H. Hergert
Facility for Rare Isotope Beams
and Department of Physics & Astronomy

MICHIGAN STATE
U N I V E R S I T Y

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Worksheet #7 (Oct 12 – Oct 16)

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1 Preparation

- Lemos, Section 1.7

- Goldstein, Sections 3.1–3.3, 3.5–3.7

2 Central-Force Problems

This week, we will start our discussion of **central-force problems**. An obvious example is the motion of a mass or particle in a spherically symmetric potential, but the importance of the following discussion extends far beyond this case. We have seen that the fundamental forces of nature like gravity or electromagnetism are pairwise interactions between objects that only depend on their relative distance, due to the fundamental symmetries of spacetime. For this reason, we will first demonstrate how such pairwise interactions can be reduced to equivalent one-body central-force problems in relative coordinates before discussing general strategies for analyzing such problems and the properties of the solutions.

2.1 Reducing Two-Body Problems to Equivalent One-Body Problems

Let us consider a system of two masses m_1 and m_2 whose positions are given by $\vec{r}_1$ and $\vec{r}_2$, respectively. Their center of mass is given by

$$\vec{R} = \frac{1}{M} \left(m_1 \vec{r}_1 + m_2 \vec{r}_2 \right), \tag{1}$$

where the total mass is $M = m_1 + m_2$. In the center-of-mass system, the coordinates of the masses are given by

$$\vec{r}_i' = \vec{r}_i - \vec{R}, \tag{2}$$

and specifically

$$\vec{r}_1' = \vec{r}_1 - \left(\frac{m_1}{M} \vec{r}_1 + \frac{m_2}{M} \vec{r}_2 \right) = \frac{m_2}{M} \left(\vec{r}_1 - \vec{r}_2 \right), \tag{3}$$

$$\vec{r}_2' = \vec{r}_2 - \left(\frac{m_1}{M} \vec{r}_1 + \frac{m_2}{M} \vec{r}_2 \right) = \frac{m_1}{M} \left(\vec{r}_2 - \vec{r}_1 \right). \tag{4}$$

Introducing the relative distance vector

$$\vec{r} \equiv \vec{r}_2 - \vec{r}_1 , \tag{5}$$

we see that

$$\vec{r}_2' - \vec{r}_1' = \frac{m_1}{M}\vec{r} - \frac{m_2}{M}(-\vec{r}) = \frac{m_1 + m_2}{M}\vec{r} = \vec{r} , \tag{6}$$

i.e., the relative distance of the two masses does not depend on our choice of coordinate system, which makes sense. We can use $\vec{r}$ to express the coordinates in the center-of-mass system as

$$\vec{r}_1' = -\frac{m_2}{M}\vec{r} , \quad \vec{r}_2' = \frac{m_1}{M}\vec{r} . \tag{7}$$

Now we can express the kinetic energy of the two masses in center-of-mass and relative coordinates. First, we have

$$
\begin{aligned}
T &= \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 \\
&= \frac{1}{2}m_1\left(\dot{\vec{r}}_1' + \dot{\vec{R}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{r}}_2' + \dot{\vec{R}}\right)^2 \\
&= \frac{1}{2}m_1\left(\dot{\vec{r}}_1'^2 + 2\dot{\vec{r}}_1' \cdot \dot{\vec{R}} + \dot{\vec{R}}^2\right) + \frac{1}{2}m_2\left(\dot{\vec{r}}_2'^2 + 2\dot{\vec{r}}_2' \cdot \dot{\vec{R}} + \dot{\vec{R}}^2\right) \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \left(m_1\dot{\vec{r}}_1' + m_2\dot{\vec{r}}_2'\right) \cdot \dot{\vec{R}} + \frac{1}{2}m_1\dot{\vec{r}}_1'^2 + \frac{1}{2}m_2\dot{\vec{r}}_2'^2 \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \left(\frac{m_1 m_2}{M}\dot{\vec{r}} - \frac{m_2 m_1}{M}\dot{\vec{r}}\right) \cdot \dot{\vec{R}} + \frac{1}{2}m_1\dot{\vec{r}}_1'^2 + \frac{1}{2}m_2\dot{\vec{r}}_2'^2 \\
&= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}m_1\dot{\vec{r}}_1'^2 + \frac{1}{2}m_2\dot{\vec{r}}_2'^2 ,
\end{aligned} \tag{8}
$$

i.e., the kinetic energy is the sum of the center-of-mass kinetic energy and the kinetic energy of the masses as expressed in the center-of-mass system — the so-called **intrinsic kinetic energy**. This result extendes to general N-particle systems. For the two-body system, the intrinsic term can be rewritten using Eq. (7) as

$$T_{\text{intr}} = \frac{1}{2}m_1\frac{m_2^2}{M^2}\dot{\vec{r}}^2 + \frac{1}{2}m_2\frac{m_1^2}{M^2}\dot{\vec{r}}^2 = \frac{1}{2}(m_1 + m_2)\frac{m_1 m_2}{M^2}\dot{\vec{r}}^2 \equiv \frac{1}{2}\mu\dot{\vec{r}}^2 , \tag{9}$$

with the **reduced mass**

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M} . \tag{10}$$

With a potential that only depends on the relative distance of the two masses, the Lagrangian now can be written as

