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PHY422/820: Classical Mechanics

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Mechanics

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_j^c + Q_j^{n.c.}\,, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$\widetilde{L}(\vec{q}, \dot{\vec{q}}, t, \vec{\lambda}) = L(\vec{q}, \dot{\vec{q}}, t) + \sum_{\alpha=1}^{N_h} \lambda_\alpha f(\vec{q}, t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{\alpha=1}^{N_h} \lambda_\alpha \frac{\partial f}{\partial q_i} + \sum_{b=1}^{N_{n.h.}} \mu \frac{\partial g}{\partial \dot{q}_i}$$

$$\left.\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\frac{dq_i}{d\epsilon} - \frac{dF}{d\epsilon}\right)\right|_{\epsilon=0} = \text{const.}$$

$$L = \frac{1}{2}\dot{\vec{\eta}}^T \boldsymbol{T}\dot{\vec{\eta}} - \frac{1}{2}\vec{\eta}^T \boldsymbol{V}\vec{\eta}\,, \qquad \left(\boldsymbol{V} - \omega^2 \boldsymbol{T}\right)\vec{\rho} = 0\,, \qquad \det\left(\boldsymbol{V} - \omega^2 \boldsymbol{T}\right) = 0$$

$$\boldsymbol{A} = \begin{pmatrix} \vec{\rho}^{\,(1)} & \cdots & \vec{\rho}^{\,(n)} \end{pmatrix}\,, \qquad \vec{\zeta} = \boldsymbol{A}\vec{\eta}\,, \qquad \vec{\eta} = \boldsymbol{A}^T\boldsymbol{T}\vec{\zeta}\,, \qquad \boldsymbol{A}^T\boldsymbol{T}\boldsymbol{A} = \mathbb{1}$$

$$\left(\vec{a},\, \vec{b}\right) = \vec{a}^T\boldsymbol{T}\vec{b} = \sum_{kl} a_k T_{kl} b_l$$

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + V(r)\,, \quad \vec{A} = \frac{\vec{p}\times\vec{l}}{m\kappa} - \frac{\vec{r}}{r}$$

$$\phi - \phi_0 = \pm\int_{r(\phi_0))}^{r(\phi)} dr'\, \frac{l}{r'^2\sqrt{2m\left(E - V_{\text{eff}}(r')\right)}}$$

$$r(\phi) = \frac{\alpha}{1 + \epsilon\cos(\phi - \phi_0)}\,, \quad \alpha = \frac{l^2}{m\kappa}\,, \quad \epsilon = |\vec{A}|$$

$$\left(\frac{d}{dt}\right)_{LF} = \left(\frac{d}{dt}\right)_{BF} + \vec{\omega}\times$$

$$\hat{I}_{ab} = \int d^3r\, \rho(\vec{r})\left(\vec{r}^{\,2}\delta_{ab} - r_a r_b\right)$$

$$\hat{I}_{ab} = \hat{I}_{ab}^{\text{com}} + M\left(\vec{R}^2\delta_{ab} - R_a R_b\right)$$

$$T_{\text{rot}} = \frac{1}{2}\vec{\omega}^T \cdot \hat{\boldsymbol{I}} \cdot \vec{\omega}\,, \quad \vec{L} = \hat{\boldsymbol{I}}\vec{\omega}$$

$$\vec{\omega} = \begin{pmatrix}\omega_{x'}\\ \omega_{y'}\\ \omega_{z'}\end{pmatrix} = \begin{pmatrix}\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\\ \dot{\phi}\cos\theta + \dot{\psi}\end{pmatrix}$$

$$\left(\frac{d\vec{L}}{dt}\right)_{LF} = \hat{\boldsymbol{I}}\dot{\vec{\omega}} + \omega \times \left(\hat{\boldsymbol{I}}\vec{\omega}\right) = \vec{N}$$

$$A\dot{\omega}_{x'} + (C-B)\omega_{y'}\omega_{z'} = N_{x'}$$
$$B\dot{\omega}_{y'} + (A-C)\omega_{z'}\omega_{x'} = N_{y'}$$
$$C\dot{\omega}_{z'} + (B-A)\omega_{x'}\omega_{y'} = N_{z'}$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j}\,, \quad H(\vec{q},\vec{p},t) = \sum_k p_k \dot{q}_k(\vec{q},\vec{p},t) - \widetilde{L}(\vec{q},\vec{p},t)$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j}\,, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$$\{f,g\} = \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k}\frac{\partial f}{\partial p_k}\right)\,, \quad \frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t}$$

$$\{fg,h\} = \{f,h\}g + f\{g,h\}\,, \quad \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$$

$$\{q_j,q_k\} = \{p_j,p_k\} = 0\,, \quad \{q_j,p_k\} = \delta_{jk}$$

Coordinate Systems

$$x = \rho\cos\phi = r\sin\theta\cos\phi\,, \quad y = \rho\sin\phi = r\sin\theta\sin\phi\,, \quad z = r\cos\theta$$

$$dV = \rho\,d\rho\,d\phi\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi = r^2\,dr\,d\Omega$$

$$\nabla = \vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z} = \vec{e}_\rho\frac{\partial}{\partial\rho} + \vec{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \vec{e}_z\frac{\partial}{\partial z} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\nabla f(r) = f'(r)\frac{\vec{r}}{r}\,, \quad r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

Rotations

$$\vec{r}\,' = \vec{r}\cos\phi + \vec{n}(\vec{n}\cdot\vec{r})(1-\cos\phi) + (\vec{n}\times\vec{r})\sin\phi$$

$$R_x(\alpha) = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{pmatrix}\,, \quad R_y(\alpha) = \begin{pmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ -\sin\alpha & 0 & \cos\alpha\end{pmatrix}\,, \quad R_z(\alpha) = \begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$A = BCD = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{pmatrix}$$

Expansions

$$\frac{1}{(1+x)^n} = 1 - nx + \frac{1}{2}n(n+1)x^2 - \frac{1}{6}n(n+1)(n+2)x^3 + O(x^4)$$

$$\sqrt{1-x} = 1 - \frac{x}{2} + O(x^2)\,, \quad x \ll 1$$

$$\sin x = x - \frac{1}{6}x^3 + O(x^5)\,, \qquad \cos x = 1 - \frac{1}{2}x^2 + O(x^4)$$

$$\ln 1 + x = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + O(x^5)\,, \quad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)\,, \quad \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

Other Mathematical Formulas

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if} \quad ijk = 123, 231, 312\,, \\ -1 & \text{if} \quad ijk = 213, 321, 132\,, \\ 0 & \text{else}\,. \end{cases} \qquad \sum_k \epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}M_{11} - a_{12}M_{12} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det \boldsymbol{A} = \det \boldsymbol{A}^T\,, \quad \det \boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}}\,, \quad \det \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_1 \cdot \ldots \cdot a_n$$

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x \qquad\qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\theta(x) = \begin{cases} 1 & \text{if} \quad x > 0\,, \\ \frac{1}{2} & \text{if} \quad x = 0\,, \\ 0 & \text{if} \quad x < 0\,. \end{cases}$$

$$\int_\alpha^\beta dx\, f(x)\delta(x - x_0) = \begin{cases} f(x_0) & \text{if } x_0 \in (\alpha,\, \beta)\,, \\ 0 & \text{else.} \end{cases}$$

$$\delta(ax) = \frac{1}{|a|}\delta(x) \qquad\qquad \delta(g(x)) = \sum_k \frac{1}{|g'(x_k)|}\delta(x - x_k)$$