatrix} \frac{1}{\sqrt{1+M/2m}} & 1 & \frac{1}{\sqrt{1+2m/M}} \\ \frac{1}{\sqrt{1+M/2m}} & 0 & -\frac{2m/M}{\sqrt{1+2m/M}} \\ \frac{1}{\sqrt{1+M/2m}} & -1 & \frac{1}{\sqrt{1+2m/M}} \end{pmatrix} \tag{128}$$

and we can compute the normal coordinates

$$\vec{\zeta} = \boldsymbol{A}^T\boldsymbol{T}\cdot\vec{\eta} \quad \Rightarrow \begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix} = \frac{1}{\sqrt{2m}}\begin{pmatrix} \frac{1}{\sqrt{1+M/2m}} & \frac{1}{\sqrt{1+M/2m}} & \frac{1}{\sqrt{1+M/2m}} \\ 1 & 0 & -1 \\ \frac{1}{\sqrt{1+2m/M}} & -\frac{2m/M}{\sqrt{1+2m/M}} & \frac{1}{\sqrt{1+2m/M}} \end{pmatrix}\begin{pmatrix}m\eta_1\\ M\eta_2\\ m\eta_3\end{pmatrix} \tag{129}$$

The three normal modes of the molecule are illustrated in Fig. 7. The mode $(\omega_1^2, \vec{\rho}^{(1)})$ is actually not vibrational, but corresponds to a rigid uniform translation of the entire molecule. This is reflected in the normal coordinate ζ_1: carrying out the matrix-vector product in Eq. (129) and plugging in the definition of the displacement coordinates in terms of the initial coordinates x_i, we have

$$\zeta_1 = \frac{m\eta_1 + M\eta_2 + m\eta_3}{\sqrt{2m+M}} = \frac{mx_1 + Mx_2 + mx_3}{\sqrt{2m+M}} = \sqrt{2m+M}X\,, \tag{130}$$

where X is the center of mass coordinate of the molecule. The equation of motion for the normal coordinate ζ_1 therefore becomes

$$\ddot{\zeta}_1 + \underbrace{\omega_1^2}_{=0}\zeta_1 = 0 \quad \Rightarrow \quad \ddot{X} = 0\,, \tag{131}$$

which is of course solved by

$$X(t) = X_0 + V_0 t\,. \tag{132}$$

In mode $(\omega_2^2, \vec{\rho}^{(2)})$, the outer masses move in opposite direction from each other while the center mass stays fixed, while in mode $(\omega_3^2, \vec{\rho}^{(3)})$ the outer masses move in sync in the opposite direction as the center mass.

4 Group Exercises

Problem G29 – Two Masses on a Circle

Consider two identical masses m that can move on a circular horizontal track of radius R (see figure). Each of the masses is connected to a fixed point by identical springs with constant $2k$, and a spring with constant k connects the masses to each other.

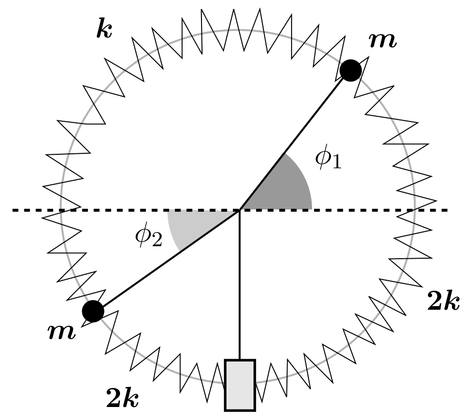
1. Construct the Lagrangian for the system in terms of the (counterclockwise) angular displacements ϕ_1 and ϕ_2 of the two masses from their equilibrium positions, as shown in the diagram.

   Hint: Distances on the circular track can be expressed in terms of arc lengths.

2. Determine the normal modes, i.e., characteristic frequencies and vectors (vectors do not need to be normalized). Sketch and interpret your solutions.

3. Now the fixed point is released, so that the system can rotate freely on the circular track. The track itself remains at rest. How will the characteristic frequencies change *qualitatively* as a result?

   Hint: A calculation is not necessary, but if you explicitly want to check, you can use that two springs connected "in series" can be replaced by a single spring with constant $k_{\text{eff}} = (\frac{1}{k_1} + \frac{1}{k_2})^{-1}$.



Problem G30 – Vertical Oscillators

Consider two equal masses m that are suspended from the ceiling using identical springs with constants k and unstretched length l. Assume that the masses are only allowed to move vertically.

1. Find the Lagrangian for the two masses in terms of the absolute positions of the two masses, as well as in terms of displacements of the system out of equilibrium. What is the advantage of the latter?

2. Show that the characteristic frequencies of the normal modes are

$$\omega_\pm^2 = \frac{(3 \pm \sqrt{5})k}{2m}\,. \tag{133}$$