H. Hergert
Facility for Rare Isotope Beams
and Department of Physics & Astronomy

MICHIGAN STATE UNIVERSITY

PHY422/820: Classical Mechanics

FS 2019

Homework #3 (Due: Sep 20)

September 14, 2019

Problem 7 – Lagrange Equations of the First and Second Kind

[**20 Points**] We consider gravity acting on a particle of mass m that glides without friction on the interior of the rotational paraboloid

$$az = x^2 + y^2 \,, \quad a = \text{const.} \tag{1}$$

1. Construct the Lagrangian $L(\rho, \phi, z)$ in cylindrical coordinates of a particle that is moving **without constraints** under the influence of gravity. Determine the Lagrange equations, and compare them to the equations resulting from Newton's Second Law in cylindrical coordinates.

2. Implement the constraint that the particle moves on the paraboloid (in cylindrical coordinates) by adding it to the Lagrangian with a Lagrange multiplier, and derive the Lagrange equations for the modified Lagrangian $\tilde{L}(r, \phi, z, \lambda)$.

3. Determine λ and use it to decouple the equations of motion.

Let us now exploit that the constraint is holonomic, and immediately use the Lagrange formalism of the second kind.

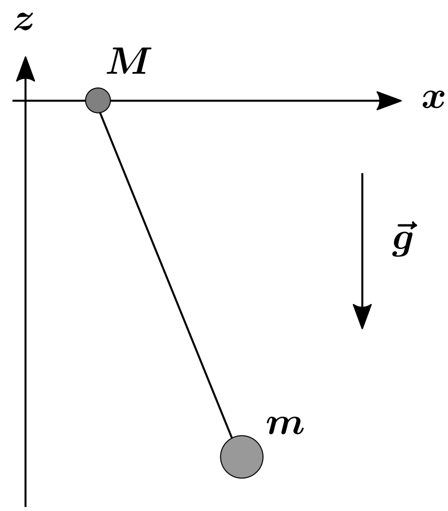
4. Use the constraint to identify suitable generalized coordinates q_i, and construct the Lagrangian $L(q_i, \dot{q}_i)$.

5. Derive the Lagrange equations for the generalized coordinates, and compare them to your results from step 3.

6. Show that the particle will move with an angular velocity $\omega = \sqrt{2g/a}$ if we restrict its trajectory to a circle at fixed height h.

Problem 8 – Pendulum with Moving Suspension

[15 Points] A mass m can swing on a string of length l around its suspension, which has mass M and can move freely in x direction itself (see figure).



1. Determine the Lagrangian and the Lagrange equations.

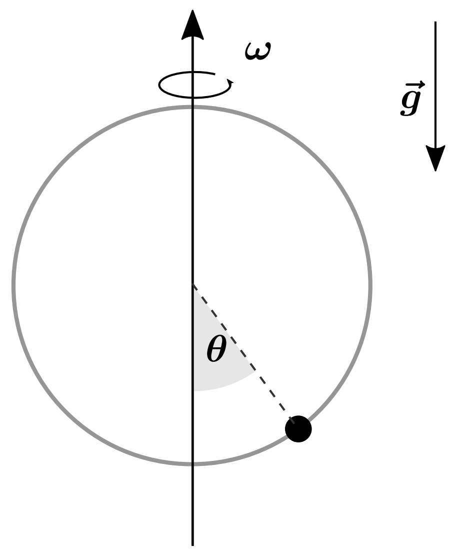
2. Which quantities are conserved?

3. Determine the frequency of the pendulum for small initial displacement. What do you obtain for $M \gg m$?

   Hint: Use the conservation law from step 2 to decouple the equations of motion.

Problem 9 – Bead on a Rotating Hoop

[15 Points] A bead of mass m can glide on a frictionless hoop with radius R under the influence of gravity. The hoop rotates around the z axis with a constant angular velocity ω (see figure). The motion of the bead can be parameterized by the angle θ.



1. Construct the Lagrangian and the equation of motion for the bead.

2. Determine the equilibrium positions of the bead on the wire, distinguishing the cases $\omega^2 > g/R$ and $\omega^2 < g/R$. Under which conditions are the equilibria stable under small perturbations $\theta \to \theta \pm \epsilon$?

3. Show that the Lagrangian and the resulting equation of motion are invariant under the transformation $\theta \to -\theta$, i.e., they possess reflection symmetry. Which of the equilibria still have this symmetry?

4. The angular velocity ω is a controllable parameter of the system. Make a qualitative sketch of the *stable* equilibria as a function of ω. What happens at the critical value $\omega_c = \sqrt{g/R}$? Have you encountered such a behavior before, perhaps in other branches of physics?

5. Finally, assume that the bead is in a stable equilibrium for $\omega < \omega_c$, and the angular velocity is then slowly increased above the critical value. What would you expect to happen in an ideal and realistic mechanical system, respectively?

Problem C3 – Variational Calculus: Visualizing the Variational Principle

[**15 Points**] You can find the Jupyter notebook with comments and code fragments in the Homework section of the course website (`http://people.nscl.msu.edu/ hergert/phy820`), or by pulling from the course material repository. Follow the procedure described in the Computation section of the website to submit your homework when you are ready.