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(\vec{r}) = L_{\text{com}} + L_{\text{intr}} \tag{11}$$

Since L only depends on $\dot{\vec{R}}$ but not on $\vec{R}$, the center-of-mass motion decouples from the intrinsic motion, and the Lagrange equations imply

$$\frac{\partial L}{\partial \vec{R}} = 0 = \frac{d}{d}\frac{\partial L}{\partial \dot{\vec{R}}} \quad \Rightarrow \quad M\dot{\vec{R}} = \text{const.}. \tag{12}$$

Thus, the center-of-mass motion is uniform and linear. The intrinsic Lagrangian is equivalent to a one-body problem in the relative coordinate $\vec{r}$.

2.2 Trajectories in the Central-Force Problem

2.2.1 General Solution of the Equations of Motion

As we have seen in the previous section, the motion of a particle in a central force field can be modeled by the Lagrangian

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r}), \tag{13}$$

where $\vec{r}$ can either indicate the coordinates of a single object of mass m in the external potential $V(\vec{r})$, or the relative coordinate of a two-body system with reduced mass $m = \mu$.

In general, a central force does not have to be spherically symmetric or conservative, but a conservative force will always be spherically symmetric. To see this, we first assume that the central force is conservative, i.e., that it can be written as

$$\vec{F}(\vec{r}) = -\vec{\nabla}V(\vec{r}) = -\left(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)V(\vec{r}), \tag{14}$$

where we have used the gradient in spherical coordinates. Since $\vec{F}(\vec{r})$ must be parallel to $\vec{e}_r$, the partial derivatives of $V(\vec{r})$ with respect to the angles θ and ϕ must vanish, implying that $V(\vec{r}) = V(|\vec{r}|) = V(r)$. Conversely, if the force is spherically symmetric, $\vec{F}(\vec{r}) = f(r)\vec{e}_r$, then

$$\begin{aligned}
\vec{\nabla} \times \vec{F}(r) &= \left(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \times \frac{f(r)}{r}\vec{r} \\
&= \vec{e}_r \times \left(\frac{f'(r)r - f(r)}{r}\vec{e}_r + \frac{f(r)}{r}\vec{e}_r\right) + \vec{e}_\theta \times \frac{f(r)}{r}\vec{e}_\theta + \vec{e}_\phi \times \frac{f(r)}{r^2\sin\theta}r\sin\theta\vec{e}_\phi \\
&= 0,
\end{aligned} \tag{15}$$

where we have used that

$$\frac{\partial\vec{r}}{\partial r} = \vec{e}_r, \quad \frac{\partial\vec{r}}{\partial\theta} = r\vec{e}_\theta, \quad \frac{\partial\vec{r}}{\partial\phi} = r\sin\theta\vec{e}_\phi. \tag{16}$$

Combining the proofs in both directions, we find that a central force is conservative if and only if it is spherically symmetric.

Assuming a conservative, spherically symmetric central force going forward, we can write the Lagrangian in spherical coordinates as

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - V(r). \tag{17}$$

The Lagrangian does not depend on ϕ, hence

$$\frac{\partial L}{\partial\phi} = 0 = \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} \quad \Rightarrow \quad \frac{\partial L}{\partial\dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi} \equiv l_z = \text{const.}, \tag{18}$$

where l_z denotes the angular momentum around the z axis of our coordinate system. However, since the Lagrangian is spherically symmetric, we can orient our coordinate system at will, so *all* components of the angular momentum vector must be conserved. Since

$$\vec{l} = \vec{r} \times \vec{p} = \text{const.}, \tag{19}$$

the motion of the mass(es) will always be confined to a plane that is perpendicular to $\vec{l}$. Choosing our coordinate system such that

$$\vec{l} = l\vec{e}_z, \tag{20}$$

we have $\theta = \frac{\pi}{2}$ in spherical coordinates, and our Lagrangian (17) simplifies to

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2 \underbrace{\dot{\theta}^2}_{=0} + r^2 \underbrace{\sin^2\theta}_{=1} \dot{\phi}^2\right) - V(r) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r). \tag{21}$$

The Lagrange equations now read

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \quad \Rightarrow \quad m\ddot{r} - mr\dot{\phi}^2 + V'(r) = 0,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad mr^2\dot{\phi} = l = \text{const.} \tag{22}$$

