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PHY422/820: Classical Mechanics

FS 2020

Homework #12 (Due: Nov 30)

November 21, 2020

Problem H23 – Normal Modes and Generalized Forces

[**10 Points**] In our discussion of coupled oscillators, we only considered the forces resulting from a potential, e.g., a spring potential, or the Taylor expansion of a general potential around equilibrium. Let us now assume that each degree of freedom η_j is also subject to a generalized external force Q_j, e.g., friction or a driving term. We can write the equations of motion componentwise as

$$\sum_k T_{jk}\ddot{\eta}_k + \sum_k V_{jk}\eta_k = Q_j\,, \quad j = 1, \ldots, N\,, \tag{1}$$

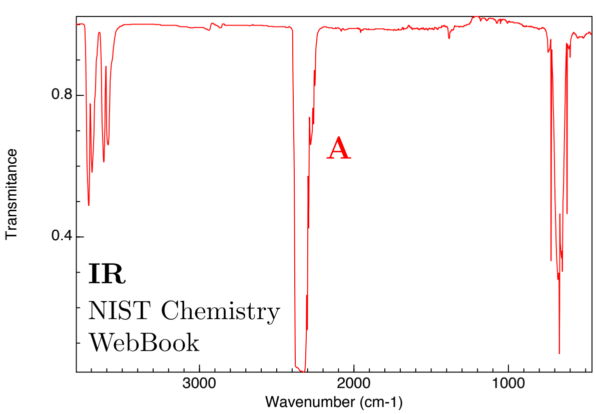
or vectorially as

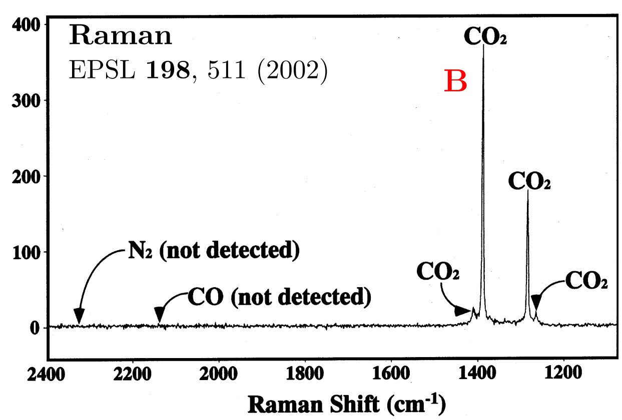
$$\boldsymbol{T}\ddot{\vec{\eta}} + \boldsymbol{V}\vec{\eta} = \vec{Q}\,. \tag{2}$$

1. Show that the equations of motion in the normal mode basis are given by

$$\ddot{\zeta}_s + \omega_s^2\zeta_s = \sum_k A_{sk}Q_k = (\mathbf{A}^T\vec{Q})_s\,, \tag{3}$$

   where the indices s and k refer to the normal mode and original bases, respectively, and $\mathbf{A}$ is the modal matrix built from the characteristic vectors.





2. Let us now consider CO_2 as a classical tri-atomic molecule. To perform molecular spectroscopy, samples are exposed to (nearly) monochromatic radiation to excite their molecular normal modes. The changing fields of the electromagnetic waves can be viewed as a periodic driving force with a (nearly) unique external frequency ω_{ext}.

   The left figure shows an *infrared absorption spectrum*. Dips in the radiation transmission indicate wave numbers $k = 2\pi/\lambda = \omega/c$ at which the molecule is absorbing high amounts of the incoming radiation, i.e., a normal mode's resonance frequency.

   The right figure shows a spectrum from *Raman scattering*, which is an approach that can excite normal modes that are not activated by simple IR irradiation.

   Which normal modes of CO_2 do the highlighted peaks A ($k_A \approx 2350\,\text{cm}^{-1}$) and B ($k_B \approx 1388\,\text{cm}^{-1}$) correspond to? Are the relative positions of the peaks (roughly) consistent with expectations, based on the mass ratio of the C and O atoms?

Problem H24 – Hamiltonian of a Particle in an Electromagnetic Field

[10 Points] We recall from problem 12 that the Lagrangian of a particle of mass m and charge q in an electromagnetic field is given (in SI units) by

$$L(\vec{r}, \dot{\vec{r}}) = \frac{1}{2} m \dot{\vec{r}}^{\,2} - q \left(\phi(\vec{r}, t) - \dot{\vec{r}} \cdot \vec{A}(\vec{r}, t) \right) . \tag{4}$$

The electric and magnetic fields are obtained from the scalar and vector potentials via

$$\vec{E}(\vec{r}, t) = -\nabla \phi(\vec{r}, t) - \frac{\partial}{\partial t} \vec{A}(\vec{r}, t), \qquad \vec{B}(\vec{r}, t) = \nabla \times \vec{A}(\vec{r}, t) . \tag{5}$$

1. Compute the canonical momentum $\vec{p}$. How is it related to the mechanical momentum $m\dot{\vec{r}}$ of the particle? Use $\vec{p}$ to construct the Hamiltonian.

2. Derive Hamilton's equations and show that they yield the familiar Lorentz force

$$\vec{F}(\vec{r}, \dot{\vec{r}}) = q\vec{E}(\vec{r}, t) + q\dot{\vec{r}} \times \vec{B}(\vec{r}, t) . \tag{6}$$

Problem H25 – Hamiltonians and Reference Frames

[10 points] We consider the Lagrangian of a mass falling in a gravitational field,

$$L(y, \dot{y}) = \frac{1}{2} m \dot{y}^2 - mgy . \tag{7}$$

1. Make a change of coordinates to an inertial frame that is moving with a constant velocity v_0, $y \to y' = y + v_0 t$. Construct the Lagrangian L' in the new frame, and show that the Lagrange equations remain unchanged.

2. Construct the Hamiltonians in the original and transformed frames, $H(y, p, t)$ and $H'(y', p', t)$, and compute their time derivatives. Are the Hamiltonians conserved? Do they correspond to the total energy of the falling mass?

   HINT: Eliminate $\dot{p}$ and $\dot{y}$ using the Hamilton equations.