H. Hergert
Facility for Rare Isotope Beams
and Department of Physics & Astronomy

MICHIGAN STATE UNIVERSITY

PHY422/820: Classical Mechanics

FS 2020

Worksheet #1 (Sep 2 – 4)

September 9, 2020

1 Reading Assignments & Other Preparation

- Lemos, Section 1.1
- Goldstein, Sections 1.1–1.2
- Give the Jupyter tutorial notebook a try if you want to brush up on your Python skills: It can be found on the course website website either under *Projects* or *Computation\Python*

2 Additional Notes on Newtonian Mechanics

2.1 Computing Work Along a Path and Potentials

Generally, we can compute the potential by the same curve integration as the work. For a conservative force that can be written as $\vec{F}(\vec{r}) = -\vec{\nabla}V(\vec{r})$, we can choose a suitable curve γ parameterized by s that connects the reference point $\vec{r}(s_i)$ and a general point $\vec{r}(s_f)$, and integrate:

$$\begin{aligned}
W &= \int_\gamma d\vec{r}\cdot\vec{F}(\vec{r}) = \int_{s_i}^{s_f} ds\,\frac{\partial\vec{r}(s)}{\partial s}\cdot\vec{F}(\vec{r}(s)) \\
&= -\int_{s_i}^{s_f} ds\,\frac{\partial\vec{r}(s)}{\partial s}\cdot\vec{\nabla}V(\vec{r}(s)) \\
&= -\int_{s_1}^{s_f} ds\,\left(\frac{\partial x}{\partial s}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial}{\partial y} + \frac{\partial z}{\partial s}\frac{\partial}{\partial z}\right)V(\vec{r}(s)) \\
&= -\int_{s_1}^{s_f} dV(\vec{r}(s)) = -\left(V(\vec{r}(s_f)) - V(\vec{r}(s_i))\right) = -W_{\text{ext}}\,. \qquad (1)
\end{aligned}$$

Going from the second to the third line, we have expanded the scalar product of $\frac{\partial\vec{r}}{\partial s}$ and $\vec{\nabla}V$, and from the third to the fourth lines, we have used that $ds\frac{\partial\vec{r}}{\partial s}\cdot\vec{\nabla}V(\vec{r})$ is the *total differential* of $V(\vec{r})$. If you are unsure about this, consider first the chain rule

$$\frac{d}{dx}f(g(x)) = \frac{\partial f}{\partial g}\frac{dg}{dx} \quad\Rightarrow\quad df = \frac{\partial f}{\partial g}dg\,. \qquad (2)$$

If we have multiple intermediate functions $g_i(x)$ that depend on the same variable x, this generalizes to

$$\frac{d}{dx}f(g_1(x),\ldots,g_n(x)) = \sum_{i=1}^{n}\frac{\partial f}{\partial g_i}\frac{dg_i}{dx} \quad\Rightarrow\quad df = \sum_{i=1}^{n}\frac{\partial f}{\partial g_i}dg_i\,. \qquad (3)$$

Note that W is the *work done by the force*, which is the negative potential energy difference (since the work is taking out of the energy stored in the potential). It is opposite in sign to the external work that we have to do *against the force* to move an object.

2.2 Work Along Closed Paths and Stokes' Theorem

You are likely familiar with Stokes' theorem in the following form:

$$\oint_{\partial A} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_A \vec{\nabla} \times \vec{F}(\vec{r}) \cdot d\vec{A} \,. \tag{4}$$

It relates the integral over a vector field $\vec{F}(\vec{r})$ along a closed loop along the boundary of an area A, indicated by ∂A, to the integral of the curl of $\vec{F}(\vec{r})$ over the interior of the area. The area itself is meant to be *orientable*, so that one can define a vectorial area element $d\vec{A} = \vec{n} dA$, where $\vec{n}$ is a unit vector that is *normal* to the area element dA. Note that $\vec{n}$ can depend on $\vec{r}$ in general, e.g., for curved surfaces: An example would be an integral over the surface of a sphere, where $\vec{n} = \pm \vec{e}_r$.

