Now consider the kinetic energy (free solid first)

$$T = \frac{1}{2}\sum_i \Delta m_i \dot{\vec{r}}_i^{\,2}$$

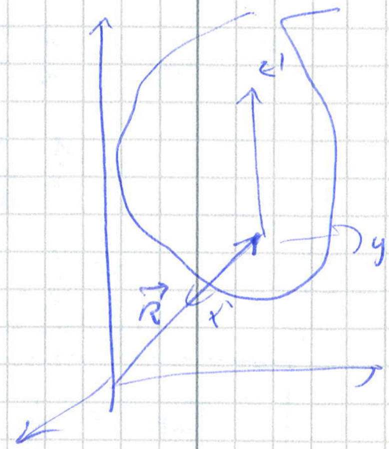
$$= \frac{1}{2}\sum_i \Delta m_i \left(\dot{\vec{R}} + \dot{\vec{r}}_i^{\,\prime}\right)^2$$

$$= \frac{1}{2}\sum_i \Delta m_i \left(\dot{\vec{R}} + \vec{\omega}\times\vec{r}_i^{\,\prime}\right)^2$$

$$= \frac{1}{2}\sum_i \Delta m_i \dot{\vec{R}}^2 + \sum_i \dot{\vec{R}}\cdot\Delta m_i\left(\vec{\omega}\times\vec{r}_i^{\,\prime}\right)$$

$$+ \frac{1}{2}\sum_i \Delta m_i \left(\vec{\omega}\times\vec{r}_i^{\,\prime}\right)\left(\vec{\omega}\times\vec{r}_i^{\,\prime}\right)$$

$$\equiv T_1 + T_2 + T_3$$




$\vec{R} = \vec{R}_{cm}$

$$T_1 = \frac{1}{2}\sum_i \Delta m_i \dot{\vec{R}}^2 = \frac{1}{2}M\dot{\vec{R}}^2 \qquad \text{CoM translation}$$

$$T_2 = \dot{\vec{R}}\cdot\left(\vec{\omega}\times\left(\sum_i \Delta m_i \vec{r}_i^{\,\prime}\right)\right) = \dot{\vec{R}}\cdot\left(\vec{\omega}\times M\vec{R}\right)$$

$= 0$ in CoM frame (cf. rot. symmetry discussion)

$$\Rightarrow T = T_3 = \frac{1}{2}\sum_i \Delta m_i \left(\vec{\omega}\times\vec{r}_i^{\,\prime}\right)\left(\vec{\omega}\times\vec{r}_i^{\,\prime}\right) \quad \text{rot. around CoM}$$

($\vec{R}$ fixed) $\quad T_1 = T_2 = 0 \quad$ ($\vec{R}$ would be on rotational axis)

$$T = T_{rot}$$

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$$\underbrace{\left(\vec{\omega}\times\vec{r}_i^{\,\prime}\right)}_{\vec{a}}\underbrace{\left(\vec{\omega}\times\vec{r}_i^{\,\prime}\right)}_{\vec{b}\times\vec{c}} = \cancel{-\vec{r}_i^{\,\prime}\left(\vec{\omega}\times\vec{r}_i^{\,\prime}\right)\vec{\omega} - \vec{r}_i^{\,\prime}\,\vec{\omega}^2}$$

$$= \vec{r}_i^{\,\prime}\left(\left(\vec{\omega}\times\vec{r}_i^{\,\prime}\right)\times\vec{\omega}\right) = -\vec{r}_i^{\,\prime}\left(\vec{\omega}\times\left(\vec{\omega}\times\vec{r}_i^{\,\prime}\right)\right)$$

$$= -\vec{r}_i^{\,\prime}\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{r}_i^{\,\prime}\right) - \omega^2\vec{r}_i^{\,\prime}\right)$$

$$= -\left(\vec{\omega}\cdot\vec{r}_i^{\,\prime}\right)^2 + \left(\omega^2\right)\left(\vec{r}_i^{\,\prime 2}\right)$$

$$\Rightarrow T_{rot} = \frac{1}{2}\sum_i \Delta m_i\left(\omega^2 r_i^{\prime 2} - \left(\vec{\omega}\cdot\vec{r}_i^{\,\prime}\right)^2\right) \qquad \omega^2 = \vec{\omega}^2 = \sum_a \omega_a^2$$

$$= \frac{1}{2}\sum_i \Delta m_i\left(\omega^2 r_i^{\prime 2} - \sum_{ab}\left(\omega_a r_{ia}^{\prime}\right)\left(\omega_b r_{ib}^{\prime}\right)\right)$$

