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MICHIGAN STATE UNIVERSITY

PHY422/820: Classical Mechanics

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Homework #12 (Due: Nov 22)

November 22, 2019

Problem 32 – Mass Densities

[10 points] Which solids are described by the following mass densities?

$$\rho_1(\vec{r}) = \frac{M}{\pi R^2}\theta\left(R-\rho\right)\delta\left(z\right)\,, \tag{1}$$

$$\rho_2(\vec{r}) = \frac{M}{\pi R^2 H}\theta\left(R-\rho\right)\theta\left(\frac{H}{2}-|z|\right)\,, \tag{2}$$

$$\rho_3(\vec{r}) = \frac{M}{\pi(a^2-b^2)H}\theta\left(a-\rho\right)\theta\left(\rho-b\right)\theta\left(\frac{H}{2}-|z|\right)\,, \tag{3}$$

$$\rho_4(\vec{r}) = \frac{M}{4\pi R^2}\delta(r-R)\,. \tag{4}$$

Perform a volume integration over three-dimensional space in appropriate coordinates to show that you obtain the mass M of the solid.

Note:

$$\int_a^b f(x)\delta(x-x_0) = \begin{cases} f(x_0) & \text{if} \quad x_0\in[a,b]\,,\\ 0 & \text{else,}\end{cases} \qquad \theta(x) = \begin{cases} 1 & \text{if} \quad x>0\,,\\ \frac{1}{2} & \text{if} \quad x=0\,,\\ 0 & \text{if} \quad x<0\,.\end{cases}$$

Problem 33 – Moment of Inertia Tensor

[15 points] A hollow cylinder of mass M and radius R can be described by the mass density

$$\rho(r,\varphi,z) = \frac{M}{2\pi RH}\theta\left(\frac{H}{2}-|z|\right)\delta(r-R)\,, \tag{5}$$

where r indicates the radial distance from the cylinder's symmetry axis, chosen to be the z axis of our coordinate system.

1. Compute the moment of inertia tensor $\hat{\boldsymbol{I}}$ of the cylinder, and determine its principal axes.

2. Use rotation matrices to determine $\hat{\boldsymbol{I}}$ in coordinate systems that are rotated by *(i)* $\frac{2\pi}{3}$ around the original z axis, and *(ii)* by $\frac{\pi}{4}$ around the original y axis.

3. Verify by explicit calculation that the kinetic energy for a rotation around the axis $\vec{\omega} = \omega \vec{e}_z$ in the original coordinate system is identical to the kinetic energy we obtain for that motion in the two rotated coordinate systems.

Note:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad R_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}, \quad R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 34 – Rotating Platelet

[**15 points**] Consider a thin rectangular platelet of mass m with side lengths a, b and a homogeneous mass distribution. Choose a coordinate system whose origin is the platelet's center of mass.

1. Express the platelet's mass density $\rho(x, y, z)$ using $\delta-$ and step functions.

2. Compute the moment of inertia tensor $\hat{\boldsymbol{I}}$ in the chosen center-of-mass frame, and determine the principal axes.

3. Determine the platelet's equations of motion in the body-fixed frame, the **Euler equations for the rigid body**, by starting from

$$\frac{d'\vec{L}'}{dt} + \vec{\omega} \times \vec{L}' = \vec{N}', \quad \vec{L}' = \hat{\boldsymbol{I}}\omega = (A\omega_\xi, B\omega_\eta, C\omega_\zeta)^T. \tag{6}$$

   A, B and C denote the principal moments of inertia.

4. Compute the torque $\vec{N}'$ that is required to make the platelet rotate with a *constant* angular velocity around its *diagonal*. What happens if the platelet is quadratic, i.e., $a = b$?