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PHY422/820: Classical Mechanics

FS 2019

Homework #8 (Due: Oct 25)

October 20, 2019

Problem 25 – Normal Modes and Generalized Forces

[10 Points] In our discussion of coupled oscillators, we only considered the forces resulting from a potential, e.g., a spring potential, or the Taylor expansion of a general potential around equilibrium. Let us now assume that each degree of freedom η_j is also subject to a generalized external force Q_j, e.g., friction or a driving term. We can write the equations of motion componentwise as

$$\sum_k T_{jk}\ddot{\eta}_k + \sum_k V_{jk}\eta_k = Q_j\,, \quad j = 1, \ldots, N\,, \tag{1}$$

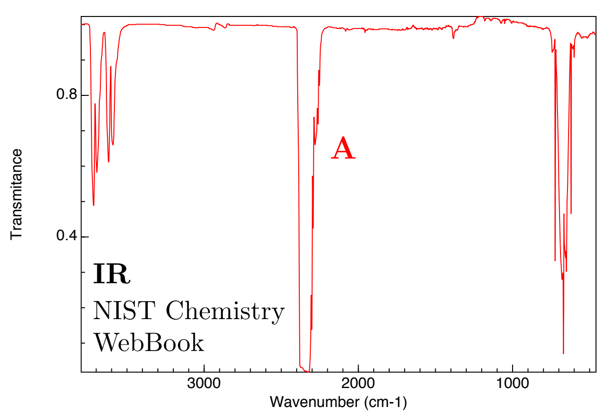
or vectorially as

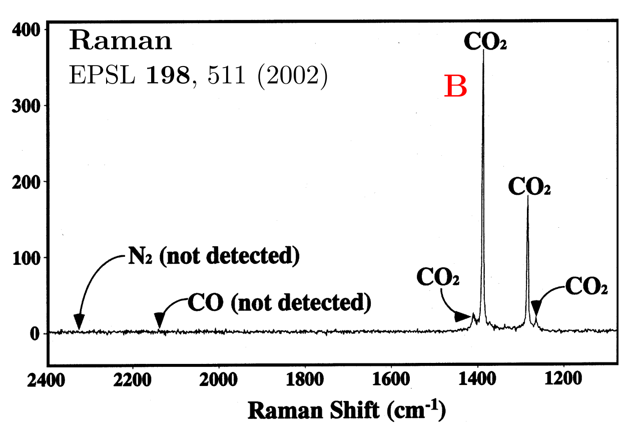
$$\boldsymbol{T}\ddot{\vec{\eta}} + \boldsymbol{V}\vec{\eta} = \vec{Q}\,. \tag{2}$$

1. Show that the equations of motion in the normal mode basis are given by

$$\ddot{\zeta}_s + \omega^2\zeta_s = \sum_k A_{sk}Q_k = (\mathbf{A}^T\vec{Q})_s\,, \tag{3}$$

   where the indices s and k refer to the normal mode and original bases, respectively, and $\mathbf{A}$ is the modal matrix built from the characteristic vectors.





2. Let us now consider CO_2 as a classical tri-atomic molecule. To perform molecular spectroscopy, samples are exposed to (nearly) monochromatic radiation to excite their molecular normal modes. The changing fields of the electromagnetic waves can be viewed as a periodic driving force with a (nearly) unique external frequency ω_{ext}.

   The left figure shows an *infrared absorption spectrum*. Dips in the radiation transmission indicate wave numbers $k = 2\pi/\lambda = \omega/c$ at which the molecule is absorbing high amounts of the incoming radiation, i.e., a normal mode's resonance frequency.

   The right figure shows a spectrum from *Raman scattering*, which is an approach that can excite normal modes that are not activated by simple IR irradiation.

   Which normal modes of CO_2 do the highlighted peaks A ($k_A \approx 2350\,\text{cm}^{-1}$) and B ($k_B \approx 1388\,\text{cm}^{-1}$) correspond to? Are the relative positions of the peaks (roughly) consistent with expectations, based on the mass ratio of the C and O atoms?

Problem 26 – Central Forces and Trajectories

[**5 Points**] In class we have shown the following relation between a central force $f(r)$ and the allowed trajectories $r(\phi)$:

$$-\frac{dV}{dr} = f(r) = \frac{l^2}{mr^4}\left[\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - r\right]. \tag{4}$$

Use Eq. (4) to find the central force that gives rise to the spiraling trajectory

$$r(\phi) = r_0 \exp(-\phi)\,. \tag{5}$$

Problem 27 – The Laplace-Runge-Lenz Vector

[**15 Points**] As shown in class, a general trajectory in the potential $V(r) = -\frac{\kappa}{r}$ is given by the *conic section* with focal parameter p and eccentricity ϵ:

$$r(\phi) = \frac{p}{1 + \epsilon\cos\phi}. \tag{6}$$

For such trajectories, there is an additional conserved quantity besides the angular momentum $\vec{l}$ and energy E, the so-called *Laplace-Runge-Lenz vector*

$$\vec{A} = \frac{\vec{p}\times\vec{l}}{m\kappa} - \vec{e}_r. \tag{7}$$

1. Show that $\dot{\vec{A}} = 0$.

2. Show that $|\vec{A}|$ is the eccentricity ϵ of the trajectory (6). To achieve this, start by computing $\vec{l}\cdot\vec{A}$ and $\vec{r}\cdot\vec{A}$, and parameterize the trajectory by defining the angular variable as $\phi \equiv \sphericalangle(\vec{r}, \vec{A})$.

Problem C8 – The Kepler Problem

[15 Points] You can find the Jupyter notebook with comments and code fragments in the Homework section of the course website (`http://people.nscl.msu.edu/ hergert/phy820`), or by pulling from the course material repository. Follow the procedure described in the Computation section of the website to submit your homework when you are ready.