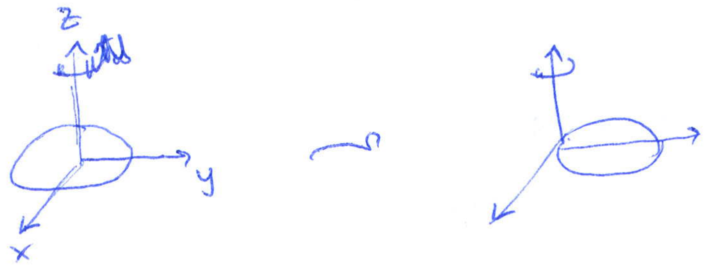
What happened here, physically?



Which moments should change?

I_{xx}, I_{zz}, but I_{yy} is physically the same!

$\Rightarrow$ parallel axis theorem / Steiner's theorem.

["MOI around a given axis $\vec{n}$ is equal to the parallel axis through the CoM plus the MoI of the body's w.r.t. ~~CoM's rotation around~~ $\vec{n}$, treated as a point mass" (sounds awkward)]

$$\hat{I}_{ab} = \left[\int dm \left(r^2 \delta_{ab} - x_a x_b\right)\right]$$

$$= \sum_i \Delta m_i \left(r_i^2 \delta_{ab} - x_{ia} x_{ib}\right) = \sum_i \Delta m_i \left(\left(\vec{r}_i' + \vec{R}\right)^2 \delta_{ab} - \left(x_{ia}' + X_a\right)\left(x_{ib}' + X_b\right)\right)$$

$$= \sum_i \Delta m_i \left(\vec{r}_i'^2 \delta_{ab} + \vec{R}^2 \delta_{ab} + 2\vec{r}_i' \cdot \vec{R}\, \delta_{ab} - x_{ia}' x_{ib}' - x_{ia}' X_b - X_a x_{ib}' - X_a X_b\right)$$

$$= \sum_i \Delta m_i \left(\vec{r}_i'^2 \delta_{ab} - x_{ia}' x_{ib}' + \vec{R}^2 \delta_{ab} - X_a X_b + 2\vec{r}_i' \cdot \vec{R}\, \delta_{ab} - x_{ia}' X_b - X_a x_{ib}'\right)$$

$$= \hat{I}_{ab}' + M\left(\vec{R}^2 \delta_{ab} - X_a X_b\right) + M\left(2\underbrace{\vec{R}'}_{=0} \cdot \vec{R}\, \delta_{ab} - M \underbrace{X_a'}_{=0} X_b - M X_a \underbrace{X_b'}_{=0}\right)$$

$$= \hat{I}_{ab}' + M\left(R^2 \delta_{ab} - X_a X_b\right)$$

Now that we have

$$L = T = \frac{1}{2} \vec{\omega}^T \hat{I} \vec{\omega}$$

let's look at EoM's.

From Newton (will do Lagrange in lecture notes – it's a bit messier b/c Euler angles)

lab / inertial frame.

$$\frac{d\vec{P}}{dt} = \vec{F} \qquad \vec{P}, \vec{L} \text{ total mom. / ang. mom.}$$

$$\frac{d\vec{L}}{dt} = \vec{N} \qquad \vec{F}, \vec{N} \text{ total force / torque}$$

$$\vec{L} = \sum_{i=1}^{A} \vec{r}_i \times \vec{p}_i = \sum_{i=1}^{A} \Delta m_i \, \vec{r}_i \times \dot{\vec{r}}_i$$

body-fixed (same origin)

$= 0$ ($\vec{r}_i$ rotates with body)

$$\dot{\vec{r}}_i = \left(\frac{d\vec{r}_i}{dt}\right)_{BF} + \vec{\omega} \times \vec{r}_i =$$

$$\Rightarrow \vec{L} = \sum_i \Delta m_i \left(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)\right) = \sum_i \Delta m_i \left(r_i^2 \vec{\omega} - (\vec{r}_i \cdot \vec{\omega}) \vec{r}_i\right)$$

$$\Rightarrow L_a = \sum_{i=1}^{A} \sum_{b=1}^{3} \Delta m_i \left(r_i^2 \delta_{ab} - x_{ib} x_{ia}\right) \omega_b = \sum_b \hat{I}_{ab} \omega_b$$

$$\Rightarrow \vec{L} = \hat{I} \vec{\omega}$$

- $\vec{L}$ is in general <u>not</u> parallel to $\vec{\omega}$!!
- It will be for rotation around a principal axis (or with appropriate symmetries, e.g. for a spherical rigid body.)