$$\omega^2 = \sum_a \omega_a^2 = \sum_{ab} \omega_a^2 \delta_{ab} = \sum_{ab} \omega_a \omega_b$$

$$\Rightarrow T_{rot} = \frac{1}{2} \sum_i \Delta m_i \sum_{ab} \left( r_i'^2 \delta_{ab} \omega_a \omega_b - r'_{ia} \omega_a r'_{ib} \omega_b \right)$$

$$= \frac{1}{2} \sum_{ab} \omega_a \left( \sum_i \Delta m_i \left( r_i'^2 \delta_{ab} - r'_{ia} r'_{ib} \right) \right) \omega_b$$

$$\boxed{\equiv \frac{1}{2} \sum_{ab} \omega_a \hat{I}_{ab} \omega_b}$$

$$\boxed{T_{rot} = \frac{1}{2} \vec{\omega}^T \hat{I} \vec{\omega}}$$

use $x_i$ for consistency?

$$\hat{I} = \sum_i \Delta m_i \left( r_i'^2 \delta_{ab} - r'_{ia} r'_{ib} \right)$$

moment of inertia tensor

$I_{aa}$: mom. of inertia around axis $a$

$I_{ab}$: deviation moment

(NB: tensors are mappings on $\mathbb{R}^N \times \mathbb{R}^N \times \dots \to \mathbb{R}$

rank 2 can be represented as a matrix

specific transformation props under rotations:

Scalar is rank 0, vectors rank 1, ...

Make it continuous:

$$I = \cancel{\sum_i \rho \, \Delta V_i}$$

$$\int_{M_{solid}} dm \left( r'^2 \delta_{ab} - r'_a r'_b \right)$$

$$= \int_{V_{solid}} d^3r' \, \rho(\vec{r}') \left( r'^2 \delta_{ab} - r'_a r'_b \right)$$

$$dm = \rho \, dV = \sigma \, dA = \lambda \, dl$$

volume density

surface/ area density (thin plane, hollow cylinder)

linear mass density (thin rod, wire)

Do these first

$\hat{I}$ transforms ~~like a [illegible]~~ under rotations ~~analog~~ in a specific way, since it's made of position vectors!

$$\hat{I}'_{ab} = \sum_{ij} R_{ai} R_{bj} \hat{I}_{ij} = \left( R \hat{I} R^T \right)_{ab}$$

What happens to $T_{rot}$? $\vec{\omega} \to R\vec{\omega}$ so

$$T_{rot} = \frac{1}{2} \vec{\omega}'^T \hat{I}' \vec{\omega}' = \frac{1}{2} \vec{\omega}^T R^T R \hat{I} R^T R \vec{\omega} = \frac{1}{2} \vec{\omega}^T \hat{I} \vec{\omega}$$

~~More general: shift in coordinates / axes.~~

$$\cancel{I_{ab} = \int dm \; r^2 \delta_{ab} - r_a r_b}$$

$$\cancel{s_a = D_a + r_a \quad / \quad \vec{s} = \vec{D} + \vec{r}}$$

$$\cancel{= \int dm \left(s^2 + D^2 + 2\vec{s}\cdot\vec{D}\right)\delta_{ab} - s_a s_b - D_a D_b + s_a D_b + s_b D_a}$$

In general, we can always diagonalize $\hat{I}$. This defines a special coordinate system whose axes are the <u>principal axes</u> of the solid:

$$\hat{I} = \begin{pmatrix} A & & \\ & B & \\ & & C \end{pmatrix} \quad \text{etc.}$$

To find the <u>moment of inertia</u> w.r.t. a fixed axis, just use

$$T_{rot} = \frac{1}{2} I \omega^2 \cancel{= \frac{1}{2} \underbrace{\sum_{ab} \left(\hat{I}_{ab} n_a n_b\right)}_{=J} \omega^2}$$

$$\vec{n} = \frac{\vec{\omega}}{|\vec{\omega}|} \quad \Rightarrow \quad \cancel{\vec{\omega}^2 = \omega^2 \sum_{a}}$$

$$\vec{\omega} = \omega \cdot \vec{n} \quad \Rightarrow \quad T_{rot} = \frac{1}{2}\omega^2 \, \vec{n}^T \cdot \hat{I} \, \vec{n}$$

$$= \frac{1}{2}\omega^2 \underbrace{\left(\sum_{ab} n_a \hat{I}_{ab} n_b\right)}_{\equiv I}$$

moment of inertia around $\vec{\omega}$

→ rotational kinetic energy is invariant under rotations

→ scalar !

