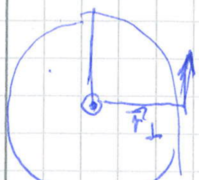
CCW rotation

$$\vec{r}' = \vec{r} \cdot \cos\theta + \vec{n}(\vec{n}\cdot\vec{r})(1-\cos\theta) + \vec{n}\times\vec{r}\,\sin\theta$$

($\vec{r}\times\vec{n}$ for CW)



infinitesimally small:

$$\vec{r}' \approx \vec{r} + \vec{n}(\vec{n}\cdot\vec{r})\cdot 0 + \vec{n}\times\vec{r}\cdot\varepsilon$$
$$= \vec{r} + \varepsilon\,\vec{n}\times\vec{r}$$

We also have for an infinitesimally small rotation matrix (cf. HW)

$$R(\varepsilon) \approx 1 + \varepsilon = \begin{pmatrix} 0 & -\varepsilon_3 & \varepsilon_2 \\ \varepsilon_3 & 0 & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 0 \end{pmatrix} \Rightarrow$$

ε's are <u>combinations</u> of Euler angles! (Try it out by expanding general rotation for small angles)

$$(1+\varepsilon)(1+\varepsilon^T) = 1 + \varepsilon + \varepsilon^T + \varepsilon\varepsilon^T \overset{!}{=} 1$$
$$\Rightarrow \varepsilon = -\varepsilon^T \quad (\text{through } O(\varepsilon))$$

So we can write

$$\vec{r}' = \vec{r} + \varepsilon(\vec{n}\times\vec{r})$$

where $\vec{n}$ is the momentary rotational axis ... namely $\vec{n} = \frac{\vec{\omega}}{|\vec{\omega}|}$

Rate of change (– setting up velocities & accel.)

$$(d\vec{g})_{LF} = (d\vec{g})_{BF} + (d\vec{g})_{rot}$$

BF vector: $(d\vec{g})_{BF} = 0$, so $\vec{r}'_{BF} = \vec{r}_{BF}$

$$(d\vec{g})_{LF} = \varepsilon(\vec{n}\times\vec{r})$$

$$\Rightarrow \left(\frac{d\vec{g}}{dt}\right)_{LF} = \left(\frac{d\vec{g}}{dt}\right)_{BF} + \vec{\omega}\times\vec{g}$$

(absorbs $|\vec{\omega}|$

Thus, we have the general rule for time derivatives:

$$\left(\frac{d}{dt}\right)_{LF} = \left(\frac{d}{dt}\right)_{BF} + \vec{\omega}\times$$

(linear operator)

Often it's most convenient to express the angular velocity vector $\vec{\omega}$ in terms of Euler angles ... this is possible because it's an infinitesimal change and infinitesimal rot's commute:

$$(1+\varepsilon_1)(1+\varepsilon_2) = 1+\varepsilon_1+\varepsilon_2 + \cancel{\mathcal{O}(\varepsilon^2)} = (1+\varepsilon_2)(1+\varepsilon_1)$$

So we can write

$$\vec{\omega} = \dot{\phi}\,\underset{LF}{\vec{e}_z} + \dot{\theta}\,\vec{e}_{x'} + \dot{\psi}\,\vec{e}_{z''} \qquad \text{in x-z convention}$$

$$= \dot{\phi}\,\cancel{A}BCB\,\vec{e}_z + \dot{\theta}\,B\,\vec{e}_x + \dot{\psi}\,\vec{e}_{z''}$$

$$\Rightarrow \vec{\omega} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}\cos\theta \end{pmatrix} + \begin{pmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

Repeat calc.

~~Lagr. Euler~~ equs. ~~in rota~~

Lagrangian for rotating frames (assume rot. invariant pot.)

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r}) = \frac{1}{2}m\left(\left(\frac{d\vec{r}'}{dt}\right)_{BF} + \vec{\omega}\times\vec{r}'\right)^2 - V(\vec{r}')$$

$$= \frac{1}{2}m\left(\dot{\vec{r}}'^2 + 2\dot{\vec{r}}'(\vec{\omega}\times\vec{r}') + (\vec{\omega}\times\vec{r}')^2\right) - V(\vec{r}')$$

$$= \frac{1}{2}m\left(\dot{\vec{r}}'^2\right.$$

Let LF and BF frame have the same origin

$$m\left(\frac{d^2\vec r}{dt^2}\right)_{LF} = \vec F$$

$$\vec v_{LF} \quad \left(\frac{d\vec r}{dt}\right)_{LF} = \left(\frac{d\vec r}{dt}\right)_{BF} + \vec\omega\times\vec r = \vec v + \vec\omega\times\vec r$$

