Multiple conventions for rotations: you only cannot rotate about the same axis in successive rotations!

Euler's theorem: A general displacement of a rigid body with one fixed point is a rotation about an axis through that point.

(generalization: Add translation of the point.)

$$\vec{r}^{\,\prime}(t) = A(t)\,\vec{r}(0) \qquad \text{(in any frame)}$$

Fixed point:

$$\vec{r}_f(t) = A(t)\,\vec{r}_f(0) = \vec{r}_f(0)$$

$\rightarrow$ EVP ... $\lambda = 1$

$$\left(A - \lambda\mathbb{1}\right)\vec{r}_f = 0 \qquad \text{(can drop } t\text{)}$$

But A is orthogonal:

$$\left(A - \cancel{\lambda}\,\mathbb{1}\right)A^T = AA^T - A^T = 1 - A^T$$

$$\cancel{\det(A-\mathbb{1})\underbrace{\det A^T}_{=\det A = 1} = 1 - \underbrace{\det A^T}_{=\det A = 1}}$$

$$\cancel{\Rightarrow \det(A - \mathbb{1}) = 1 - 1 = 0}$$

~~Eigenvalue~~

$$\det\left[(A - \mathbb{1})A^T\right] = \det(1 - A^T)$$

$$\det(A - \mathbb{1})\underbrace{\det A^T}_{=\det A = 1} = \det(1 - A^T) = \det\left((1-A)^T\right)$$

$$= \det(\mathbb{1} - A)$$

$$\Rightarrow \quad \det(A - \mathbb{1}) = \det(\mathbb{1} - A) = \det\left(-(A - \mathbb{1})\right)$$

$$= \underbrace{\det(-\mathbb{1})}_{=(-1)^n \text{ in } \mathbb{R}^n}\det(A - \mathbb{1})$$

$n = 3$:

$$\det(A - \mathbb{1}) = -\det(A - \mathbb{1})$$

$$\Rightarrow \quad \det(A - \mathbb{1}) = 0 \qquad \text{EVP}$$

Note: for 2D ; $\det(A - \mathbb{1}) = \det(\mathbb{1} - A)$ and the theorem doesn't hold : no vector unaltered in $\mathbb{R}^2$! Rotation must be around an axis that's outside of the space (perpendicular to the plane)

↳ Can now determine other ~~angles~~ EV (see Goldstein), ~~[illegible]~~

(Choose coordinate system with $\vec{e}_z \parallel$ fixed rotation axis)

- direction of rotation axis ~~[illegible]~~ is not fixed;
- also, rotation by $-\Phi$ satisfies equs. because

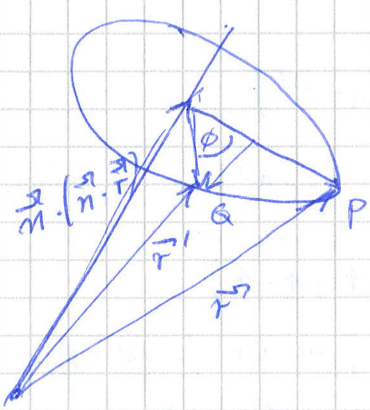
$$A(-\phi) = A^T(\phi)$$

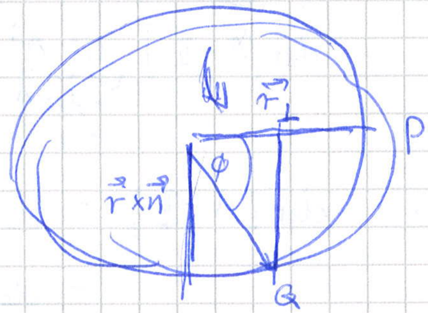
→ need a convention – RHS coord. system.

Single rotation about fixed axis.

- 3 params: angle of rotation, plus 2 angles specifying direction of axis in space

(properties matches with Euler: direction angls are $[0,\pi]$, $[0,2\pi)$, rot angle $[0,2\pi[$)

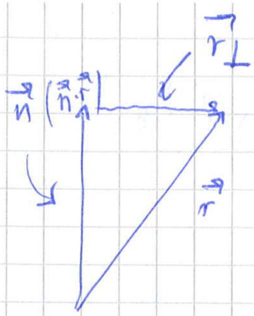




$$\vec{r}\times\vec{n} = \vec{r}_\perp \times \vec{n}$$

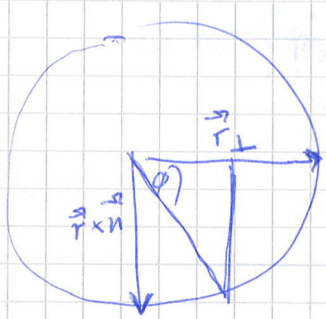
$$\Rightarrow \vec{r}\,' = \vec{r}\cos\phi + \vec{n}(\vec{n}\cdot\vec{r})(1-\cos\phi) + (\vec{r}\times\vec{n})\sin\phi$$

$$= \vec{n}(\vec{n}\cdot\vec{r}) - \vec{n}\times(\vec{r}\times\vec{n})\cos\phi + (\vec{r}\times\vec{n})\sin\phi$$



$$\vec r = \vec r_\parallel + \vec r_\perp$$
$$= \vec n(\vec n\cdot\vec r) + \vec r_\perp$$

$$\vec r \times \vec n = \vec r_\perp \times \vec n$$



$$\vec r_\perp = \vec n \times (\vec r \times \vec n)$$

$$\Rightarrow \vec r\,' = \vec n(\vec n\cdot\vec r) - \vec n\times(\vec r\times\vec n)\cos\phi$$

stays fixed

$$+ \vec r \times \vec n \sin\phi$$

$$\vec n\times(\vec r\times\vec n) = \vec r(\vec n^2) - \vec n(\vec n\cdot\vec r) = \vec r_\perp$$
$$\vec n\times(\vec r\times\vec n)\cos\phi = \vec r\cos\phi - \vec n(\vec n\cdot\vec r)\cos\phi$$
$$\Rightarrow \vec r_\perp \cdot \cos\phi$$