Expressing $\dot{\phi}$ by the conserved angular momentum l, and introducing $f(r) = -V'(r)$, we can write the radial equation of motion as

$$m\ddot{r} - \frac{l^2}{mr^3} - f(r) = 0. \tag{23}$$

Since the force field we consider here is conservative, the total energy of the moving mass must be conserved. Alternatively, we can note that $\frac{\partial L}{\partial t} = 0$ since L does not explicitly depend on time, which implies conservation of the total energy (cf. worksheet #4). Starting from the energy conservation law

$$E = T + V$$
$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) = \text{const.}, \tag{24}$$

we can solve for $\dot{r}$,

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\left(E - \frac{l^2}{2mr^2} - V(r)\right)}, \tag{25}$$

where the sign depends on the initial or boundary conditions, and must be chosen so that $r \geq 0$ at all times. Separating the variables, we obtain

$$dt = \pm\frac{dr}{\sqrt{\frac{2}{m}\left(E - \frac{l^2}{2mr^2} - V(r)\right)}} \tag{26}$$

which can be integrated to give

$$t - t_0 = \pm\int_{r_0}^{r(t)} \frac{dr'}{\sqrt{\frac{2}{m}\left(E - \frac{l^2}{2mr'^2} - V(r')\right)}}. \tag{27}$$

Analogously, we can separate the variables in the angular momentum conservation law (22) and integrate to obtain

$$\phi - \phi_0 = \int_{t_0}^{t} \frac{l}{mr(t')^2} dt'. \tag{28}$$

Together, Eqs. (27) and (28) constitute the general solution of any central force problem. As we might guess, these equations can be solved analytically only under very special circumstances.

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However, we can continue the discussion of the general solutions qualitatively if we introduce the **effective potential**

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + V(r) \tag{29}$$

and write Eq. (27) as

$$t - t_0 = \pm \int_{r_0}^{r(t)} \frac{dr'}{\sqrt{\frac{2}{m}\left(E - V_{\text{eff}}(r')\right)}} \, . \tag{30}$$

In this form, the integral equation is equivalent to the general solution for one-dimensional motion of an object in the effective potential V_{eff} - here, this is one-dimensional **radial** motion. The object is performing **angular** motion around the center of the potential due to the angular-momentum dependent term in V_{eff}.

Figure (1) shows the effective potential for the Kepler problem,

$$V(r) = -\frac{\kappa}{r}, \quad \kappa > 0, \tag{31}$$

as well as the main types of radial trajectories as a function of the energy E. We can distinguish the following cases:

- $E = V_{\text{eff}}(r_{\text{min}})$: According to the figure, the potential has a global minimum at $r = r_{\text{min}}$, and for a trajectory with $E = V_{\text{eff}}(r_{\text{min}})$, the entire energy resides in the potential term. Thus, the radial kinetic energy vanishes, which implies that there is no radial motion and the radius of the trajectory stays fixed at all times:

$$T_r = \frac{1}{2}m\dot{r}^2 = 0 \quad \Rightarrow r = r_{\text{min}} = \text{const.} \tag{32}$$

  This means that the trajectory of the object is a circle around the center of the potential.

  We also note that trajectories with $E < V(r_{\text{min}})$ are physically forbidden because they would require a negative kinetic energy.

- $V(r_{\text{min}}) < E < 0$: In this case, the trajectory is *bounded* radially, and the distance of the mass from the origin of the potential varies periodically between the turning points of the motion, which are defined by the conditions $E = V_{\text{eff}}(r_p) = V_{\text{eff}}(r_a)$. Borrowing terminology that was originally introduced specifically for the Kepler problem, we refer to the turning point at the minimum distances as the **periapsis** (from Greek *peri-*, "near", and *apsis*, "orbit") and **apoapsis** (*apo-*, "away from"). At any point $r_p < r < r_a$, the difference between E and $V_{\text{eff}}(r)$ corresponds to the radial kinetic energy of the motion,

