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PHY422/820: Classical Mechanics

FS 2019

Homework #13 (**Due: Dec 2**)

November 25, 2019

Problem 35 – Racing Solids

[**15 points**] A solid sphere, a solid cylinder and a thin-walled, hollow cylinder (both of height H) with equal masses M and radii R are rolling down an inclined plane with inclination angle α.

1. Compute the solids' moments of inertia with respect to rotations around principal axes through the center of mass that are relevant for the rolling motion studied here. **You do not have to compute the complete moment of inertia tensor.**

2. Describe the motion of the solids as a superposition of center-of-mass translation and rotation around the center of mass. Construct the Lagrangian, determine the equations of motion and state their general solutions.

3. Now describe the motion of the solids as a rotation around a principal axis through the point where the solid touches the inclined plane. Construct the Lagrangian for this case and show that you obtain the same equations of motion as in the previous part of the problem.

4. Which of the three solids will reach the bottom of the inclined plane in the shortest amount fo time after being released from rest at the top?

Problem 36 – Legendre Transformations

[**15 points**] Consider the following situation: We know a function $f(x, y)$ that relates two quantities x and y, but we would rather have a relationship $\tilde{f}(x, z)$ where $z = \frac{\partial f}{\partial y}$ is of greater interest to us, e.g., because it can be measured more readily, or because it might be conserved (an example would be $L(q, \dot{q})$, with $p = \frac{\partial L}{\partial \dot{q}}$ being a conserved momentum).

1. **The naive approach:** Consider the family of functions

$$f_a(x, y) = x^2 + (y - a)^2 \,. \tag{1}$$

   Determine $\tilde{f}_a(x, z) = f_a(x, y(x, z))$ by computing $z = \frac{\partial f_a}{\partial y}$, solving the resulting equation for y, and plugging this $y(x, z)$ into the original function. Show that the relationship between f_a and $\tilde{f}_a$ is not unique, which means that we cannot invert the procedure to obtain f_a from $\tilde{f}_a$.

2. **The Legendre transformation:** Compute $z = \frac{\partial f_a}{\partial y}$ as before, and use it to construct the Legendre transform

$$g_a(x,z) \equiv zy(x,z) - f_a(x, y(x,z))\,. \tag{2}$$

Show that the transformation is invertible, i.e., that we can compute $y = \frac{\partial g_a}{\partial z}$ and obtain the original function from

$$f_a(x,y) = yz(x,y) - g_a(x, z(x,y))\,. \tag{3}$$

How are $\frac{\partial g_a}{\partial x}$ and $\frac{\partial f_a}{\partial x}$ related?

Have a Happy (and Relaxing) Thanksgiving!