Now

$$\left(\frac{d\vec{L}}{dt}\right)_{LF} = \left(\frac{d\vec{L}}{dt}\right)_{BF} + \vec{\omega} \times \vec{L}$$

$$= \hat{I} \dot{\vec{\omega}} + \vec{\omega} \times \vec{L} = \hat{I} \dot{\vec{\omega}} + \vec{\omega} \times (\hat{I} \vec{\omega}) = \vec{N}$$

$\hat{I}$ does not depend on time in the BF fixed frame, but it <u>will</u> in lab frame (since rot. matrix to BF frame depends on time-dep. angles!)

~~Do this before angular motion / Euler eqn.s~~

Thus we have

$$\left(\overleftrightarrow{I}\,\dot{\vec{\omega}}\right)_{BF} + \vec{\omega}\times\vec{L} = \vec{N}$$

$$\Rightarrow \quad \vec{\omega} = \omega_{x'}\vec{e}_{x'} + \omega_{y'}\vec{e}_{y'} + \omega_{z'}\vec{e}_{z'}$$

$$\vec{L} = A\omega_{x'}\vec{e}_{x'} + B\omega_{y'}\vec{e}_{y'} + C\omega_{z'}\vec{e}_{z'}$$

(with $\vec{\omega}$ components as shown before/exercise)

$$\Rightarrow \quad N_{x'} = A\dot{\omega}_{x'} + (C-B)\,\omega_{y'}\omega_{z'}$$

$$N_{y'} = B\dot{\omega}_{y'} + (A-C)\,\omega_{x'}\omega_{z'}$$

$$N_{z'} = C\dot{\omega}_{z'} + (B-A)\,\omega_{y'}\omega_{x'}$$

Euler's eqn.s for rigid body (in the body-fixed frame / principal axis)

General procedure:

- first determine external forces that give torque $\vec{N}$ in lab frame
- transform to BF frame (ideally principal axis system) using rotational matrices with Euler angles.

Free Top

$\vec{N} = 0$ $\Rightarrow$ Note that this need not mean that there are no forces! E.g. if CoM fixed in lab frame

$$M\vec{R}\times\vec{g} = 0$$

$$\Rightarrow \quad A\dot{\omega}_{x'} + (C-B)\,\omega_{y'}\omega_{z'} = 0$$

$$B\dot{\omega}_{y'} + (A-C)\,\omega_{x'}\omega_{z'} = 0$$

$$C\dot{\omega}_{z'} + (B-A)\,\omega_{y'}\omega_{x'} = 0$$

Conserved quantities:

- multiply each equation by the ω_i and add:

$$A\dot{\omega}_{x'}\omega_{x'} + (C-B)\omega_{y'}\omega_{z'}\omega_{x'}$$
$$+ B\dot{\omega}_{y'}\omega_{y'} + (A-C)\omega_{x'}\omega_{z'}\omega_{y'}$$
$$+ C\dot{\omega}_{z'}\omega_{z} + (B-A)\omega_{x'}\omega_{y'}\omega_{z'} = 0$$
$$= A\dot{\omega}_{x'}\omega_{x'} + B\dot{\omega}_{y'}\omega_{y} + C\dot{\omega}_{z'}\omega_{z} = \frac{d}{dt}\frac{1}{2}\left(A\omega_{x'}^2 + B\omega_{y'}^2 + C\omega_{z'}^2\right)$$
$$= \frac{d}{dt}T_{rot} = 0$$