$$\left.\frac{1}{2}m\dot{r}^2\right|_r = E - V_{\text{eff}}(r). \tag{33}$$

- $E \geq 0$: The trajectories are *unbounded* in radially direction, i.e., the object approaches the potential center up to some closest distance r_p that can be determined from $E = V_{\text{eff}}(r_p)$, but will then move to $r \to \infty$ and outside of the potential's influence, in general. Thus, the motion corresponds to a potential **scattering process** . The shape of the scattering trajectory depends on the potential, and may be distinct for $E = 0$ and $E > 0$. For the Kepler potential, a trajectory with $E = 0$ is a parabola, while trajectories with $E > 0$ are hyperbolas (cf. homework problem H12). .
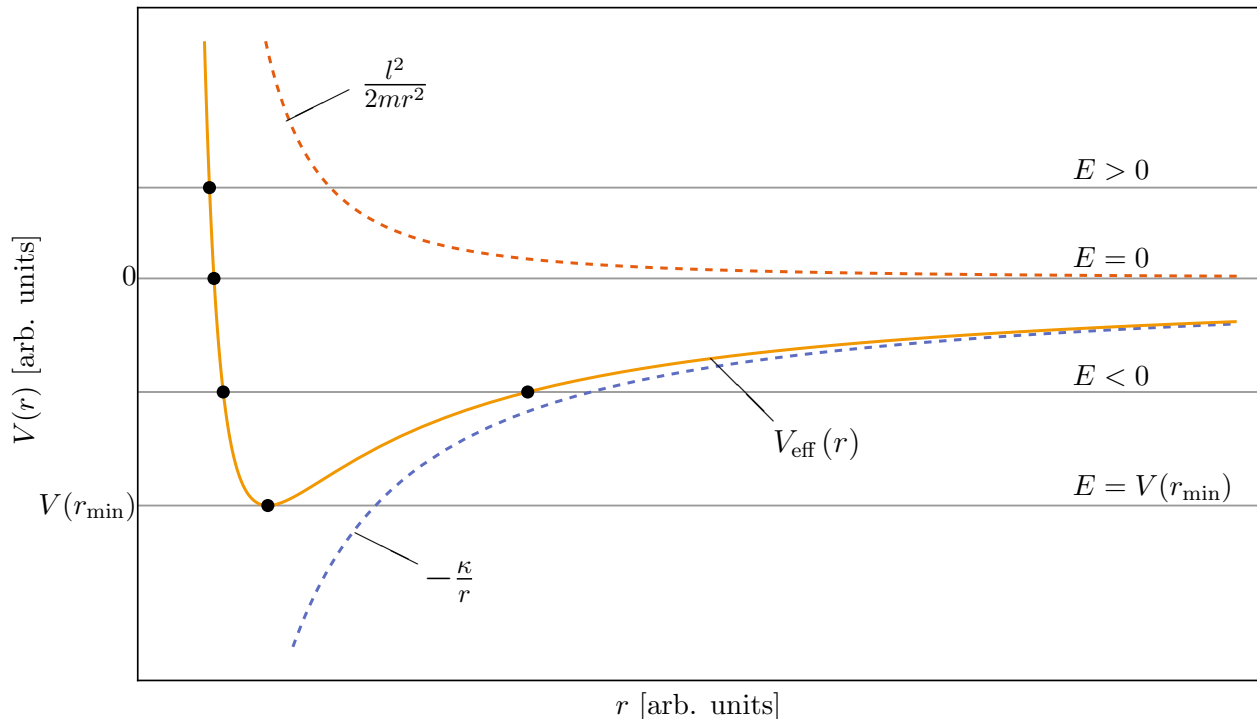
Figure 1: Effective potential as a function of r and trajectories with fixed energy E.

It is understood that the nature of the radial trajectories depends on the form of the effective potential, and not all types of orbits need to be possible for each V_{eff}. Figure 2 shows additional examples of (more or less) physical radial potentials. Figure 2a is an isotropic oscillator, which only has bounded solutions, while Fig. 2b is an inverted parabolic potential which leads to an effective potential that only admits scattering solutions. Highly singular inverse power law potentials like the one shown in Fig. 2c appear in effective theories of the strong interaction, and inverse power laws with alternating signs (Fig. 2d) are used to model molecular interactions. The latter two examples are particularly interesting because whether a trajectory with a given energy E is a bound or scattering solution depends on the region of the potential an object is located.

2.3 Geometry of Central-Force Trajectories

The general solutions (30) and (28) allow us to determine the trajectories of objects in central force fields as functions of time by determining $r(t)$ from (30) and plugging it into Eq. (28) to find $\phi(t)$. Very often, however, we are only interested in determining the *geometry* of trajectories for a given force field— or, conversely, in finding what type of force generates a particular observed trajectory. For central forces, a unique relationship between the force law and the shape of a trajectory exists, which we want to derive now.

First, consider the time derivatives of the radial coordinate. Using the chain rule, we can write

