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MICHIGAN STATE UNIVERSITY

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Mechanics

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_j^c + Q_j^{n.c.}\,, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$S = \int_{t_1}^{t_2} dt\, L(q,\dot{q},t)\,, \quad \delta S = 0$$

$$\widetilde{L}(q,\dot{q},t,\vec{\lambda}) = L(q,\dot{q},t) + \sum_{\alpha=1}^{N_h} \lambda_\alpha f_\alpha(q,t)\,, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{\alpha=1}^{N_h} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} + \sum_{\beta=1}^{N_{n.h.}} \mu_\beta \frac{\partial g_\beta}{\partial \dot{q}_i}$$

$$q_i'(t') = q_i(t) + \epsilon \eta_i(t)\,, \quad t' = t + \epsilon\tau(t)\,, \quad L' = L + \frac{dF}{dt}$$

$$J = \sum_i \frac{\partial L}{\partial \dot{q}_i}\left(\dot{q}_i \tau - \eta_i\right) - L\tau + F = \text{const.}$$

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + V(r)\,, \quad \vec{A} = \frac{\vec{p}\times\vec{l}}{m\kappa} - \frac{\vec{r}}{r}$$

$$\phi - \phi_0 = \pm \int_{r(\phi_0))}^{r(\phi)} dr' \, \frac{l}{r'^2\sqrt{2m\left(E - V_{\text{eff}}(r')\right)}}$$

$$r(\phi) = \frac{\alpha}{1 + \epsilon\cos(\phi - \phi_0)}\,, \quad \alpha = \frac{l^2}{m\kappa}\,, \quad \epsilon = |\vec{A}|$$

$$I_{ab} = \int d^3r\, \rho(\vec{r})\left(\vec{r}^2\delta_{ab} - r_a r_b\right)$$

$$I_{ab} = I_{ab}^{\text{CoM}} + M\left(\vec{R}^2\delta_{ab} - R_a R_b\right)$$

$$T_{\text{rot}} = \frac{1}{2}\vec{\omega}\cdot\boldsymbol{I}\cdot\vec{\omega}\,, \quad \vec{L} = \boldsymbol{I}\vec{\omega}$$

$$\vec{\omega} = \begin{pmatrix}\omega_{x'}\\ \omega_{y'}\\ \omega_{z'}\end{pmatrix} = \begin{pmatrix}\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\\ \dot{\phi}\cos\theta + \dot{\psi}\end{pmatrix}$$

$$\left(\frac{d\vec{L}}{dt}\right)_{LF} = \left(\frac{d}{dt}\left(\boldsymbol{I}\vec{\omega}\right)\right)_{BF} + \vec{\omega}\times\left(\boldsymbol{I}\vec{\omega}\right) = \vec{N}$$

$$\frac{d}{dt}\left(A\omega_{x'}\right) + (C-B)\omega_{y'}\omega_{z'} = N_{x'}$$
$$\frac{d}{dt}\left(B\omega_{y'}\right) + (A-C)\omega_{z'}\omega_{x'} = N_{y'}$$
$$\frac{d}{dt}\left(C\omega_{z'}\right) + (B-A)\omega_{x'}\omega_{y'} = N_{z'}$$

$$L = \frac{1}{2}\dot{\vec{\eta}}\cdot\boldsymbol{T}\cdot\dot{\vec{\eta}} - \frac{1}{2}\vec{\eta}\cdot\boldsymbol{V}\cdot\vec{\eta}, \qquad \left(\boldsymbol{V}-\omega^2\boldsymbol{T}\right)\vec{\rho} = 0\,, \qquad \det\left(\boldsymbol{V}-\omega^2\boldsymbol{T}\right) = 0$$

$$\boldsymbol{A} = \begin{pmatrix}\vec{\rho}^{(1)} & \cdots & \vec{\rho}^{(n)}\end{pmatrix}, \qquad \vec{\eta} = \boldsymbol{A}\vec{\zeta}, \qquad \vec{\zeta} = \boldsymbol{A}^T\boldsymbol{T}\vec{\eta}, \qquad \boldsymbol{A}^T\boldsymbol{T}\boldsymbol{A} = \mathbb{1}$$

$$\left(\vec{a},\ \vec{b}\right) = \vec{a}\cdot\boldsymbol{T}\cdot\vec{b} = \sum_{kl} a_k T_{kl} b_l$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j}\,, \quad H(q,p,t) = \sum_k p_k \dot{q}_k(q,p,t) - \widetilde{L}(q,p,t)$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j}\,, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$$\left\{f,g\right\} = \sum_k\left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k}\frac{\partial f}{\partial p_k}\right)\,, \quad \frac{df}{dt} = \left\{f,H\right\} + \frac{\partial f}{\partial t}$$

$$\left\{fg,h\right\} = \left\{f,h\right\}g + f\left\{g,h\right\}, \quad \left\{f,\left\{g,h\right\}\right\} + \left\{g,\left\{h,f\right\}\right\} + \left\{h,\left\{f,g\right\}\right\} = 0$$

$$\left\{q_j,q_k\right\} = \left\{p_j,p_k\right\} = 0\,, \quad \left\{q_j,p_k\right\} = \delta_{jk}$$

