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PHY422/820: Classical Mechanics

FS 2019

Homework #14 (Due: Dec 6)

November 30, 2019

Problem 37 – Hamiltonian of a Particle in an Electromagnetic Field

[**10 Points**] We recall from problem 12 that the Lagrangian of a particle of mass m and charge q in an electromagnetic field is given (in SI units) by

$$L(\vec{r}, \dot{\vec{r}}) = \frac{1}{2}m\dot{\vec{r}}^{\,2} - q\left(\phi(\vec{r}, t) - \dot{\vec{r}}\cdot\vec{A}(\vec{r}, t)\right)\,. \tag{1}$$

The electric and magnetic fields are obtained from the scalar and vector potentials via

$$\vec{E}(\vec{r}, t) = -\nabla\phi(\vec{r}, t) - \frac{\partial}{\partial t}\vec{A}(\vec{r}, t), \qquad \vec{B}(\vec{r}, t) = \nabla\times\vec{A}(\vec{r}, t)\,. \tag{2}$$

1. Compute the canonical momentum $\vec{p}$. How is it related to the mechanical momentum $m\dot{\vec{r}}$ of the particle? Use $\vec{p}$ to construct the Hamiltonian.

2. Derive Hamilton's equations and show that they yield the familiar Lorentz force

$$\vec{F}(\vec{r}, \dot{\vec{r}}) = q\vec{E}(\vec{r}, t) + q\dot{\vec{r}}\times\vec{B}(\vec{r}, t)\,. \tag{3}$$

Problem 38 – Hamiltonians and Reference Frames

[**10 points**] We consider the Lagrangian of a mass falling in a gravitational field,

$$L(y, \dot{y}) = \frac{1}{2}m\dot{y}^2 - mgy\,. \tag{4}$$

1. Make a change of coordinates to an inertial frame that is moving with a constant velocity v_0, $y \to y' = y + v_0 t$. Construct the Lagrangian L' in the new frame, and show that the Lagrange equations remain unchanged.

2. Construct the Hamiltonians in the original and transformed frames, $H(y, p, t)$ and $H'(y', p', t)$, and compute their time derivatives. Are the Hamiltonians conserved? Do they correspond to the total energy of the falling mass?

   Hint: Eliminate $\dot{p}$ and $\dot{y}$ using the Hamilton equations.

Problem 39 – Poisson Brackets

[**15 points**] We consider a system with n generalized coordinates q_j and canonical momenta p_j, that is described by the Hamiltonian $H(q_j, p_j, t)$. The **Poisson bracket** of two functions $f(q_j, p_j, t), g(q_j, p_j, t)$ is defined by

$$\{f, g\} \equiv \sum_{k=1}^{n} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right) . \tag{5}$$

Prove the following properties:

1. $\{q_i, p_j\} = \delta_{ij}$.
2. $\{f, g\} = -\{g, f\}$.
3. $\{fg, h\} = \{f, h\}g + f\{g, h\}$.
4. Show that the Poisson brackets satisfy the **Jacobi identity**

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 . \tag{6}$$

   HINT: Compute the sum of two of the three terms, e.g., $\{f, \{g, h\}\} + \{g, \{h, f\}\}$, and compare your result to the remaining term of the identity.

5. Use Hamilton's equations to show that the total time derivative of an arbitrary function $f(q_j, p_j, t)$ can be written as

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} . \tag{7}$$

   What happens if f is a conserved quantity?

6. Show that the Poisson brackets can be used to write Hamilton's equations as

$$\dot{q}_i = \{q_i, H\} , \quad \dot{p}_i = \{p_i, H\} . \tag{8}$$

Problem C14 – Nonlinear Dynamics

[**ungraded**] You can find the Jupyter notebook with comments and code fragments in the Homework section of the course website (`http://people.nscl.msu.edu/ hergert/phy820`), or by pulling from the course material repository.