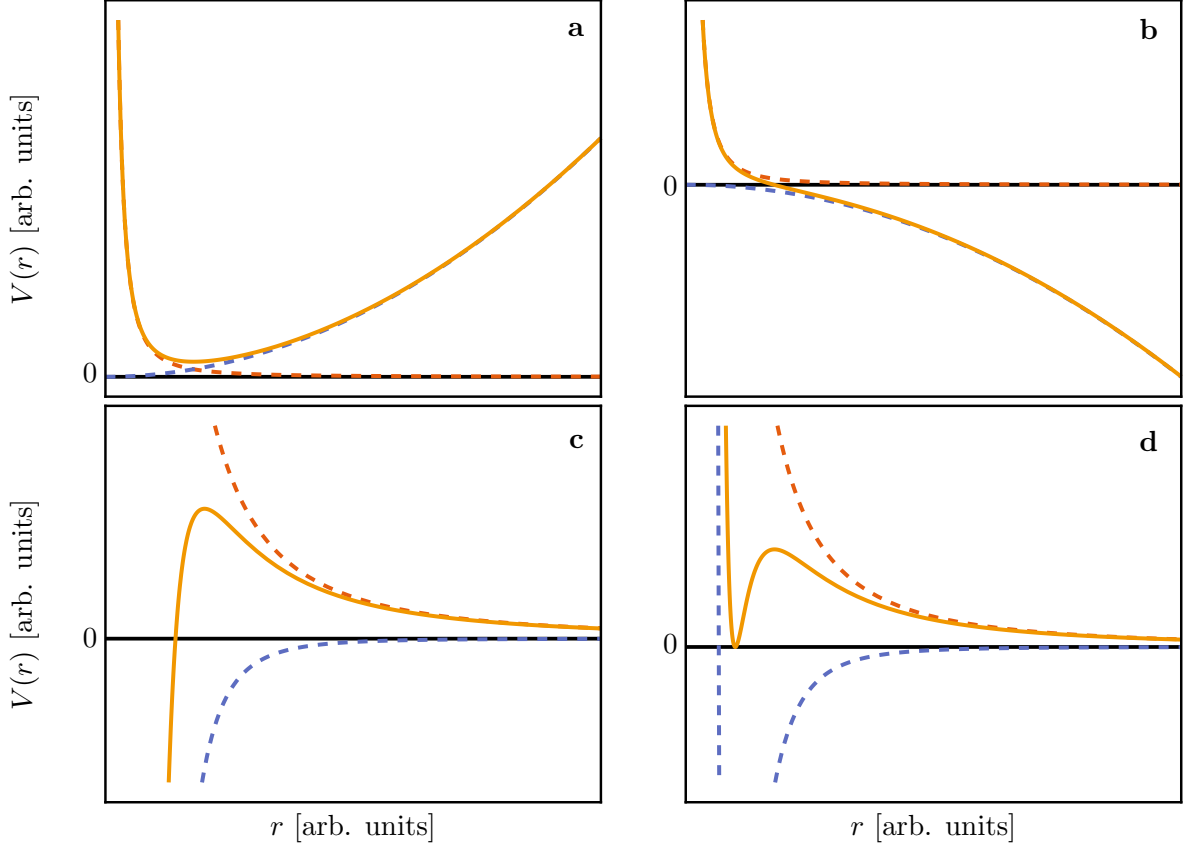$$\dot{r} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\dot{\phi}, \tag{34}$$

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Figure 2: Different types of effective potentials.

$$\ddot{r} = \frac{d^2r}{dt^2} = \frac{d^2r}{d\phi^2}\dot{\phi}^2 + \frac{dr}{d\phi}\ddot{\phi}. \tag{35}$$

From angular momentum conservation (Eq. (22)), we obtain

$$\frac{d}{dt}mr^2\dot{\phi} = 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = 0 \quad \Rightarrow \quad \ddot{\phi} = -\frac{2}{r}\dot{r}\dot{\phi} = -\frac{2}{r}\frac{dr}{d\phi}\dot{\phi}^2, \tag{36}$$

and plugging this into Eq. (35), we have

$$\ddot{r} = \frac{d^2r}{d\phi^2}\dot{\phi}^2 - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2\dot{\phi}^2. \tag{37}$$

The angular momentum conservation law (22) also allows us to express $\dot{\phi}$ in terms of l,

$$\dot{\phi}^2 = \frac{l^2}{m^2r^4} \tag{38}$$

and plugging both this relation and Eq. (37) into the radial equation of motion, we find

$$f(r) = m\ddot{r} - \frac{l^2}{mr^3} = m\left(\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2\right)\frac{l^2}{mr^4} - \frac{l^2}{mr^3} \tag{39}$$

and after minor rearrangement

$$f(r) = \left(\frac{d^2 r}{d\phi^2} - \frac{2}{r} \left(\frac{dr}{d\phi} \right)^2 - r \right) \frac{l^2}{mr^4} \,. \tag{40}$$

For a given orbit $r(\phi)$, Eq. (40) immediately yields the underlying force law.

To determine the geometry of the orbit for a given force or potential, we can derive an integral equation by noting that $\frac{dr}{d\phi}$ is related to the ratio of the energy and angular momentum conservation laws:

$$\frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} = \frac{\pm\sqrt{\frac{2}{m}\left(E - V_{\text{eff}}(r)\right)}}{l/mr^2} = \pm r^2 \frac{\sqrt{2m}}{l} \sqrt{E - V_{\text{eff}}(r)} \,. \tag{41}$$

Separating the variables, we have

$$d\phi = \pm \frac{l\,dr}{r^2 \sqrt{2m\left(E - V_{\text{eff}}(r)\right)}} \,, \tag{42}$$

and integrating, we obtain

$$\phi - \phi_0 = \pm \int_{r(\phi_0)}^{r(\phi)} dr' \frac{l}{r'^2 \sqrt{2m\left(E - V_{\text{eff}}(r')\right)}} \tag{43}$$