Function	**Transformation**	**Simplest Case**	
$F_1(q,Q,t)$	$p_i = \frac{\partial F_1}{\partial q_i}\,, \quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = \sum_i q_i Q_i$	$Q_i = p_i, P_i = -q_i$
$F_2(q,P,t)$	$p_i = \frac{\partial F_2}{\partial q_i}\,, \quad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = \sum_i q_i P_i$	$Q_i = q_i, P_i = p_i$
$F_3(p,Q,t)$	$P_i = -\frac{\partial F_3}{\partial Q_i}\,, \quad q_i = -\frac{\partial F_3}{\partial p_i}$	$F_3 = \sum_i p_i Q_i$	$Q_i = -q_i, P_i = -p_i$
$F_4(p,P,t)$	$q_i = -\frac{\partial F_4}{\partial p_i}\,, \quad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = \sum_i p_i P_i$	$Q_i = p_i, P_i = -q_i$

Coordinate Systems

$$x = \rho\cos\phi = r\sin\theta\cos\phi\,, \quad y = \rho\sin\phi = r\sin\theta\sin\phi\,, \quad z = r\cos\theta$$

$$dV = \rho\, d\rho\, d\phi\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi = r^2\, dr\, d\Omega$$

$$\nabla = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} = \vec{e}_\rho \frac{\partial}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \vec{e}_z \frac{\partial}{\partial z} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\nabla f(r) = f'(r)\frac{\vec{r}}{r}\,, \quad r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

Rotations

$$\vec{r}\,' = \vec{r}\cos\phi + \vec{n}(\vec{n}\cdot\vec{r})(1-\cos\phi) + (\vec{n}\times\vec{r})\sin\phi$$

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}\,, \quad R_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}\,, \quad R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = BCD = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{pmatrix}$$

Expansions

$$\frac{1}{(1+x)^n} = 1 - nx + \frac{1}{2}n(n+1)x^2 - \frac{1}{6}n(n+1)(n+2)x^3 + O(x^4)$$

$$\sqrt{1-x} = 1 - \frac{x}{2} + O(x^2)\,, \quad x \ll 1$$

$$\sin x = x - \frac{1}{6}x^3 + O(x^5)\,, \qquad \cos x = 1 - \frac{1}{2}x^2 + O(x^4)$$

$$\ln 1 + x = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + O(x^5)\,, \quad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)\,, \quad \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

Other Mathematical Formulas

Linear Algebra

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if} \quad ijk = 123, 231, 312\,, \\ -1 & \text{if} \quad ijk = 213, 321, 132\,, \\ 0 & \text{else}\,. \end{cases} \qquad \sum_k \epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}M_{11} - a_{12}M_{12} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det \boldsymbol{A} = \det \boldsymbol{A}^T\,, \quad \det \boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}}\,, \quad \det \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_1 \cdot \ldots \cdot a_n$$

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

Properties of Trigonometric Functions

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(2x) = 2 \sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

$$\int_0^{2\pi} dx\ \sin x = \int_0^{2\pi} dx\ \cos x = 0 \qquad \int_0^{2\pi} dx\ \sin^2 x = \int_0^{2\pi} dx\ \cos^2 x = \pi$$

Heaviside Step Function and Dirac Delta Function

$$\theta(x) = \begin{cases} 1 & \text{if} \quad x > 0\,, \\ \frac{1}{2} & \text{if} \quad x = 0\,, \\ 0 & \text{if} \quad x < 0\,. \end{cases} \qquad \int_{-\infty}^{\infty} dx\, \theta(a-x)\theta(x-b) f(x) = \int_a^b dx\, f(x)$$

$$\int_\alpha^\beta dx\, f(x)\delta(x - x_0) = \begin{cases} f(x_0) & \text{if } x_0 \in (\alpha,\, \beta)\,, \\ 0 & \text{else.} \end{cases}$$

$$\delta(ax) = \frac{1}{|a|}\delta(x) \qquad \delta(g(x)) = \sum_k \frac{1}{|g'(x_k)|}\delta(x - x_k)$$

Miscellaneous

$$ax^2 + bx + c = 0 \quad \Rightarrow \qquad x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$