Physics 905

Fundamentals of Accelerator Physics

Final Exam – Timed, Take Home.

Handout / Distribute :	Friday, May 6, 2016 10:00 am	S.M. Lund Office: 1118 or Course Web Site.
Due : (Cannot be late)	Saturday, May 7, 2016 5:00 pm	S.M. Lund Office: 1118

Comprehensive. All work must be individual by the student alone. No help from other students, scientists, professors, etc. Clarification questions allowed to S.M. Lund.

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Email/phone will be monitored during the exam. You are free to use all course notes, books, online materials, etc., insofar as you do not ask others to provide input, for full consistency with an individual effort.

Steve Lund

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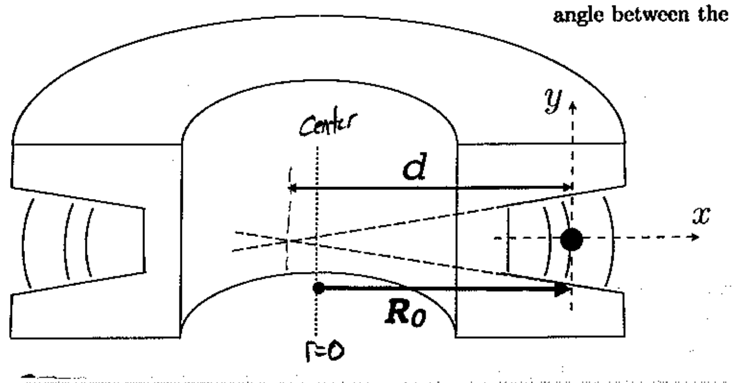
Problem #1 Transverse Focusing and Weak Focusing Synchrotron / Betatron

25 pts

Early accelerators shaped magnetic dipole pole faces to provide both bending and acceleration as idealized below for a continuous ring. Seperated function bend + focus lattices with stronger focusing are now more common. These older focusing schemes are now called "weak focusing".

n is determined by adjusting the opening angle between the poles



$$B = B_0 \left(\frac{R_0}{r} \right)^n$$

$$n \approx \frac{R_0}{d} \qquad B_0 = \text{const}$$

n = "field index"

R_0 = Radius particle ref orbit.

10pts

a) Take

$$B = B_y(y=0) = B_0 \left(\frac{R_0}{r} \right)^n$$

$$r = R_0 + x$$

and find leading order field expressions in $|x|/R_0$ for B_x and B_y about the reference orbit. The fields should satisfy the vacuum Maxwell equations $\nabla \times \vec{B} = 0$, $\nabla \cdot \vec{B} = 0$. Curvature terms can be neglected here and take

$$\vec{B} = B_x(x,y)\hat{x} + B_y(x,y)\hat{y}$$

10 pts

b) Using class equations for the particle orbits in bent coordinate systems show that a particle with rigidity $(B\rho)$ evolves according to (leading order)

$$\frac{d^2x}{ds^2} + \frac{1-n}{R_0^2} x = 0$$

$$\frac{d^2y}{ds^2} + \frac{n}{R_0^2} y = 0$$

Hint: use formulation in 04.lecture.pdf with B_x, B_y of a), for a bent orbit and expand equations to linear order.

5pts

c) What range of n will give focusing in x and y? Why? For what value of n are the x- and y-focusing strengths equal?