2.3.1 Stability of Circular Orbits

As discussed in Sec. 2.2.1, a circular orbit is obtained at an extremum of the effective potential

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + V(r) \,. \tag{44}$$

Thus, the radius R of the circular orbit is obtained by solving

$$V'_{\text{eff}}(R) = -\frac{l^2}{mR^3} + V'(R) = 0 \,. \tag{45}$$

Whether the extremum is a minimum, maximum, or saddle point can be determined by considering the second derivative, which is given by

$$V''_{\text{eff}}(R) = \frac{3l^2}{mR^4} + V''(R) = \frac{3}{R}V'(R) + V''(R) \,. \tag{46}$$

Comparing with the radial equation of motion (cf. Eq. (23))

$$m\ddot{r} = \frac{l^2}{mr^3} - V'(r) \,, \tag{47}$$

we see that the extremum of the effective potential corresponds to a radial equilibrium position, since

$$m\ddot{R} = 0 \,. \tag{48}$$

Naturally, we can only have a circular trajectory with fixed radius R if there is no acceleration in radial direction.

Let us now consider a perturbation of the circular orbit,

$$r(t) = R + \epsilon(t),$$ (49)

and plug it into Eq. (23). We obtain

$$
\begin{aligned}
0 &= m\ddot{\epsilon} - \frac{l^2}{m(R+\epsilon)^3} - f(R+\epsilon) \\
&= m\ddot{\epsilon} - \frac{l^2}{mR^3}\left(1 - 3\frac{\epsilon}{R} + O\left(\frac{\epsilon^2}{R^2}\right)\right) - \left(f(R) + f'(R)\epsilon + O(\epsilon^2)\right) = 0 \\
&\approx m\ddot{\epsilon} \underbrace{- \frac{l^2}{mR^3} - f(R)}_{=0} + \left(\frac{3l^2}{mR^4} - f'(R)\right)\epsilon = 0.
\end{aligned}
$$ (50)

This is the equation of motion of a harmonic oscillator if

$$\omega^2 \equiv \frac{1}{m}\left(\frac{3l^2}{mR^4} - f'(R)\right) = \frac{1}{m}\left(\frac{3l^2}{mR^4} + V''(R)\right) > 0.$$ (51)

or

$$\omega^2 = \frac{1}{m}V''_{\text{eff}}(R) > 0.$$ (52)

Thus, an orbit with radius R will be stable if V_{eff} has a minimum at R. For $V(r)$, the condition (51) implies that we must have

$$\frac{3l^2}{mR^4} + V''(R) = \frac{3}{R}V'(R) + V''(R) > 0,$$ (53)

where we have used Eq. (48). In terms of the forces, the inequality reads

$$\frac{3}{R}f(R) + f'(R) < 0.$$ (54)

Example: Power-Law Potentials

As an example, we consider power-law potentials of the form

$$V(r) = Ar^n, \quad n \in \mathbb{N}.$$ (55)

Thus, the radial force field and its derivative are

$$f(r) = -nAr^{n-1}, \quad f'(r) = -n(n-1)Ar^{n-2}.$$ (56)

Plugging these functions into Eq. (54), we find that n has to satisfy

$$\frac{3}{R}nAR^{n-1} + n(n-1)AR^{n-2} > 0,$$ (57)

i.e.,

$$3n + n(n-1) > 0 \quad \Rightarrow \quad n > -2.$$ (58)

Thus, only power-law potentials with $n > -2$ can support stable circular orbits.

2.3.2 Closed Orbits and Bertrand's Theorem

Let us now discuss the conditions under which an orbit is closed, which will lead us to **Bertrand's theorem**. It states that the Kepler and isotropic oscillator potentials are the only potentials for which *all* orbits are closed, regardless of the initial conditions — i.e., regardless of the specific values of E and l.

We start from the integral equation (43) for the trajectory. For a full period of the motion, the orbiting mass moves from the the periapsis r_p to the apoapsis r_a and back, advancing by the angle

$$\Delta\phi = \pm 2 \int_{r_p}^{r_a} dr \, \frac{l}{r^2 \sqrt{2m\left(E - V_{\text{eff}}(r)\right)}} \,. \tag{59}$$

Thus, an orbit will only be closed if

$$m\Delta\phi = n \cdot 2\pi \,, \quad m, n \in \mathbb{N} \,. \tag{60}$$

To solve the integral, we substitute

$$u = \frac{1}{r}, \quad du = -\frac{1}{r^2}dr, \tag{61}$$

obtaining

$$\Delta\phi = 2 \int_{u_p}^{u_a} du \, \frac{l}{\sqrt{2m\left(E - W(u)\right)}} \,, \tag{62}$$

with

$$W(u) \equiv \frac{l^2 u^2}{2m} + V\left(\frac{1}{u}\right) \,. \tag{63}$$

We focus on the positive branch for simplicity (the sign only matters if we need to know whether the angular motion is clockwise or counter-clockwise).

