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# PHY422/820: Classical Mechanics

FS 2020
Worksheet #10 (Nov 2 – Nov 6)

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## 1 Preparation

- Lemos, Chapter 3 and Sections 4.1-4.8
- Goldstein, Chapter 4 (skip 4.5) and Sections 5.1-5.6

## 2 Rigid Body Kinematics

### 2.1 Euler Angles

We can characterize a general rotation in three dimension by three degrees of freedom: the usual azimuthal and polar angles $\phi$ and $\theta$ of a spherical coordinate system, which characterize the direction of the axis of rotation, and a third angle $\psi$ that specifies *how far* we rotate the coordinate system. From this argument, we can infer the ranges of the angles: $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, as usual in spherical coordinates, and $\psi \in [0, 2\pi]$.

To construct a general rotation, we consider the basic $SO(3)$ matrices which describe rotations around the axes of a Cartesian coordinate system, spanned by the right-handed triad of unit vectors $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$:

$$\boldsymbol{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \; \boldsymbol{R}_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}, \; \boldsymbol{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

A general rotation can be described by a product of three such matrices as long as we do not rotate around the same axis twice, because then we could replace the consecutive rotations by a single rotation with the combined angle, and we would not used all three degrees of freedom. In physics, the most frequently used convention is to perform consecutive rotations around the $z$, $x$, and $z$ axes by the angles $\phi, \theta$ and $\psi$, which are referred to as **Euler angles** in this context. Importantly, we are rotating the basis vectors, which requires that we consider $\boldsymbol{R}_i$ with angles $-\alpha$, and implies a *passive* view of the rotation (see Sec. **??**).

The conventional sequence of rotations that transforms between two coordinate systems is illustrated in Fig. **??**:

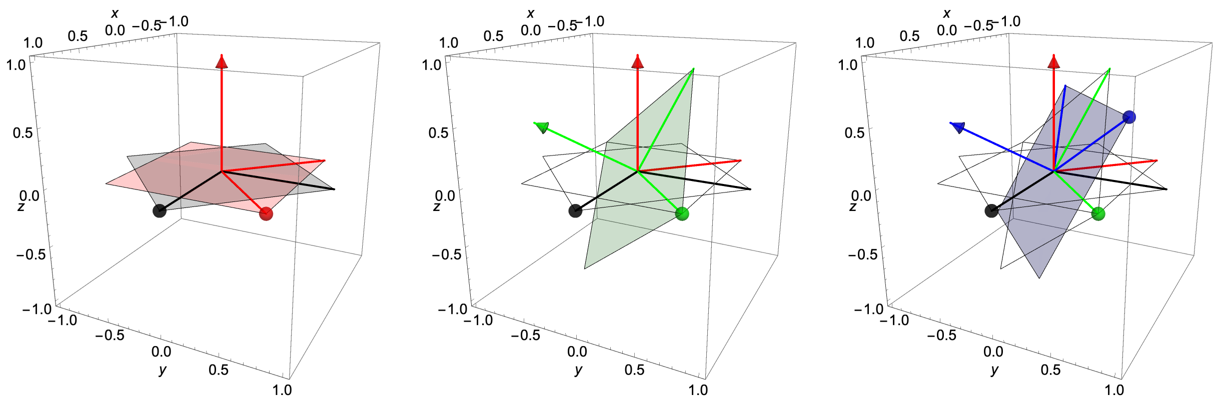


Figure 1: Transformation between the space-fixed (black) and body-fixed (blue) coordinate systems by a sequence of rotations with the Euler angles $\phi, \theta, \psi$.

- We start by rotating the initial coordinate system around the $z$ axis by an angle $\phi$ using the rotation matrix

$$\boldsymbol{D} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2}$$

  Note that this rotation leaves the $z$ axis invariant, while the $x$ and $y$ axes are tilted by the angle $\phi$ compared to their previous directions.

- In the resulting intermediate coordinate system, we rotate around the intermediate $x'$ axis by an angle $\theta$, which tilts the entire coordinate system against the $z'$ axis. The rotation matrix is given by

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} , \tag{3}$$

  but note that this representation refers to the *intermediate* coordinate system, not to the initial one.

- Finally, we rotate around the new $z''$ axis by the angle $\psi$ to obtain the final coordinate system, using

$$\boldsymbol{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{4}$$

  which is now represented in the basis of the doubly primed coordinate system.

The complete rotation between the two coordinate system can be expressed as a single rotation given by

$$\boldsymbol{A} = \boldsymbol{BCD} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & \cos\theta\sin\psi\cos\phi + \cos\psi\sin\phi & \sin\theta\sin\psi \\ -\cos\theta\cos\psi\sin\phi - \sin\psi\cos\phi & \cos\theta\cos\psi\cos\phi - \sin\psi\sin\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} . \tag{5}$$

It is straightforward to verify that $\boldsymbol{A}$ is an $SO(3)$ matrix.

[To be continued...]

