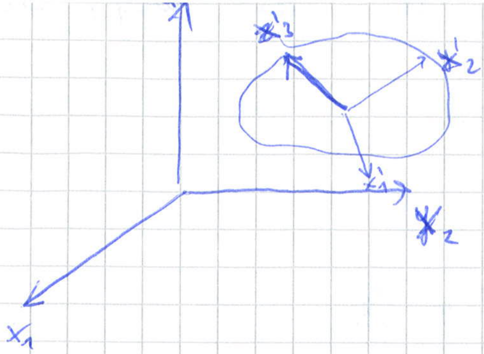



lab / space fixed

vs body-fixed

$$\vec{a} = \sum_i \underbrace{(\vec{a}\cdot\vec{e}_i)}_{= a_i}\,\vec{e}_i = \sum_{j} \underbrace{(\vec{a}\cdot\vec{e}_j{}')}_{a_i'}\,\vec{e}_j{}'$$

How are $a_i$ and $a_j'$ related? Basis transformation.

$$\cancel{a_i' = \vec{a}\cdot\vec{e}_i{}' = \sum_j (a_j\vec{e}_j{}')\cdot\vec{e}_i = \sum_j a_j(\vec{e}_j{}'\cdot\vec{e}_i)}$$

$$a_i' = (\vec{a}\cdot\vec{e}_i{}') = \left(\sum_j a_j\vec{e}_j\right)\cdot\vec{e}_i{}' = \sum_j a_j\,(\vec{e}_j\cdot\vec{e}_i{}')$$

components ~~express~~ of $\vec{e}_j$ in terms of $\vec{e}_i{}'$

$\overset{A}{\phi}_{ij} \equiv \vec{e}_i{}'\cdot\vec{e}_j$ → cosine of the angle between $\vec{e}_j$, $\vec{e}_i{}'$ or the axes they indicate

$\overset{A}{\phi}_{ij}$ cannot be independent:

$$\vec{a}\cdot\vec{a} = \sum_i a_i a_i = \sum_i a_i' a_i' = a^2$$

$$\sum_i a_i' a_i' = \sum_{ijk}\left(\overset{A}{\phi}_{ij} a_j\right)\left(\overset{A}{\phi}_{ik} a_k\right) = \sum_{ijk}\left(\overset{A}{\phi}_{ij}\overset{A}{\phi}_{ik}\right) a_j a_k$$

$$= \sum_{ijk}\left(\overset{A}{\phi}{}^T_{ji}\overset{A}{\phi}_{ik}\right) a_j a_k \overset{!}{=} \sum_j a_j a_j$$

$$\Rightarrow \underline{\underline{\sum_i\left(\overset{A}{\phi}{}^T_{ji}\overset{A}{\phi}_{ik}\right) = \delta_{jk}}}$$

<u>orthogonal matrices</u> : called so because they preserve angles between vectors, i.e. scalar products – that's how we found them just now!

$$\left( \Rightarrow \begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = \overset{A}{\phi}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right)$$

cf. coupled oscillators

we can just do the connection the other way, which will imply

$$\sum_i O_{ji} O^T_{ik} = \delta_{jk}$$

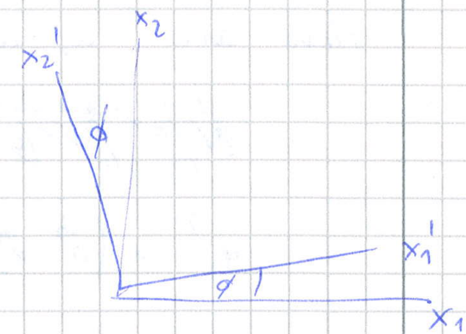
for the inverse transformation.

$$\Rightarrow \quad O^T O = O O^T = \mathbb{1}$$

~~standard scalar product~~ implies $O^T = O^{-1}$.

Example: in plane,

$$x_1' = x_1 \cos\phi + x_2 \sin\phi$$
$$x_2' = -x_1 \sin\phi + x_2 \cos\phi$$




$$\Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$ , $\phi$ specifies trafo.

$$\Rightarrow O^T = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \quad \Rightarrow O^T O = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ \cos\phi\sin\phi - \sin\phi\cos\phi & \cos^2\phi + \sin^2\phi \end{pmatrix}$$

$$= \mathbb{1}$$

Active vs. passive transformation:

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = O \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$ alternatively:

new vector, coordinate sys. fixed — <u>active</u>

same vector, coordinate system transformed — <u>passive</u>

Orthogonal matrices (or, more abstractly, orthogonal transformations) form a <u>group</u>:

$$a_i'' = \sum_j A_{ij} a_j' = \sum_{jk} A_{ij} B_{jk} a_k$$

$$= \sum_k \Big( \sum_j A_{ij} B_{jk} \Big) a_k = \sum_k C_{ik} a_k$$

~~intuitively clear;~~

Is $C$ ~~[illegible]~~ orthogonal?

$$C = AB \Rightarrow C^T = B^T A^T$$

$$\Rightarrow C^T C = B^T \underbrace{A^T A}_{= \mathbb{1}} B = \underbrace{B^T B}_{= \mathbb{1}} = \mathbb{1}$$

Intuitively clear: two successive rotations are still a rotation.