Now consider a circular orbit, which occurs at an extremum of the effective potential, as discussed in the Sec. 2.3.1. To describe the near-circular orbit, we write it as a circular orbit with an added perturbation ($U = 1/R$):

$$u(\phi) = U + \epsilon(\phi) \,. \tag{64}$$

Note that $\epsilon(\phi)$ is an inverse length here. Next, we expand the energy difference in the square root denominator. Denoting the energy of the circular orbit by E_c, we have

$$E - W(u) = E_c + \Delta E - \left(W\left(U\right) + W'(U)\epsilon + \frac{1}{2}W''(U)\epsilon^2\right) = \Delta E - \frac{1}{2}W''(U)\epsilon^2 \,, \tag{65}$$

where we have used that

$$E_c = W(U) \,, \quad W'(U) = 0 \,. \tag{66}$$

In terms of u, the relations for the first and second derivatives of the effective potential for circular orbits read

$$W'(U) = \frac{l^2}{m}U - \frac{1}{U^2}V'\left(\frac{1}{U}\right) = \frac{1}{R}\left(\frac{l^2}{m} - R^3 V'(R)\right) = 0 \tag{67}$$

and

$$W''(U) = \frac{l^2}{m} + \frac{2}{U^3}V'\left(\frac{1}{U}\right) + \frac{1}{U^4}V''\left(\frac{1}{U}\right) = 3R^3 V'(R) + R^4 V''(R) = R^4 V_{\text{eff}}''(R) \,, \tag{68}$$

(recall that derivatives of $W(u)$ are with respect to u, those of $V_{\text{eff}}(r)$ with respect to r).

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The integral can now be written as

$$\Delta\phi = \frac{2l}{\sqrt{mW''(U)}} \int_{u_p}^{u_a} du \frac{1}{\sqrt{\frac{2\Delta E}{W''(U)} - \epsilon^2}} . \tag{69}$$

The perturbation is going to cause oscillatory motion around the circular trajectory which can be parameterized as a function of the *polar angle* ϕ that characterizes the *circular trajectory* instead of time. For each angle ϕ on the circular trajectory, the near-circular orbit will be

$$u(\phi) = U + \epsilon(\phi) = U - \epsilon_0 \cos(\phi) , \tag{70}$$

where we have made the assumption that

$$u_p = U - \epsilon_0 , \quad u_a = U + \epsilon_0 . \tag{71}$$

From energy conservation for the perturbed orbit at the periapsis, we have

$$\underbrace{E - W(U)}_{=0} + \Delta E - \underbrace{W'(U)}_{=0}(-\epsilon_0) - \frac{1}{2}W''(U)\epsilon_0^2 = 0 , \tag{72}$$

which means that

$$\epsilon_0^2 = \frac{2\Delta E}{W''(U)} . \tag{73}$$

Plugging everything into the integral and changing the integration variable from u to the angle,

$$\frac{du}{d\phi'} = \epsilon_0 \sin \phi' \tag{74}$$

we have

$$\Delta\phi = \frac{2l}{\sqrt{mW''(U)}} \int_0^\pi d\phi' \frac{\epsilon_0 \sin \phi'}{\sqrt{\epsilon_0^2 (1 - \cos^2 \phi')}} = \pm \frac{2l}{\sqrt{mW''(U)}} \int_0^\pi d\phi' = \pm \frac{2\pi l}{\sqrt{mW''(U)}} . \tag{75}$$

Using Eq. (68), we obtain

$$\Delta\phi = \pm \frac{2\pi l}{R^2 \sqrt{mV''_{\text{eff}}(R)}} = \pm 2\pi \sqrt{\frac{V'(R)}{3V'(R) + RV''(R)}} . \tag{76}$$

We see that $\Delta\phi$ diverges for a saddle point, or becomes complex for a maximum ($V''_{\text{eff}}(R) < 0$) — as discussed in the previous section, the circular orbit would be unstable in these cases, any perturbation would grow exponentially and prevent a closed orbit.

According to the closed-orbit condition (60), we must have

$$\Delta\phi = \pm 2\pi \sqrt{\frac{V'(R)}{3V'(R) + RV''(R)}} = \frac{n}{m} \cdot 2\pi . \tag{77}$$

Squaring and inverting the equation, we have

$$\frac{V'(R)}{3V'(R) + RV''(R)} = \frac{n^2}{m^2} . \tag{78}$$

Switching to forces and rearranging, we obtain

$$\frac{R}{f(R)}f'(R) = \frac{m^2}{n^2} - 3 \equiv \beta^2 - 3 \,. \tag{79}$$

Since we can vary R smoothly by varying the angular momentum l (cf. Eq. (45)) this equation should hold for general r, so we obtain the differential equation

$$\frac{r}{f(r)}f'(r) = \frac{r}{f}\frac{df}{dr} = \beta^2 - 3 \,. \tag{80}$$

Separating the variables, we have

$$\frac{1}{f}df = \left(\beta^2 - 3\right)\frac{1}{r}dr \,, \tag{81}$$

and integrating, we obtain

$$\ln f(r) - \ln f(r_0) = \left(\beta^2 - 3\right)\left(\ln r - \ln r_0\right) \,. \tag{82}$$