# 3 Group Exercises

## Problem G24 – Mass Densities

Which solids are described by the following mass densities?

$$\rho_1(\vec{r}) = \frac{M}{\pi R^2}\Theta\left(R-\rho\right)\delta\left(z\right)\,, \tag{6}$$

$$\rho_2(\vec{r}) = \frac{M}{\pi R^2 H}\Theta\left(R-\rho\right)\Theta\left(\frac{H}{2}-|z|\right)\,, \tag{7}$$

$$\rho_3(\vec{r}) = \frac{M}{\pi(a^2-b^2)H}\Theta\left(a-\rho\right)\Theta\left(\rho-b\right)\Theta\left(\frac{H}{2}-|z|\right)\,, \tag{8}$$

$$\rho_4(\vec{r}) = \frac{M}{4\pi R^2}\delta(r-R)\,. \tag{9}$$

Perform a volume integration over three-dimensional space in appropriate coordinates to show that you obtain the mass $M$ of the solid.

Hint:

$$\int_a^b f(x)\delta(x-x_0) = \begin{cases} f(x_0) & \text{if} \quad x_0 \in [a,b]\,, \\ 0 & \text{else,} \end{cases} \qquad \Theta(x) = \begin{cases} 1 & \text{if} \quad x>0\,, \\ \frac{1}{2} & \text{if} \quad x=0\,, \\ 0 & \text{if} \quad x<0\,. \end{cases}$$

## Problem G25 – Moment of Inertia Tensor

A hollow cylinder of mass $M$ and radius $R$ can be described by the mass density

$$\rho(r,\varphi,z) = \frac{M}{2\pi R H}\Theta\left(\frac{H}{2}-|z|\right)\delta(r-R)\,, \tag{10}$$

where $r$ indicates the radial distance from the cylinder's symmetry axis, chosen to be the $z$ axis of our coordinate system.

1. Compute the moment of inertia tensor $\boldsymbol{I}$ of the cylinder, and determine its principal axes.
2. Use rotation matrices to determine $\boldsymbol{I}$ in coordinate systems that are rotated by *(i)* $\frac{2\pi}{3}$ around the original $z$ axis, and *(ii)* by $\frac{\pi}{4}$ around the original $y$ axis.
3. Verify by explicit calculation that the kinetic energy for a rotation around the axis $\vec{\omega} = \omega\vec{e}_z$ in the original coordinate system is identical to the kinetic energy we obtain for that motion in the two rotated coordinate systems.

Hint:

$$\boldsymbol{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}\,, \quad \boldsymbol{R}_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}\,, \quad \boldsymbol{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Problem G26 – Racing Solids

A solid sphere, a solid cylinder and a thin-walled, hollow cylinder (both of height $H$) with equal masses $M$ and radii $R$ are rolling down an inclined plane with inclination angle $\alpha$.

1. Compute the solids' moments of inertia with respect to rotations around principal axes through the center of mass that are relevant for the rolling motion studied here. **You do not have to compute the complete moment of inertia tensor.**

2. Describe the motion of the solids as a superposition of center-of-mass translation and rotation around the center of mass. Construct the Lagrangian, determine the equations of motion and state their general solutions.

3. Now describe the motion of the solids as a rotation around a principal axis through the point where the solid touches the inclined plane. Construct the Lagrangian for this case and show that you obtain the same equations of motion as in the previous part of the problem.

4. Which of the three solids will reach the bottom of the inclined plane in the shortest amount fo time after being released from rest at the top?

## Problem G27 – Rotating Cuboid

Consider a homogenous rotating cuboid with side lengths $a, b, c$ and mass $M$.

1. Compute the principal moments of inertia with respect to the cuboid's center of mass.

   Hint: The diagonalization of the moment-of-inertia tensor can be avoided through an appropriate choice of coordinate system.

2. Determine the cuboid's equations of motion in the body-fixed frame, the **Euler equations for the rigid body**, by starting from

$$\frac{d'\vec{L}'}{dt} + \vec{\omega} \times \vec{L}' = \vec{N}' , \quad \vec{L}' = \boldsymbol{I}\vec{\omega} = (A\omega_{x'}, B\omega_{y'}, C\omega_{z'})^T , \tag{11}$$

   where $\frac{d'}{dt}$ is the time-derivative in the body-fixed frame, and $A, B$ and $C$ denote the principal moments of inertia.

3. Consider the force-free rotation of the cuboid around a principal axis, e.g., $\vec{\omega}_0 = (\omega_0, 0, 0)^T =$ const. Under which conditions is a rotation around this axis stable?

   Hint: Assume a small perturbation of the rotational axis,

$$\vec{\omega} = \vec{\omega}_0 + \vec{\epsilon} = \vec{\omega}_0 + (\epsilon_{x'}, \epsilon_{y'}, \epsilon_{z'})^T , \tag{12}$$

   and determine the conditions under which the amplitude of the perturbation $\vec{\epsilon}$ remains small. Omit terms of order $O(\epsilon^2)$ and higher.