H. Hergert
Facility for Rare Isotope Beams
and Department of Physics & Astronomy

MICHIGAN STATE UNIVERSITY

PHY422/820: Classical Mechanics

FS 2019

Homework #7 (Due: Oct 18)

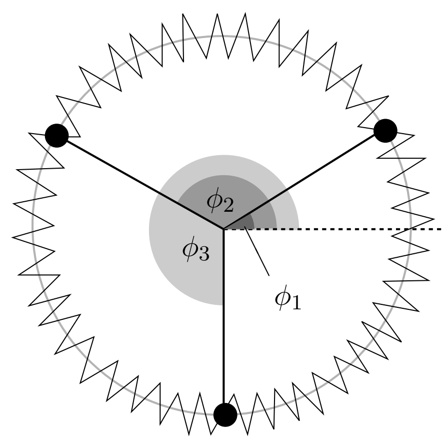
October 15, 2019

Problem 23 – Coupled Oscillators on a Circle

[15 Points] Consider three identical masses m that can move on a circular track of radius R (see figure). Each of the masses is coupled to its neighbors by identical springs. Assume that the resulting potential between masses i and j can be written as

$$V_{ij} = \frac{D}{2}\left(\phi_i - \phi_j\right)^2\,, \tag{1}$$

where ϕ_i indicates the position of mass i on the track.



1. Construct the Lagrangian of the system and derive the equations of motion for the angles ϕ_i.

2. Determine the normal modes, i.e., characteristic frequencies and vectors (vectors do not need to be normalized). Interpret your solutions.

Problem 24 – Linear Chain of Oscillators

[15 Points] The Lagrangian

$$L = \frac{1}{2}m\sum_{l=1}^{N-1}\dot{\eta}_l^2 - \frac{1}{2}m\omega_0^2\sum_{l=0}^{N-1}\left(\eta_{l+1}-\eta_l\right)^2 \tag{2}$$

describes a linear chain of oscillators whose ends are fixed by the boundary conditions

$$\eta_0 = 0\,, \eta_N = 0\,. \tag{3}$$

1. State the $\boldsymbol{T}$ and $\boldsymbol{V}$ matrices.

2. Find the equations of motion for $\eta_1,\ldots,\eta_{N-1}$ and show that they are solved by the following normal modes:

$$\eta_l^{(s)} = \rho_l^{(s)}\cos(\omega_s t + \phi_s)\,,\quad s,l = 1,\ldots,N-1\,, \tag{4}$$

where the characteristic frequencies and vectors are given by

$$\omega_s \equiv 2\omega_0 \sin\frac{s\pi}{2N}\,,\quad \rho^{(s)}_l \equiv C^{(s)}\sin\frac{sl\pi}{N}\,, \tag{5}$$

and $C^{(s)}$ are normalization constants.

HINT:

$$\begin{aligned}\sin(\alpha\pm\beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta\\ \cos(\alpha\pm\beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta\end{aligned}$$

3. Show that all frequencies ω_s are distinct, which means that the characteristic vectors for different frequencies are orthogonal in the inner product induced by $\boldsymbol{T}$.

   HINT: Consider the range of values that the argument of the sine function in ω_s can take.

4. Show that all characteristic vectors $\vec{\rho}^{(s)}$ are normalized by the same factor

$$C^{(s)} = \sqrt{\frac{2}{Nm}}\,. \tag{6}$$

5. Combine your previous results to prove the following trigonometric identity:

$$\sum_{l=1}^{N-1}\sin\frac{rl\pi}{N}\sin\frac{sl\pi}{N} = \frac{N}{2}\delta_{rs}\,,\quad r,s=1,\ldots,N-1\,. \tag{7}$$

Note: Recall that the inner product induced by $\boldsymbol{T}$ is defined as

$$\left(\vec{\rho}^{(r)},\ \vec{\rho}^{(s)}\right) \equiv \sum_{k,l=1}^{N-1}\rho^{(r)}_k\, T_{kl}\, \rho^{(s)}_l\,. \tag{8}$$

Problem C7 – Coupled Oscillators

[15 Points] You can find the Jupyter notebook with comments and code fragments in the Homework section of the course website (`http://people.nscl.msu.edu/ hergert/phy820`), or by pulling from the course material repository. Follow the procedure described in the Computation section of the website to submit your homework when you are ready.