Logic:

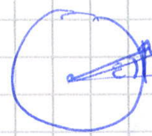
$$\vec r = \vec r_\parallel + \vec r_\perp = \vec n(\vec n\cdot\vec r) + (\vec r - \vec n(\vec n\cdot\vec r))$$
$$= \vec n(\vec n\cdot\vec r) + \underbrace{(\vec r(\vec n^2 - \vec n(\vec n\cdot\vec r))}_{= \vec n\times(\vec r\times\vec n)}$$

$$\Rightarrow \boxed{\begin{aligned} \vec r\,' &= \vec r_\parallel + \cos\phi\,\vec r_\perp + \sin\phi\,(\vec r\times\vec n) \\ &= \vec n(\vec n\cdot\vec r) + \vec r\cos\phi - \vec n(\vec n\cdot\vec r)\cos\phi + (\vec r\times\vec n)\sin\phi \end{aligned}}$$

Try infinitesimal rot.:

$$\vec{r}' = \vec{n}(\vec{n}\cdot\vec{r}) - \vec{n}\times(\vec{r}\times\vec{n})\,O(\varepsilon^2) + (\vec{r}\times\vec{n})\,\varepsilon$$

$$= \underbrace{\vec{n}(\vec{n}\cdot\vec{r})}_{\approx\vec{r}} + (\vec{r}\times\vec{n})\,\varepsilon$$



$(\vec{r}\times\vec{n})\,\varepsilon$

?

Infinitesimal rotations commute.

$$\vec{r}' = \cancel{(1+\varepsilon)\vec{r}}\ \vec{r} + \varepsilon\,\vec{r}$$

inf. rotation matrix

~~[illegible]~~

$$(1+\varepsilon_1)(1+\varepsilon_2) = 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2$$
$$= 1 + \varepsilon_1 + \varepsilon_2 + O(\varepsilon^2)$$

$$\Rightarrow (1+\varepsilon_2)(1+\varepsilon_1) = 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_2\varepsilon_1$$
$$= 1 + \varepsilon_1 + \varepsilon_2 + O(\varepsilon^2)$$

Euler matrix

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 - \cancel{\psi\phi\psi} & \phi + \psi & \cancel{\psi\theta} \\ -\cancel{\psi} - \phi & -\cancel{\psi\phi} + 1 & \theta \\ \cancel{\theta\phi} & -\theta & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & \phi+\psi & 0 \\ -(\phi+\psi) & 1 & \theta \\ 0 & -\theta & 1 \end{pmatrix}$$

$$A^{-1} = 1 - \varepsilon \Rightarrow AA^{-1} = (1+\varepsilon)(1-\varepsilon)$$
$$= 1 + \varepsilon - \varepsilon + \mathcal{O}(\varepsilon^2) = 1$$

But also

$$A^T = 1 + \varepsilon^T = A^{-1} = 1 - \varepsilon$$

$$\Rightarrow \varepsilon^T = -\varepsilon$$

Comparing with rot. matrix, we can say

$$\varepsilon = \begin{pmatrix} 0 & d\omega_3 & -d\omega_2 \\ -d\omega_3 & 0 & d\omega_1 \\ d\omega_2 & -d\omega_1 & 0 \end{pmatrix} \Rightarrow \vec{r}\,' = (1+\varepsilon)\vec{r}$$

$$\vec{r}\,' = \vec{r} \cdot \cos\theta + \vec{n}(\vec{n}\cdot\vec{r})(1-\cos\theta) + \vec{r}\times\vec{n}\sin\theta$$
$$\approx \vec{r} + \vec{n}(\vec{n}\cdot\vec{r})(1-1) + (\vec{r}\times\vec{n})\theta$$
$$= \vec{r} + (\vec{r}\times\vec{n})\theta = \vec{r} - (\vec{n}\times\vec{r})\theta$$

$$\vec{r}\,' - \vec{r} = \underset{\uparrow\ \text{inf.}}{\varepsilon}\,\vec{r} \Leftrightarrow \left.\begin{aligned} dx_1 &= x_2 d\omega_3 - x_3 d\omega_2 \\ dx_2 &= x_3 d\omega_1 - x_1 d\omega_3 \\ dx_3 &= x_1 d\omega_2 - x_2 d\omega_1 \end{aligned}\right\} = d\vec{r}\,' = \vec{r}\times d\vec{\omega}$$

Note: not differential - just a inf. small vector!

$$\vec{r}\,' - \vec{r} = (\vec{r}\times\vec{n})\,\varepsilon_\theta$$