Same origin: $(\vec r)_{LF} = (\vec r)_{BF}$

$$\left(\frac{d^2\vec r}{dt^2}\right)_{LF} = \left(\frac{d}{dt}\vec v_{LF}\right)_{LF} = \left(\frac{d\vec v_{LF}}{dt}\right)_{BF} + \vec\omega\times\vec v_{LF}$$

$$= \left(\frac{d\vec v}{dt}\right)_{BF} + \left(\frac{d\vec\omega}{dt}\right)_{BF}\times\vec r + \overbrace{2\vec\omega\times\vec v}^{} + \vec\omega\times(\vec\omega\times\vec r)$$

$$= \vec\omega\times\left(\frac{d\vec r}{dt}\right)_{BF} + \vec\omega\times\left(\frac{d\vec r}{dt}\right)_{BF}$$

from $\frac{d}{dt}\vec v_{LF}$ from $\vec\omega\times\vec v_{LF}$

$$\left(\frac{d\vec\omega}{dt}\right)_{LF} = \left(\frac{d\vec\omega}{dt}\right)_{BF} + \underbrace{\vec\omega\times\vec\omega}_{=0} \rightarrow \text{can drop subscript.}$$

$$\Rightarrow m\ddot{\vec r}_{BF} = \vec F + \underbrace{2m\vec v\times\vec\omega}_{\text{Coriolis}} + \underbrace{m\vec\omega\times(\vec r\times\vec\omega)}_{\text{centrifugal}} + \underbrace{m\vec r\times\frac{d\vec\omega}{dt}}_{\text{Euler}}$$

is the force in the BF frame (exercise: get this from Lagrangian, cf. 3B problem 8, using point-trafo invariance of Lagrange eqns.)

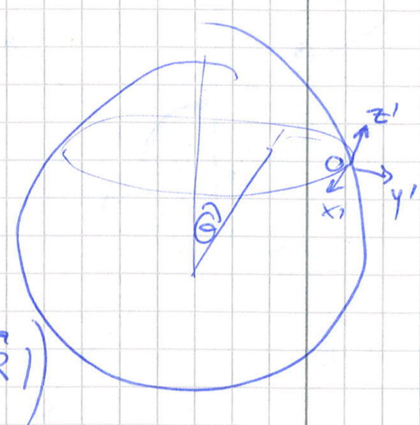
Example: rotating Earth

- Origin of inertial frame at center of earth ~~at centre of the Earth~~

$$\vec a_{O'} = m\left(\frac{d^2\vec R}{dt^2}\right)_{LF} =$$

$$m\left(\frac{d}{dt}\right)_{LF}\left(\underbrace{\left(\frac{d\vec R}{dt}\right)_{BF}}_{=0} + \vec\omega\times\vec R\right)$$

$$= m\left(\underbrace{\frac{d\vec\omega}{dt}}_{=0}\times\vec R + \vec\omega\times(\vec\omega\times\vec R)\right)$$



$$m\vec{a} = \vec{F}_{NG} + m\vec{g} + \underbrace{m\vec{\omega}\times(\vec{R}\times\vec{\omega})}_{\text{rot. of origin}} + \underbrace{2m\vec{v}\times\vec{\omega} + m\vec{\omega}\times(\vec{r}\times\vec{\omega})}_{\text{rot. of particle not sitting at origin}}$$

$\vec{F} = \vec{F}_{NG} + m\vec{g}$

$$\leadsto \quad m\vec{a} = \vec{F}_{NG} + m\vec{g}_{eff} + 2m\vec{v}\times\vec{\omega} + m\vec{\omega}\times(\vec{r}\times\vec{\omega})$$

$\vec{g}_{eff} = -\omega\cos\lambda\,\vec{e}_{x'} + \omega\sin\lambda\,\vec{e}_{z'}$ $\qquad$ latitude $\lambda = \pi/2 - \theta$

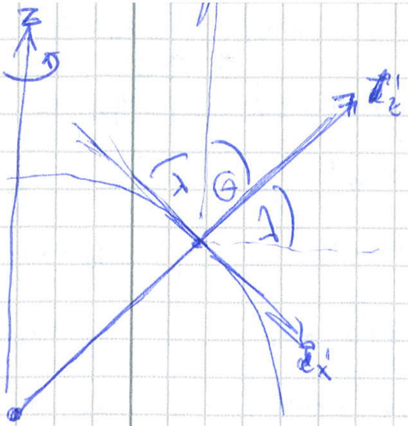
$\rightarrow$ $\vec{g}_{eff}$ does not point exactly to the center of the Earth! $(\sim \vec{e}_{z'})$

$\rightarrow$ reduction of $\vec{g}$ ~~as well~~

Small effects:

$$\omega = \frac{2\pi}{24\cdot 3600\,s} \approx 7.3\times 10^{-5}/s$$

$$\leadsto R\,\omega^2 \approx (6400\text{ km})\cdot\left(7.3\times 10^{-5}/s\right)^2 \approx 3\cdot 10^{-2}\,m/s^2 \qquad (\text{vs. } 9.8\,m/s^2)$$



$$\vec\omega = -\omega\cos\lambda\,\vec e_x' + \omega\sin\lambda\,\vec e_z'$$

$$\vec\omega\times(\vec R\times\vec\omega) = \vec R\,\omega^2 - \vec\omega(\vec R\cdot\vec\omega)$$

$$= R\vec e_z'\,\omega^2 - (-\omega\cos\lambda\,\vec e_x' + \omega\sin\lambda\,\vec e_z')\,R\omega\sin\lambda$$

$$= R\omega^2(1-\sin^2\lambda)\,\vec e_z' + R\omega^2\sin\lambda\cos\lambda\,\vec e_x'$$

$$\Rightarrow \vec g_{eff} = -\vec e_z'(+g - R\omega^2\cos^2\lambda) + R\omega^2\sin\lambda\cos\lambda\,\vec e_x'$$

max. at $\lambda = 0$