($\hat{=}$ total energy if we have no translation or forces)

Multiply by $A\omega_{x'}$, $B\omega_{y'}$, $C\omega_{z'}$:

$$A^2\dot{\omega}_{x'}\omega_{x'} + A(C-B)\omega_{y'}\omega_{z'}\omega_{x'}$$
$$+ B^2\dot{\omega}_{y'}\omega_{y'} + B(A-C)\omega_{x'}\omega_{z'}\omega_{y'}$$
$$+ C^2\dot{\omega}_{z'}\omega_{z} + C(B-A)\omega_{x'}\omega_{y'}\omega_{z'} = 0$$
$$= A^2\dot{\omega}_{x'}\omega_{x'} + B^2\dot{\omega}_{y'}\omega_{y} + C\dot{\omega}_{z'}\omega_{z} = 0$$
$$= \frac{d}{dt}\frac{1}{2}\left(A^2\omega_{x'}^2 + B^2\omega_{y'}^2 + C^2\omega_{z'}^2\right) = \frac{d}{dt}\left(\frac{1}{2}\vec{L}^2\right).$$

$$\vec{L} = \hat{I}\vec{\omega}$$
$$\vec{L}^2 = \left(\hat{I}\vec{\omega}\right)^2$$
$$\vec{\omega}^T \cdot \underbrace{\hat{I}^T \cdot \hat{I}}_{\hat{I}^2 = \begin{pmatrix} A^2 & & \\ & B^2 & \\ & & C^2 \end{pmatrix}} \vec{\omega}$$

$$\Rightarrow \frac{d|\vec{L}|^2}{dt} = 0$$

$\rightarrow$ absolute value of $\vec{L}$ is conserved.

To preserve $\vec{L}$ direction:

$$A\dot{\omega}_{x'} = B\dot{\omega}_{y'} = C\dot{\omega}_{z'} = 0$$

$$\mapsto \quad (C-B)\omega_{y'}\omega_{z'} = 0$$
$$(A-C)\omega_{x'}\omega_{z'} = 0$$
$$(B-A)\omega_{x'}\omega_{y'} = 0$$

- If $A=B=C$ that's trivially true
- If $A \neq B \neq C$: $\omega_{x'}=0, \omega_{y'}=0$

$\rightarrow$ $\vec{L}, \vec{\omega} \uparrow\uparrow$ principal axis

Stability:

Let's assume rotation around z' axis, plus perturbation

$$\omega_{z'} = \omega_0 + \varepsilon_{z'}$$
$$\omega_{x'} = \varepsilon_{x'}$$
$$\omega_{y'} = \varepsilon_{y'}$$

$$\Rightarrow \quad A\dot{\varepsilon}_{x'} + (C-B)\,\varepsilon_{y'}(\omega_0 + \cancel{\varepsilon_{z'}}) = 0$$
$$B\dot{\varepsilon}_{y'} + (A-C)\,\varepsilon_{x'}(\omega_0 + \cancel{\varepsilon_{z'}}) = 0$$
$$C\dot{\varepsilon}_{z'} + \underbrace{(B-A)\,\varepsilon_{y'}\varepsilon_{x'}}_{=O(\varepsilon^2)} = 0 \quad \Rightarrow \quad C\dot{\varepsilon}_{z'} = 0$$
$$\Rightarrow \varepsilon_z = \text{const.}$$

$$\Rightarrow \quad B\ddot{\varepsilon}_{y'} + (A-C)\,\dot{\varepsilon}_x(\omega_0 + \cancel{\varepsilon_{z'}}) = 0 \quad \Rightarrow \quad \dot{\varepsilon}_{x'} = -\frac{B}{(A-C)\omega_0}\ddot{\varepsilon}_{y'}$$

in first eqn.

