Liouville's Theorem

We have rewritten mechanics in terms of a flow in phase space; a time evolution generated by H.

Consider an infinitesimal volume

$$dV = dq_1 \cdots dq_n \, dp_1 \cdots dp_n$$

$$\Rightarrow \text{ for } \quad q_i \to q_i + \dot{q}_i \, dt = q_i + \frac{\partial H}{\partial p_i} dt = q_i'$$

$$p_i \to p_i + \dot{p}_i \, dt = p_i - \frac{\partial H}{\partial q_i} dt = p_i'$$

$$\Rightarrow \quad dV' = dq_1' \cdots dq_n' \, dp_1' \cdots dp_n' = \det M \cdot dV$$

Recall $\quad M \mathbb{J} M^T = \mathbb{J} \quad \Rightarrow \quad \det (M \mathbb{J} M^T) = \det \mathbb{J}$

$$\det M \cdot \det J \cdot \det M^T = \det J$$

$$(\det M)^2 \cdot \det J = \det J$$

$$\to (\det M)^2 = 1$$

This leaves the possibilities $\det M = \pm 1$, but we must have $M \to \mathbb{1}$ for infinitesimal canonical trafos, so $\det M = 1$. This means canonical transformations preserve the volume element (or general volumes) in phase space, and time evolution via H does so in particular! This is <u>Liouville's theorem</u>.

Can use it to evolve <u>distributions</u> $\rho(q, p, t)$:

- e.g. single system, but we don't know precise values of q, p:

$$\int \rho(q_i, p_i, t) \, dp_1 \cdots dp_n \, dq_1 \cdots dq_n = 1$$

- large number of particles (Stat. Mech.!)

$$\int \rho(q_i, p_i, t) \, dp_1 \cdots dp_n \, dq_1 \cdots dq_n = N$$

$\Rightarrow$ particles / probabilities must be conserved, so

$$\frac{d\rho}{dt} = \{\rho, H\} + \frac{\partial \rho}{\partial t} = 0 \qquad \underline{\text{Liouville's equation}}$$