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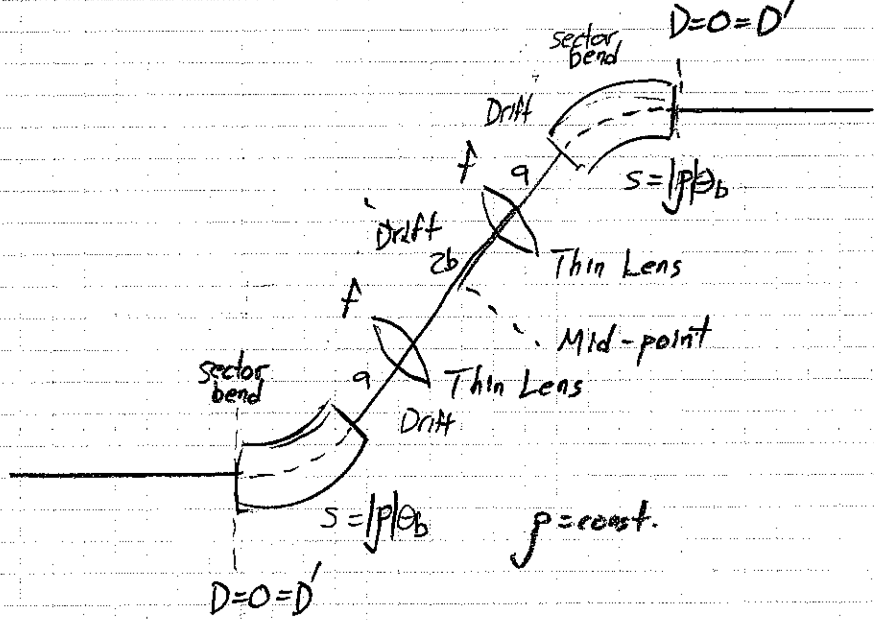
Comment: The FRIB front end uses $90°$ bending dipoles with slanted poles to provide additional focusing. This problem illustrates the essential physics in an idealized form. Johnathan: you might think of exploiting this in your simulation analysis.

Dispersion & Momentum Compaction

Problem #2 Beam Transfer Line + Synchrotron 35 pts

A transfer line that translates the beam to the side while being nondispersive at exit can be constructed as:



f = Thin lens focus strength

a, b drift lengths.

Recall from problem sets; 3x3 transfer matrices for dispersion are:

Drift

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_s = \begin{pmatrix} 1 & s-s_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{s_i}$$

Thin Lens

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{s_i^+} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{s_i^-}$$

Sector Bend $\rho > 0$ here.

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_s = \underbrace{\begin{pmatrix} \cos(s/\rho) & \rho \sin(s/\rho) & \rho(1-\cos(s/\rho)) \\ -\frac{\sin(s/\rho)}{\rho} & \cos(s/\rho) & \sin(s/\rho) \\ 0 & 0 & 1 \end{pmatrix}}_{M_B} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{s_i}$$

for a full bend of angle θ_b:

$$\bar{M}_B = \begin{pmatrix} \cos\theta_b & \rho\sin\theta_b & \rho(1-\cos\theta_b) \\ -\frac{\sin\theta_b}{\rho} & \cos\theta_b & \sin\theta_b \\ 0 & 0 & 1 \end{pmatrix}$$

Comment: This is a system you may really use in the lab.

25 pts

a) Show that the value of f that results in zero dispersion at the exit ($D=0=D'$) when there is zero initial dispersion ($D=0=D'$), is

$$F \equiv \frac{f}{\rho}, \quad A = \frac{a}{\rho}, \quad B = \frac{b}{\rho}$$

$$F = \frac{B(1-\cos\theta_b + A\sin\theta_b)}{1-\cos\theta_b + (A+B)\sin\theta_b}$$

Hints: • By symmetry, must have at mid-point (middle 2b drift)

$$D = 0, \quad D' \neq 0.$$

• Use this to write a transfer matrix constraint for the half-system (initial to mid 2b drift). Then examine what element should be zero for $D=0$ at the mid-point to obtain the needed constraint.

• No need for symbolic math packages if you take hints. Algebra is short if setup right.

10 pts

b) The momentum compaction (see 11.lecture.pdf) is defined by

$$\frac{\Delta L}{L_0} = \alpha_c \frac{\delta p}{p_0} \qquad \alpha_c \equiv \frac{\int_{s_1}^{s_2} \frac{D(s)}{\rho(s)} ds}{\int_{s_1}^{s_2} ds}$$

$$\int_{s_1}^{s_2} ds = 2\rho\theta_b + 2a + 2b \qquad \begin{array}{l} s_1 = \text{start 1st bend} \\ s_2 = \text{end 2nd bend.} \end{array}$$

Show for this system that

$$\alpha_c = \frac{\rho\theta_b - \rho\sin\theta_b}{2\rho\theta_b + 2a + 2b} = \frac{1}{2}\,\frac{\theta_b - \sin\theta_b}{\theta_b + A + B}$$

Which implies non-isochronous for θ_b finite.

Hints • $\rho(s) \to \infty$ in straight.

• Both D and ρ change sign in the 2nd Bend so the contributions to α_c are the same.