$$-\frac{AB}{(A-C)\omega_0}\ddot{\varepsilon}_y + (C-B)\,\varepsilon_{y'}\omega_0 = 0$$

$$\ddot{\varepsilon}_y - \frac{(A-C)(C-B)}{AB}\omega_0^2\,\varepsilon_{y'} = 0$$

Analogously,

$$A\ddot{\varepsilon}_{x'} + (C-B)\,\dot{\varepsilon}_{y'}\omega_0 = 0 \quad \Rightarrow \quad \dot{\varepsilon}_{y'} = -\frac{A}{(C-B)\omega_0}\ddot{\varepsilon}_{x''}$$

in second eq.

$$\Rightarrow \quad -\frac{AB}{(C-B)\omega_0}\ddot{\varepsilon}_{x''} + (A-C)\,\omega_0\,\varepsilon_{x'} = 0$$

$$\ddot{\varepsilon}_{x''} - \frac{(A-C)(C-B)}{AB}\omega_0^2\,\varepsilon_{x''} = 0$$

Define

$$\Omega^2 = \frac{(A-C)(B-C)}{AB}:$$

$$\ddot{\varepsilon}_{x''} + \Omega^2\varepsilon_{x'} = 0$$
$$\ddot{\varepsilon}_{y''} + \Omega^2\varepsilon_{y'} = 0$$

stable solutions for $\Omega^2 > 0$, so $A > C$, $B > C$ or $A < C$ & $B < C$

i.e., C is smallest or largest principal MoI for $B < C < A$ or $A < C < B$ relation is unstable.

Intermediate Axis Theorem

Free symmetric top:

$A = B \neq C$

$$\Rightarrow \quad A\dot{\omega}_{x'} + (C-A)\omega_{y'}\omega_{z'} = 0 \qquad \text{i}$$

$$A\dot{\omega}_{y'} + (A-C)\omega_{x'}\omega_{z'} = 0 \qquad \text{ii}$$

$\omega_{0\parallel}$?

$$C\dot{\omega}_{z'} = 0 \quad \Rightarrow \quad \omega_{z'} = \text{const} \equiv \omega_{0z'}$$

$$\Rightarrow \quad A\dot{\omega}_{x'} + (C-A)\omega_{0z'}\omega_{y'} = 0 \qquad \text{i}$$

$$A\dot{\omega}_{y'} + (A-C)\omega_{0z'}\omega_{x'} = 0 \qquad \text{ii}$$

$$\Rightarrow \quad A\ddot{\omega}_{x'} + (C-A)\omega_{0z'}\dot{\omega}_y = 0 \quad \Rightarrow \quad \dot{\omega}_{y'} = \frac{-A}{(C-A)\omega_0}\ddot{\omega}_{x'}$$

in ii

$$\cancel{\frac{A^2\ddot{\omega}_{x'}}{(C-A)\omega_0} + (A-C)\omega_0\omega_x}$$

$$\frac{A^2}{(A-C)\omega_0}\ddot{\omega}_{x'} + (A-C)\omega_{0z'}\omega_{x'} = 0$$

$$\ddot{\omega}_{x'} + \frac{(A-C)^2}{A^2}\omega_{0z'}^2\omega_{x'} = 0$$

Analogously

$$\ddot{\omega}_{y'} + \underbrace{\frac{(C-A)^2}{A^2}\omega_{0z'}^2}_{\equiv \Omega^2}\omega_{y'} = 0$$

can skip – same as intermediate axis

$$\Rightarrow \quad \omega_{x'}(t) = \omega_{0\perp}\, \cancel{\cos}\ \sin(\Omega t + \beta)$$

$$\omega_{y'}(t) = \omega_{0\perp}\, \cancel{\sin}\ \cos(\Omega t + \beta')$$

since ω_x and ω_y are related. $\omega_{0\perp}$, β must be the same

$\Rightarrow$
- $\omega_{0z'}$ is const.
- $\omega = \sqrt{\omega_{0\perp}^2 + \omega_{0z'}^2} = \text{const}.$
- $\vec{\omega}_\perp(t) = (\omega_{x'}(t), \omega_{y'}(t))$ is a circle. in the BF xy plane