We can rewrite this as

$$\ln \frac{f(r)}{f(r_0)} = \left(\beta^2 - 3\right)\ln \frac{r}{r_0} = \ln \left(\frac{r}{r_0}\right)^{\beta^2 - 3} \tag{83}$$

and exponentiate, which yields

$$\frac{f(r)}{f(r_0)} = \left(\frac{r}{r_0}\right)^{\beta^2 - 3} \,. \tag{84}$$

Thus, only power-law forces of the form

$$f(r) = Ar^{\beta^2 - 3} \tag{85}$$

will satisfy the conditions for closed orbits. This means that the potential either has a power-law form as well,

$$V_\alpha(r) \equiv -\frac{A}{\alpha}r^\alpha \equiv -\frac{A}{\beta^2 - 2}r^{\beta^2 - 2} \,, \tag{86}$$

or it is logarithmic

$$V_0(r) \equiv A \ln \frac{r}{r_0} \tag{87}$$

where we have ignored irrelevant constants. Note that for the power-law potentials, we have $\alpha > -2$ for a stable near-circular orbit, which agrees with our result from Sec. 2.3.1.

Plugging the logarithmic potential into Eq. (76), we obtain

$$\Delta\phi = \pm 2\pi\sqrt{\frac{A/R}{3A/R - RA/R^2}} = \pm 2\pi\frac{1}{\sqrt{2}} \,, \tag{88}$$

which is not rational and will therefore not yield a closed orbit. For the power-law potentials, we find

$$\Delta\phi = \pm 2\pi\sqrt{\frac{-AR^{\alpha - 1}}{-3AR^{\alpha - 1} - (\alpha - 1)AR^{\alpha - 1}}} = \pm 2\pi\sqrt{\frac{1}{2 + \alpha}} \,. \tag{89}$$

[Finish argument.]

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3   Group Exercises

Problem G16 – Kepler's Second Law

Kepler's Second Law states that a line segment joining the sun and a planet in the solar system sweeps out equal areas dA during equal amounts of time dt, implying $dA \sim dt$. In fact, this is a property of general central-force problems if we consider the line segment from the center of the potential to the moving object.

Show that the law can be stated in differntial form as

$$\frac{dA}{dt} = \frac{l}{2m}, \tag{90}$$

where m and l are the object's mass and angular momentum, respectively. What does this imply for the connection between Kepler's Second Law and angular momentum conservation?

Problem G17 – Stability of Orbits

[**10 Points**] A particle of mass m is moving in the central force field

$$\vec{F}(\vec{r}) = \left(-\frac{\alpha}{r^2} - \frac{\beta}{r^3}\right)\vec{e}_r, \quad \alpha, \beta > 0. \tag{91}$$

1. Compute the potential $V(r)$, choosing any integration constants such that the potential vanishes for $r \to \infty$. Sketch $V(r)$ and indicate which term dominates at short and long distances, respectively.

2. Now consider the effective potential. How must $V_{\text{eff}}(r)$ behave at short distances to support stable orbits, i.e., orbit that do not reach the origin or infinity? What is the critical angular momentum l_c that an object must have to move in a stable orbit? Sketch $V_{\text{eff}}(r)$ for $l < l_c$ and $l > l_c$.

3. For $l > l_c$, determine the radius and energy of *circular* orbits.

4. For what *range* of energies E will an orbit with $l \geq l_c$ be bound? Determine the turning points $r_{\min}$ and $r_{\max}$ of such bound orbits as a function of α, β, E and l.

Problem G18 – The Laplace-Runge-Lenz Vector

A general trajectory in the potential $V(r) = -\frac{\kappa}{r}$ is given by the *conic section* with focal parameter p and eccentricity ϵ:

$$r(\phi) = \frac{p}{1 + \epsilon \cos \phi}. \tag{92}$$

For such trajectories, there is an additional conserved quantity besides the angular momentum $\vec{l}$

and energy E, the so-called *Laplace-Runge-Lenz vector*

$$\vec{A} = \frac{\vec{p} \times \vec{l}}{m\kappa} - \vec{e}_r. \tag{93}$$

1. Show that $\dot{\vec{A}} = 0$.

2. Show that $|\vec{A}|$ is the eccentricity ϵ of the trajectory (92). To achieve this, start by computing $\vec{l} \cdot \vec{A}$ and $\vec{r} \cdot \vec{A}$, and parameterize the trajectory by defining the angular variable as $\phi \equiv \sphericalangle(\vec{r}, \vec{A})$.