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PHY422/820: Classical Mechanics

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Homework #11 (Due: Nov 15)

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Problem 30 – Vectors and Antisymmetric Matrices in Three Dimensions

[15 points] In three dimensions, one can construct a unique mapping between vectors and anti-symmetric (or skew-symmetric) matrices:

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^{3\times 3} : \quad \Phi(\vec{v}) = \Phi\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}, \tag{1}$$

or in components

$$[\Phi(\vec{v})]_{ij} = -\epsilon_{ijk} v_k\,, \quad \epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312 \quad \text{(cyclic permutations)}\,, \\ -1 & \text{if } ijk = 213, 132, 321 \quad \text{(anticyclic permutations)}\,, \\ 0 & \text{else}\,, \end{cases} \tag{2}$$

where ϵ is the usual Levi-Civita tensor.

1. Show that the mapping is *linear*, i.e.,

$$\Phi\left(\alpha\vec{u} + \beta\vec{v}\right) = \alpha\Phi(\vec{u}) + \beta\Phi(\vec{v})\,. \tag{3}$$

2. Show that the usual scalar product can be written as

$$\vec{u}\cdot\vec{v} = \frac{1}{2}\mathrm{tr}\,\left[\Phi(\vec{u})^T\Phi(\vec{v})\right]\,. \tag{4}$$

3. Show that the vector product can be obtained from

$$\vec{u}\times\vec{v} = \Phi(\vec{u})\vec{v} \tag{5}$$

or, alternatively,

$$\Phi(\vec{u}\times\vec{v}) = \Phi(\vec{u})\Phi(\vec{v}) - \Phi(\vec{v})\Phi(\vec{u}) = \left[\Phi(\vec{u}), \Phi(\vec{v})\right] \tag{6}$$

where we have introduced the commutator

$$\left[A, B\right] = AB - BA\,. \tag{7}$$

Problem 31 – Infinitesimal Rotations and SO(3) Generators

[15 points] A counter-clockwise rotation by an angle ϕ around the axis $\vec{n}$ can be expressed in vector form as

$$\vec{r}' = \vec{r}\cos\phi + \vec{n}(\vec{n}\cdot\vec{r})(1-\cos\phi) + (\vec{n}\times\vec{r})\sin\phi\,. \tag{8}$$

1. Show that for infinitesimal angles

$$\vec{r}' = \vec{r} + (\epsilon\vec{n})\times\vec{r} = (\mathbb{1}+\boldsymbol{\epsilon})\,\vec{r}\,, \tag{9}$$

where we have defined

$$\boldsymbol{\epsilon} \equiv \Phi(\epsilon\vec{n})\,. \tag{10}$$

2. Use the mapping between vectors and antisymmetric matrices to show that

$$\boldsymbol{\epsilon} = \epsilon n_x\,\boldsymbol{L_x} + \epsilon n_y\,\boldsymbol{L_y} + \epsilon n_z\,\boldsymbol{L_z}\,, \tag{11}$$

where

$$\boldsymbol{L}_x = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\,,\quad \boldsymbol{L}_y = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}\,,\quad \boldsymbol{L}_z = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\,, \tag{12}$$

are the so-called **generators** of infinitesimal rotations.

3. Show that the generators satisfy

$$\left[\boldsymbol{L_x},\boldsymbol{L_y}\right] = \boldsymbol{L_z}\,,\quad \left[\boldsymbol{L_y},\boldsymbol{L_z}\right] = \boldsymbol{L_x}\,,\quad \left[\boldsymbol{L_z},\boldsymbol{L_x}\right] = \boldsymbol{L_y}\,, \tag{13}$$

where the commutator was defined in Eq. (7).

4. The generators can be used to construct arbitrary antisymmetric matrices. Show that the matrix exponential of any antisymmetric matrix is a rotation matrix, i.e.,

$$\left(e^{\boldsymbol{A}}\right)^T = \left(e^{\boldsymbol{A}}\right)^{-1}\,,\quad \det e^{\boldsymbol{A}} = 1\,. \tag{14}$$

The matrix exponential is defined by the series

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty}\frac{1}{k!}\boldsymbol{A}^k\,. \tag{15}$$