From Eq. (4), it is readily apparent that if everywhere in the area A

$$\vec{\nabla} \times \vec{F}(\vec{r}) = 0 \,, \tag{5}$$

we will automatically have

$$\oint_{\partial A} \vec{F}(\vec{r}) \cdot d\vec{r} = 0 \,. \tag{6}$$

Potential issues can arise whenever $\vec{F}(\vec{r})$ is not smooth, i.e., when the derivatives may be ill defined. A concrete example is discussed in homework # 1.

2.3 General Solution of the Equation of Motion for a Single Degree of Freedom

Consider the motion of a mass m in one spatial dimension in the presence of a conservative force field $F(x)$. Per Newton's Second Law, the equation of motion is

$$m\ddot{x} = F(x) = -\frac{dV}{dx} \,, \tag{7}$$

with the potential $V(x)$ that corresponds to the force. To solve Eq. (7), we multiply by $\dot{x}$, obtaining

$$m\dot{x}\ddot{x} = -\dot{x}\frac{dV}{dx} \,. \tag{8}$$

The left and right-hand sides of this equation are the time derivatives of the kinetic energy,

$$\frac{d}{dt} T(\dot{x}(t)) = \frac{1}{2}\frac{d}{dt}\left(m\dot{x}^2\right) = m\dot{x} \cdot \ddot{x} \tag{9}$$

and the potential energy,

$$-\frac{d}{dt} V(x(t)) = -\frac{dV}{dx}\dot{x} \,, \tag{10}$$

respectively, where we have used the chain rule. Bringing both terms to the same side, we immediately obtain the conservation theorem for the energy $E = T + V$,

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + V(x)\right) = 0 \;\Rightarrow\; \frac{1}{2}m\dot{x}^2 + V(x) = E = \text{const} \,. \tag{11}$$

We can now use energy conservation to determine the velocity of the mass, since

$$\frac{dx}{dt} = \dot{x} = \pm\sqrt{\frac{2}{m}(E - V(x))}\ . \tag{12}$$

The sign can be fixed by convention, e.g., by agreeing that a positive (negative) sign corresponds to motion to the right (left). Equation (12) is a first order differential equation that can be integrated to yield $x(t)$. Let us separate the variables and bring dt and dx to separate sides of the equation:

$$dt = \pm\frac{dx}{\sqrt{\frac{2}{m}(E - V(x))}}\,. \tag{13}$$

Integrating both sides, we have

$$t - t_0 = \pm\int_{x_0}^{x} dx' \frac{1}{\sqrt{\frac{2}{m}(E - V(x'))}}\,, \tag{14}$$

and by evaluating the integral, we obtain $x(t)$ as a function of the parameters t_0 and E. The starting location x_0 is *not* an independent parameter here, since it must be consistent with $x(t_0) = x_0$. Equation (12) requires that $E \geq V(x)$ for physically relevant solutions; trajectories with $E < V(x)$ are *forbidden*[1]. For $V(x) = E$, we have $\dot{x} = 0$, which corresponds to a *turning point* of the trajectory: The velocity changes sign as the particle changes its direction.

2.4 Kinematics in Curvilinear Coordinates

The description of mechanical processes can often be simplified by an appropriate choice of coordinate systen, which may then turn out to rely on curvilinear coordinates. A prominent example is the motion on a circle with fixed radius in a plane, which is more readily parameterized in terms of a radius r and an angle ϕ instead of Cartesian coordinates x and y, especially since the radius will be constrained to be $r = R =$ const. — this latter point will be one of the primary motivators for switching from a Newtonian treatment of dynamics to the Lagrangian formulation.

Here, we want to briefly discuss how we would set up the tools for describing motion in curvilinear coordinates, from the basis vectors to derivatives and integration measures. We will use cylindrical coordinates in three dimensions as an example, and then summarize the procedure for general coordinates.