Group properties: (1) If $A, B$ are elements of $G$, so is $C = AB$

(2) $A(BC) = (AB)C$ associativity

(3) $G$ contains an identity element so that

$$A \mathbb{1} = \mathbb{1} A = A$$

(4) For each $A \in G$, there is a unique $B \in G$ satisfying $AB = BA = \mathbb{1}$. $B$ is called the inverse.

(1), (4) are clear / proved, (2) follows from associativity of matrix mult, (3) ~~from $AA^{-1}$~~ obvious.

$\Rightarrow$ Orthogonal matrices are a group called $O(3)$, orth. transformations of $\mathbb{R}^3$.

Determinant of $O$:

$$\det O \det O^T = \det(OO^T) = 1 \Rightarrow (\det O)^2 = 1$$

$\Leftrightarrow \det O = \pm 1$.

Simple orthogonal matrix with $\det O = -1$:

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = -\mathbb{1} \Rightarrow \underline{\text{reflection}}\ !$$

Multiplication with $-\mathbb{1}$ does not correspond to an actual displacement / rotation – you cannot rotate a left-handed into a right-handed coordinate system! Thus, only orthogonal matrices with $\det O = +1$ ~~[illegible]~~ <u>are rotations!</u>

Rotations form a subgroup of $O(3)$; called $SO(3)$
special

$$\det \mathbb{C} = \underbrace{\det A}_{=+1} \ \underbrace{\det B}_{=+1} = +1$$

$$\left( \det \mathbb{C} = \underbrace{\det A}_{=-1} \ \underbrace{\det B}_{=-1} \neq -1 \right)$$

Can easily show that

$$(O\vec{a}) \cdot (O\vec{b}) = \vec{a} \cdot \vec{b} \qquad \text{(see above)}$$

as well as

$$(O\vec{a}) \times (O\vec{b}) = O(\vec{a} \times \vec{b})$$

One issue: group $SO(3)$ is not commutative / non-Abelian

$$A \cdot B \neq BA$$

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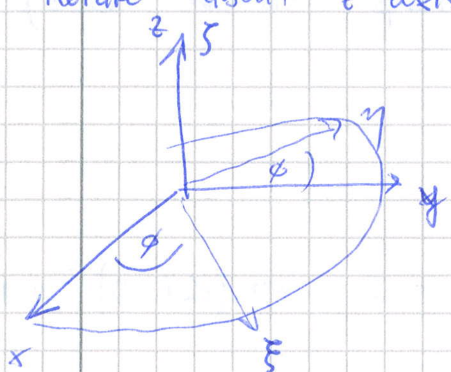
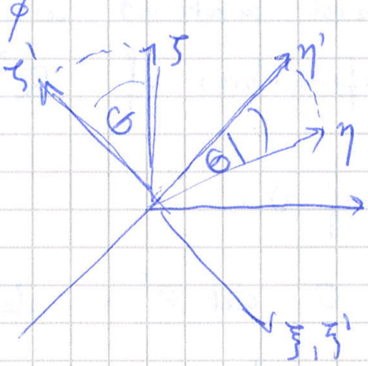
Euler angles.

(1) Rotate about $z$ axis by $\phi$




(2) rotate about $\xi$ by $\theta$

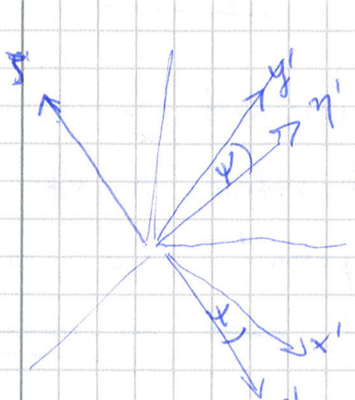



acts on $x,y,z$

$$D = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Don't have $x,y,z$, need to act on $\xi,\eta,\zeta$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$




$$B = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

acts on $\xi', \eta', \zeta'$

$$\Rightarrow A = BCD$$

$0 \leq \phi \leq 2\pi$

$0 \leq \theta \leq \pi$