→ $\hat{I}$ is a rank 2 tensor! $\vec{\omega}$ is a vector ~~(axial)~~

• $\hat{I}$ depends on choice of tagged point.
(→ will lead to Steiner's theorem)

$$\hat{I}_{ab} = \int \lambda \left( r'^2 \delta_{ab} - r'_a r'_b \right) dl \qquad \lambda = \frac{M}{2\pi R}$$

$\vec{R}$ (happens to be CoM)

$x' = R \cos\phi'$

$y' = R \sin\phi'$

$z' = 0$

$$\Rightarrow I_{x'x'} = \frac{M}{2\pi R} \int_0^{2\pi} \left( R^2 - R^2 \cos^2\phi \right) R \, d\phi =$$

$$= \frac{M}{2\pi} R^2 \int_0^{2\pi} \sin^2\phi \, d\phi = M R^2 \cdot \underbrace{\frac{1}{2\pi} \int_0^{2\pi} \sin^2\phi \, d\phi}_{=\frac{1}{2}} = \frac{1}{2} M R^2$$

analogous for $I_{y'y'}$ ;

$$I_{z'z'} = \frac{M}{2\pi} R^2 \int_0^{2\pi} 1 \, d\phi' = M R^2$$

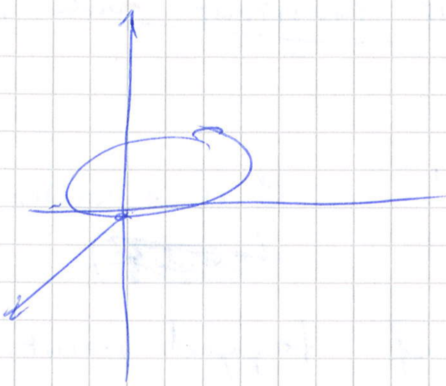
(→ cf. ~~analogous for~~ hollow cylinder)

$$I_{x'y'} = \frac{M}{2\pi R} R \int_0^{2\pi} \left( - R^2 \underbrace{\sin\phi \cos\phi}_{= \frac{1}{2} \sin 2\phi} \right) d\phi = 0$$

$I_{x'z'}$ , $I_{y'z'}$ trivially zero

$$\hat{I} = \begin{pmatrix} \frac{1}{2} M R^2 & & 0 \\ & \frac{1}{2} M R^2 & \\ 0 & & M R^2 \end{pmatrix}$$

→ diag because of symmetry & CoM system.



rotate around point on a rim.

$\hat{I}$: $x' = R\cos\phi'$

$y' = R\sin\phi' + R$

$z' = 0$

$\Rightarrow r'^2 = 2R^2(1+\sin\phi)$

$$\hat{I}_{xx} = \frac{m}{2\pi R} R \int_0^{2\pi} d\phi \left[2R^2(1+\sin\phi) - R^2\cos^2\phi\right]$$

$$= \frac{m}{2\pi}\left(\underbrace{\int_0^{2\pi} d\phi\, R^2(1-\cos^2\phi)}_{=\pi R^2}\right.$$

$R^2\cos^2\phi + R^2\sin^2\phi + 2R^2\sin\phi + R^2$

$$\left. + \int_0^{2\pi} d\phi\, R^2(1+\sin\phi)\right)$$

$$= m/2\pi\left(\pi R^2 + 2\pi R^2 + R^2\underbrace{\int_0^{2\pi} d\phi\,\sin\phi}_{=0}\right) = \frac{3}{2}mR^2$$

$$I_{yy} = \cancel{\tfrac{1}{2}MR^2}\ \frac{m}{2\pi R}\int_0^{2\pi} d\phi\, 2R^2(1+\cancel{\sin\phi}) - \left(R^2\sin^2\phi + \cancel{2R^2\sin\phi} + R^2\right)$$

$$= \frac{m}{2\pi R}\int_0^{2\pi} d\phi\, R^2(1-\sin^2\phi) = \frac{1}{2}MR^2 \quad \text{as before}$$

$I_{zz} = 2MR^2$

$I_{xy} = I_{yz} = I_{xz} = 0$

$$\rightarrow \hat{I} = \begin{pmatrix} 3/2\,MR^2 & & \\ & 1/2\,MR^2 & \\ & & 2MR^2 \end{pmatrix} = \hat{I}_0 + \begin{pmatrix} MR^2 & & \\ & 0 & \\ & & MR^2 \end{pmatrix}$$

will get back to that