2.4.1 Cylindrical Coordinates

Cylindrical coordinates ρ, ϕ, z in $\mathbb{R}^3$ can be introduced implicitly by expressing the Cartesian coordinates x, y, z as

$$x = \rho\cos\phi\,, \tag{15}$$

$$y = \rho\sin\phi\,, \tag{16}$$

$$z = z\,, \tag{17}$$

or conversely,

$$\rho = \sqrt{x^2 + y^2}\,, \tag{18}$$

[1]Note: This requirement no longer applies in Quantum mechanics!

$$\phi = \arctan \frac{y}{x}\,, \tag{19}$$

$$z = z\,, \tag{20}$$

Thus, a general position vector can be written as

$$\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z = \rho\cos\phi\vec{e}_x + \rho\sin\phi\vec{e}_y + z\vec{e}_z\,. \tag{21}$$

Basis Vectors

We can achieve a more compact representation of $\vec{r}$ by introducing basis vectors associated with the cylindrical coordinates. The key observation is that the basis vectors are *tangential* to the lines of the coordinate mesh, which suggests a relationship between the basis vectors and derivatives of $\vec{r}$ with respect to ρ, ϕ, z. Indeed, we can proceed to define

$$\vec{e}_\rho = \frac{1}{h_\rho}\frac{\partial\vec{r}}{\partial\rho}\,,\quad h_\rho = \left|\frac{\partial\vec{r}}{\partial\rho}\right| = 1\,, \tag{22}$$

$$\vec{e}_\phi = \frac{1}{h_\phi}\frac{\partial\vec{r}}{\partial\rho}\,,\quad h_\phi = \left|\frac{\partial\vec{r}}{\partial\phi}\right| = \rho\,, \tag{23}$$

$$\vec{e}_z = \frac{1}{h_z}\frac{\partial\vec{r}}{\partial z}\,,\quad h_z = \left|\frac{\partial\vec{r}}{\partial\phi}\right| = 1\,, \tag{24}$$

where $h_{\rho,\phi,z}$ are simply norms of the coordinate derivatives of $\vec{r}$. In terms of the original Cartesian unit vectors, we have

$$\vec{e}_\rho = \cos\phi\vec{e}_x + \sin\phi\vec{e}_y\,, \tag{25}$$

$$\vec{e}_\phi = -\sin\phi\vec{e}_x + \cos\phi\vec{e}_y\,, \tag{26}$$

$$\vec{e}_z = \vec{e}_z\,, \tag{27}$$

so the orientation of the basis vectors changes with the coordinate ϕ, unlike $\vec{e}_{x,y,z}$, which remain fixed at all times. Focusing on a coordinate circle around the z axis, $\vec{e}_\rho$ points radially away from the circle, and $\vec{e}_\phi$ is tangential to it (as expected by its definition).

We can readily verify that the basis vectors are orthonormal, i.e.,

$$\vec{e}_\rho\cdot\vec{e}_\phi = \vec{e}_\rho\cdot\vec{e}_z = \vec{e}_\phi\cdot\vec{e}_z = 0\,, \tag{28}$$

and

$$\vec{e}_\rho\cdot\vec{e}_\rho = \vec{e}_\phi\cdot\vec{e}_\phi = \vec{e}_z\cdot\vec{e}_z = 1\,. \tag{29}$$

We also note that the basis vectors form a right-handed coordinate system if we arrange them in the order $\{\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z\}$:

$$\vec{e}_\rho\times\vec{e}_\phi = \vec{e}_z,\quad \vec{e}_\phi\times\vec{e}_z = \vec{e}_\rho,\quad \vec{e}_z\times\vec{e}_\rho = \vec{e}_\phi\,. \tag{30}$$

Inverting the system of equations (25)–(27), we can express an arbitray coordinate vector as

$$\vec{r} = \rho\vec{e}_\rho + z\vec{e}_z\,. \tag{31}$$

When we are computing time (or other) derivatives of $\vec{r}$, it is now important to keep in mind that the basis vectors are coordinate dependent as well, and we need to apply the product and chain rules as needed (see homework #1).