Problem #3 Emittance Evolution 40 pts

5 pts

a) Consider a beam composed of particles evolving with an equation of motion

$$x''(s) + \alpha(s)\, x^n(s) = 0$$

$n > 0$ $\qquad\qquad$ $' \equiv \frac{d}{ds}$

$x = \perp$ coord

$s =$ longitudinal (ref) coord. $\qquad$ $\alpha(s) =$ some function (nonlinear amplitude)

Define an rms emittance

$$\varepsilon_{x,rms} \equiv \left[\langle x^2\rangle\langle x'^2\rangle - \langle x x'\rangle^2\right]^{1/2}$$

where

$\langle \cdots \rangle$ = statistical avg over a distribution of particles.

Use the particle equation of motion to derive an evolution equation for $\frac{d}{ds}\varepsilon^2_{x,rms}$ in terms of $\langle x^2\rangle \langle xx'\rangle$, $\langle x' x^n\rangle$, and $\langle x^{n+1}\rangle$?

5 pts

b) For what value of n is $\frac{d}{ds}\varepsilon^2_{x,rms} = 0$?

Does this make sense if $\varepsilon_{x,rms}$ is a statistical measure of phase-space area? Why?

20 pts

c) For Hill's equation, $n=1$ and $\alpha(s) = \kappa(s)$ periodic with period L_p, we have

$$x''(s) + \kappa(s)\, x(s) = 0 \qquad \kappa(s+L_p) = \kappa(s)$$

Take a phase-amplitude form of the particle orbit with:

$$x = A_i\, w(s) \cos\psi(s) \qquad \psi(s) = \int_{s_i}^{s} \frac{d\tilde{s}}{w(\tilde{s})} + \psi_i$$

A_i = const = particle amplitude $\qquad$ $w''(s) + \kappa(s) w(s) - \frac{1}{w^3(s)} = 0$

ψ_i = const = particle initial phase $\qquad$ $w(s+L_p) = w(s)$

$w(s) > 0$

Show that if particles within the beam are uniformly distributed in initial phase ψ_i that we have

Courant-Snyder "Parameters"

$$\beta \equiv w^2$$
$$\gamma \equiv \frac{1}{w^2} + w'^2 = \frac{1+\alpha^2}{\beta}$$
$$\alpha \equiv w w'$$

$$\langle x^2 \rangle = \frac{\beta}{2} \langle A_i^2 \rangle$$

$$\langle x'^2 \rangle = \left(\frac{1+\alpha^2}{2\beta}\right) \langle A_i^2 \rangle = \frac{\gamma}{2} \langle A_i^2 \rangle$$

$$\langle x x' \rangle = \frac{-\alpha}{2} \langle A_i^2 \rangle$$

Here, $\langle \cdots \rangle$ denotes an average over the particle phase-space of the full beam,

$$\frac{\int_{-\pi}^{\pi} \int_0^{\infty} \cdots\cdots f(A_i)\, A_i\, dA_i\, d\psi_i}{\int_{-\pi}^{\pi} \int_0^{\infty} f(A_i)\, A_i\, dA_i\, d\psi_i}$$

$f(A_i) =$ # particles with initial amplitude A_i
No initial phase dependance.

You only need to show enough work on term averages to convince me that you know which terms vanish.

5pts

d) Using results from part c) relate the rms emittance to the average particle amplitude

$$\mathcal{E}_{x,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2} = \frac{1}{2} \langle A_i^2 \rangle$$

5pts

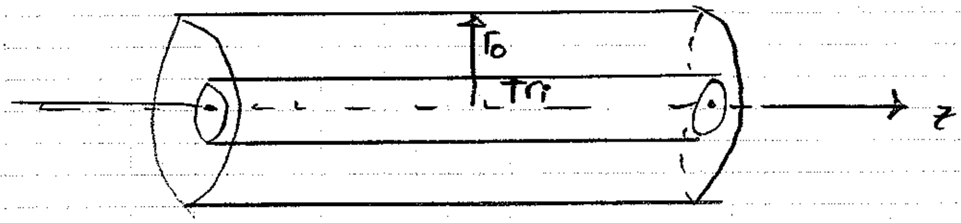
e) If the initial particle amplitudes are uniformly distributed to A_{max} and we take

$$\pi \mathcal{E}_{x,max} = \pi A_{max}^2$$

with $\mathcal{E}_{x,max}$ the max single particle emittance, relate $\mathcal{E}_{x,max}$ to $\mathcal{E}_{x,rms}$.

