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PHY422/820: Classical Mechanics

FS 2019

Homework #5 - **Update #6** (Due: Oct 7)

October 3, 2019

Problem 13 – Constraint on a Curve

[**5 Points**] A bead of mass m is sliding with velocity v along a curve $y = f(x)$ in the xy plane (i.e., the horizontal plane). Use the Lagrange formalism of the first kind to determine the constraint force acting on the bead.

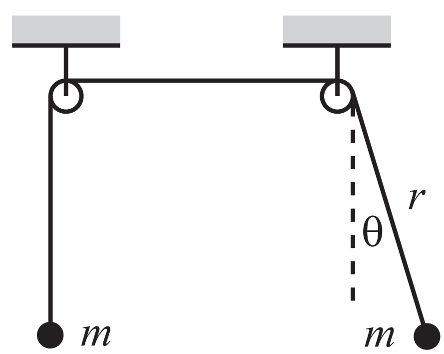
Problem 14 – Two Masses, One Swinging

[**10 Points**] Two equal masses m that are connected by a massless string of length l hang over two ideal massless and frictionless pulleys. The left mass is guided and can only move in a vertical line, but the right mass can swing.

1. Show that the Lagrangian of the sytem is given by

$$L = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr(1 - cos\theta)\,, \qquad (1)$$

   where r and θ are defined as shown in the figure.



2. Derive the equations of motion.

3. Assume the left mass starts at rest, and the right mass is undergoing oscillations with a small amplitude ϵ. What is the average acceleration $\ddot{r}$ over a few periods of the oscillation, and what does this imply for the motion of the left mass?

Problem 15 – Atwood Machines

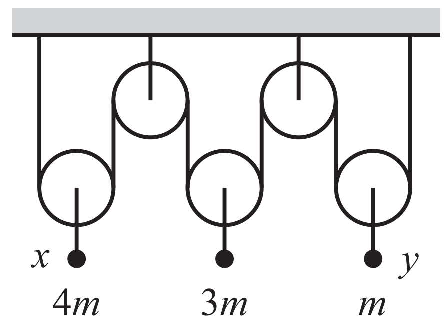
[**5 Points**] Consider the Atwood machine shown in the figure, consisting of the indicated masses, several ideal pulleys and a string of fixed length l.

1. Show that the Lagrangian of the machine is given by

$$L = \frac{7}{2}m\dot{x}^2 + 3m\dot{x}\dot{y} + 2m\dot{y}^2 - mg(x - 2y), \qquad (2)$$

   where x and y are the lengths indicated in the figure. (Note that there is some flexibility in the definition of the lengths.)

2. Show that the Lagrangian is invariant under the transformation $x \to x + 2\epsilon$ and $y \to y + \epsilon$, and use Noether's theorem to compute the conserved momentum.



Problem 16 – Atwood Machines II

[**5 Points**] Consider the Atwood machine shown in the figure, consisting of two masses m_1, m_2, an ideal pulley and a string of fixed length l.

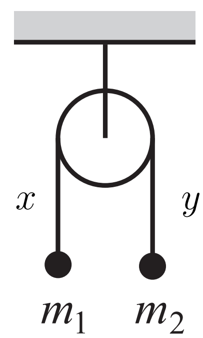
1. Show that the Lagrangian of the machine is given by

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_1 gx + m_2 gy, \qquad (3)$$

   where x and y are the lengths indicated in the figure.

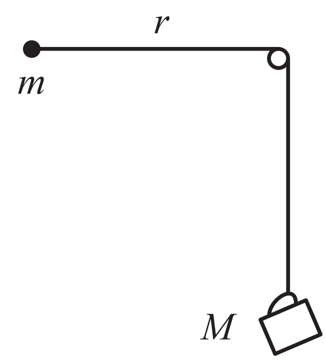
2. Use the Lagrange formalism of the first kind to show that the tension in the string is

$$T = \frac{2m_1m_2}{m_1 + m_2}g\,. \qquad (4)$$



Problem 17 – Coffee Cup and Mass

[**5 Points**] A coffee cup of mass M is connected to a mass m by a string of length l. The cup hangs over a frictionless pulley of negligible size, and the mass is initially held with the string horizontal, as shown in the figure. The mass m is then released. Assume that m somehow does not run into the string holding the cup up.