Exercise 2.1: Basis Vectors in Spherical Coordinates

Show that the basis vectors for a spherical coordinate system r, θ, ϕ, defined through

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta \,, \tag{32}$$

are given by

$$\begin{aligned} \vec{e}_r &= \sin\theta \cos\phi \vec{e}_x + \sin\theta \sin\phi \vec{e}_y + \cos\theta \vec{e}_z \,, & (33)\\ \vec{e}_\theta &= \cos\theta \cos\phi \vec{e}_x + \cos\theta \sin\phi \vec{e}_y - \sin\theta \vec{e}_z \,, & (34)\\ \vec{e}_\phi &= -\sin\phi \vec{e}_x + \cos\phi \vec{e}_y \,. & (35) \end{aligned}$$

Prove that the vectors are orthonormal, and that $\{\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi\}$ is a right-handed coordinate system.

Derivatives and Differentials

We will frequently need the gradient operator, which is given in Cartesian coordinates by

$$\vec{\nabla} \equiv \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} \,. \tag{36}$$

Using the chain rule, we have

$$\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \,, \tag{37}$$

and

$$\begin{aligned} \frac{\partial \rho}{\partial x} &= \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} = \frac{\rho\cos\phi}{\rho} = \cos\phi \,, & (38)\\ \frac{\partial \phi}{\partial x} &= \frac{\partial}{\partial x}\arctan\frac{y}{x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{-y}{x^2} = -\frac{1}{1+\tan^2\phi}\frac{\sin\phi}{\rho\cos^2\phi} = -\cos^2\phi\frac{\sin\phi}{\rho\cos^2\phi} = -\frac{1}{\rho}\sin\phi \,. & (39) \end{aligned}$$

Analogously, we find

$$\begin{aligned} \frac{\partial \rho}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \frac{\rho\sin\phi}{\rho} = \sin\phi \,, & (40)\\ \frac{\partial \phi}{\partial y} &= \frac{\partial}{\partial y}\arctan\frac{y}{x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \cos^2\phi\frac{1}{\rho\cos\phi} = \frac{1}{\rho}\cos\phi \,. & (41) \end{aligned}$$

Using these partial derivatives and the definition of the cylindrical-basis vectors, we obtain

$$\begin{aligned} \vec{\nabla} &= \vec{e}_x\left(\cos\phi\frac{\partial}{\partial\rho} - \frac{1}{\rho}\sin\phi\frac{\partial}{\partial\phi}\right) + \vec{e}_y\left(\sin\phi\frac{\partial}{\partial\rho} + \frac{1}{\rho}\cos\phi\frac{\partial}{\partial\phi}\right) + \vec{e}_z\frac{\partial}{\partial z}\\ &= \vec{e}_\rho\frac{\partial}{\partial\rho} + \vec{e}_\phi\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}\right) + \vec{e}_z\frac{\partial}{\partial z} \,. \end{aligned} \tag{42}$$

Note that this can also be written as

$$\vec{\nabla} = \vec{e}_\rho\left(\frac{1}{h_\rho}\frac{\partial}{\partial\rho}\right) + \vec{e}_\phi\left(\frac{1}{h_\phi}\frac{\partial}{\partial\phi}\right) + \vec{e}_z\left(\frac{1}{h_z}\frac{\partial}{\partial z}\right) \,. \tag{43}$$

Using this representation of $\vec{\nabla}$ in the cylindrical coordinates and basis vectors, we can readily evaluate other vectorial derivatives like the divergence and the curl.

Exercise 2.2: Divergence and Curl in Cylindrical Coordinates

Show that the divergence and curl of a vector field $\vec{A}(\rho, \phi, z)$ in cylindrical coordinates are given by

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}\,, \tag{44}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho}(\partial_\phi A_z - \rho \partial_z A_\phi)\vec{e}_\rho + (\partial_z A_\rho - \partial_\rho A_z)\vec{e}_\phi + \frac{1}{\rho}(\partial_\rho(\rho A_\phi) - \partial_\phi A_\rho)\vec{e}_z\,. \tag{45}$$

Exercise 2.3: Vector Derivatives in Spherical Coordinates

Show that the gradient operator and the divergence and curl of a vector field $\vec{A}(r, \theta, \phi)$ in spherical coordinates are given by