1. Show that the Lagrangian of the system is given by

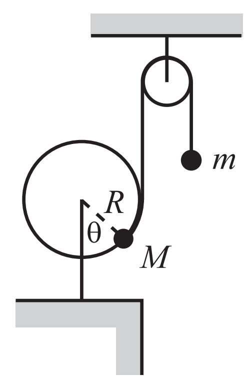
$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}M\dot{r}^2 + mgr\sin\theta + Mg(l-r), \tag{5}$$

   where r is the length of string between m and the pulley and θ (the angle that the string to m makes with the horizontal).

2. Determine the equations of motion. Show that the coffee cup will initially fall, but eventually reach a lowest point and then rise back up.

Problem 18 – Hoop and Pulley

[**5 Points**] A mass M is attached to a massless hoop (of radius R) which lies in a vertical plane. The hoop is free to rotate about its fixed center. M is tied to a string which winds part way around the hoop, then rises vertically up and over a massless pulley. A mass m hangs on the other end of the string (see figure).



1. Show that the Lagrangian of the machine is given by

$$L = \frac{1}{2}(M+m)R^2\dot{\theta}^2 + MgR\cos\theta + mgR\theta\,. \tag{6}$$

2. Find the equation of motion for the angle of rotation of the hoop. What is the frequency of small oscillations around the equilibrium? Assume that m moves only vertically, and that $M > m$.

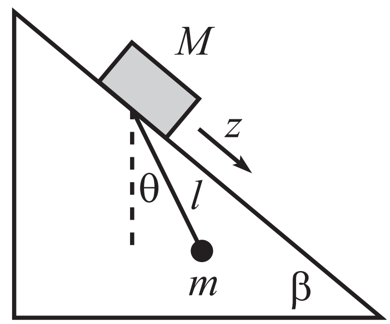
Problem 19 – Pendulum on Inclined Plane

[**10 Points**] A mass M is free to slide down a frictionless plane inclined at an angle β. A pendulum of length l and mass m hangs from M (see figure) (assume that M extends a short distance beyond the side of the plane, so the pendulum can hang down).

1. Show that the Lagrangian of the machine is given by

$$L = \frac{1}{2}M\dot{z}^2 + \frac{1}{2}m\left(\dot{z}^2 + l^2\dot{\theta}^2 + 2l\dot{z}\dot{\theta}\cos(\theta+\beta)\right) + Mgz\sin\beta + mg\left(z\sin\beta + l\cos\theta\right), \qquad (7)$$

   where z is the distance traveled on the plane and θ is the angle between the pendulum and the vertical axis.



2. Find the equations of motion and determine the equilibrium positions. Solve them for small displacements from equilibrium. (Equivalently, you can determine the normal modes and frequencies.)

Problem 20 – Friction

[**10 Points**] A dumbbell consisting of two equal masses that are connected by a massless rod of length l can move in a horizontal plane. The dumbbell is subject to a frictional force that is linear in the velocity.

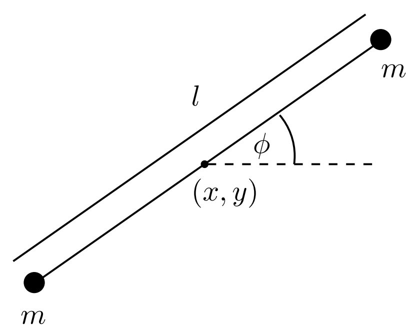
1. Show that the Lagrangian for the motion without friction is

$$L = m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{4}ml^2\dot{\phi}^2 \qquad (8)$$

   and Rayleigh's dissipation function

$$D = \beta\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{4}\beta l^2\dot{\phi}^2, \qquad (9)$$

   where x, y are the coordinates of the center of mass and ϕ is the angle indicated in the figure.



2. Compute the generalized forces Q_x, Q_y and Q_ϕ for the case with friction, and derive the equations of motion. State the general solutions in terms of the initial values for the coordinates (x_0, y_0, ϕ_0) and velocities $(v_{x0}, v_{y0}, \omega_0)$.