$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) + \vec{e}_\phi \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\,, \tag{46}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial \rho}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}\,, \tag{47}$$

$$\begin{aligned}\vec{\nabla} \times \vec{A} = {} & \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right)\vec{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\vec{e}_\theta \\ & + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\vec{e}_\phi\,. \end{aligned} \tag{48}$$

3 Group Exercises

Problem G1 – Conservative Forces

Let $\vec{F} = C\left(k_1 xy,\, x^2 + k_2 z^2,\, yz\right)^T$ be a force field, with $C > 0$ having the units of a force, and k_1, k_2 being initially unspecified parameters.

1. For which values of the constants k_1 and k_2 does the force have a potential?

Use these values k_1 and k_2 that result in a conservative force $\vec{F}$ in the following.

2. Compute the work required to move a point mass m from the origin of the coordinate system to $\vec{a} = (a_x, a_y, a_z)^T$. Under which conditions does this transport require and generate energy, respectively?

3. Determine the potential and check that it indeed yields the correct expression for the force field.

Problem G2 – Motion in a One-Dimensional Potential

A point mass m is moving in the one-dimensional harmonic oscillator potential

$$V(x) = \frac{1}{2} m\omega^2 x^2. \tag{49}$$

1. What are the physically allowed energies for this potential? For which values of E is the trajectory of the mass bounded?

2. Determine the turning points $x_\pm$ of the bounded trajectories.

3. Integrate Eq. (14) to determine the trajectory $x(t)$ with initial conditions, $x(0) = x_+$ and $\dot{x}(0) = 0$, i.e., for a mass that is released from rest at the positive turning point.
   HINT:
   $$\int dx \,\frac{1}{\sqrt{a^2 - x^2}} = \arctan \frac{x}{\sqrt{a^2 - x^2}} + c\,, \quad \text{for} \quad a > 0\,. \tag{50}$$

4. Solve the equation of motion for the mass m with the same initial conditions as in the previous part, and show that the two solutions are consistent.

Problem G3 – Properties of the Two-Body Problem

Two particles with masses m_1 and m_2 interact with each other and are also subject to external forces. We denote the interparticle force $\vec{F}_{12} = -\vec{F}_{21}$, and the external forces $\vec{F}^{(e)}_{1,2}$. Assume that we describe the dynamics of the particles from an inertial frame.

1. Use the equations of motion for the individual masses to show that the equation of motion for the center of mass is

$$\frac{d\vec{P}}{dt} = M\frac{d^2\vec{R}}{dt^2} = \vec{F}_1^{(e)} + \vec{F}_2^{(e)}\,, \tag{51}$$

where $M = m_1 + m_2$.

2. Analogously, show that the equation of motion for the relative degree of freedom is

$$\frac{d\vec{p}}{dt} = \mu\frac{d^2\vec{r}}{dt^2} = \vec{F}_{12} + \mu\left(\frac{\vec{F}_1^{(e)}}{m_1} - \frac{\vec{F}_2^{(e)}}{m_2}\right)\,, \tag{52}$$

where

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2\,, \quad \mu \equiv \frac{m_1 m_2}{m_1 + m_2}\,. \tag{53}$$

3. Show that the total kinetic energy and total angular momentum of the two-particle system can be expressed as

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 \tag{54}$$

and

$$\vec{L} = M\vec{R}\times\dot{\vec{R}} + \mu\vec{r}\times\dot{\vec{r}}\,. \tag{55}$$

4. Assume that the interparticle force is central, i.e., $\vec{F}_{12} = F_{12}(r)\vec{e}_r$ with $r = |\vec{r}|$. Show that

$$\frac{d\vec{L}}{dt} = \vec{N}^{(e)}\,, \tag{56}$$

where $\vec{L}$ is the total angular momentum and $\vec{N}^{(e)}$ is the torque exerted on the particles by the external forces:

$$\vec{N}^{(e)} \equiv \vec{r}_1\times\vec{F}_1^{(e)} + \vec{r}_2\times\vec{F}_2^{(e)}\,. \tag{57}$$