Problem #4 Coaxial Half-Wave Resonator 70 pts

Consider a coaxial transmission line with inner conductor radius r_i and outer radius r_o:



5 pts a) Between the conductors an EM wave satisfies:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0$$

Examine the equation for $\vec{E}$ and show that an EM wave with a radial electric field in traveling wave form

$$E_r(r,\theta,z,t) = E_r(r)\, e^{\pm i k z + i\omega t}$$

$i = \sqrt{-1}$
$\omega = \text{const}$ Angular Freq.
$k = \text{const}$ Axial Wavenumber

is supported. Show that $k = \pm\frac{\omega}{c}$ and $E_r = \text{const}/r$ satisfies the wave equation, i.e.

$$E_r(r,\theta,z,t) = \frac{C_1}{r}\, e^{\pm i \frac{\omega}{c} z + i\omega t} \qquad C_1 = \text{const}.$$

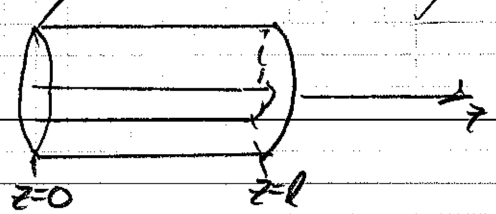
Does this satisfy needed boundary conditions at $r = r_i, r_o$ if $\vec{E} = E_r \hat{r}$ if the inner and outer conductors are perfectly conducting?

2 pts b) A forward and backward traveling wave E_r can be superimposed in a cavity to obtain:

$$E_r(r,\theta,z,t) = \frac{-V}{r}\sin\left(\frac{\omega}{c} z\right)\cos(\omega t + \phi)$$

$V = \text{const}$ RF Amplitude parameter
$\phi = \text{const}$ RF phase

For a cavity with axial length ℓ and coordinates:

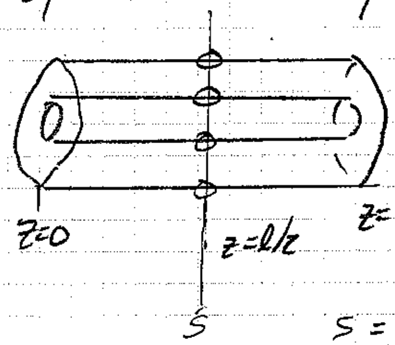


Show that $\omega = \frac{p\pi c}{\ell}$, $p = 1, 2, 3, \dots$ to meet $\vec{E}$ boundary conditions.

10 pts. c) Find a corresponding form of $\vec{B} = B_\theta \hat{\theta} = B_\theta(r,z,t)\hat{\theta}$ consistent with the EM standing wave $E_r(r,\theta,z,t)$ in part b) and show it meets needed cavity boundary conditions and satisfies all Maxwell equations between conductors.

3 pts. d) For a cavity $l = 1\,m$ long what is the lowest allowed resonant frequency $f_{rf} = \omega/(2\pi)$. How many RF wavelengths long is the structure? Is there any dependance of the resonant frequency on r_i and r_o?

5 pts. e) Imagine the cavity has an apeture hole cut through at $z = l/2$, and is operated in the $p=1$ mode.



Assume apeture small and does not perturb cavity field.

s = axial accelerator coordinate.

Sketch what the axial electric field E_s should look like at time $\omega t + \phi = 0$. Let $s=0$ correspond to $r=0$.

How should the RF phase advance be timed so that the particle will gain energy in each half ($s=-r_o \rightarrow s=-r_i$ and $s=r_i \rightarrow s=r_o$) of the RF structure? Justify your answer.

5 pts. f) For $P=1$, will a particle traversing the gaps be significantly deflected by the magnetic field B_θ for any RF phase? Justify your answer.

10 pts. g) How would you define a transit-time-factor for the two-gap cavity. Give a workable expression for

$$\Delta W = q E_0 L T \cos\phi \qquad \text{Panofsky Eqn.}$$

Define E_0, L, and T to make this work.

No need to simplify answer for T and take:

$$E_0 \equiv \int_{\text{gaps},\ \omega t+\phi=0} |E_s|\, ds = 2V \ln(r_o/r_i)$$

to account for the sign change between gaps.

5pts

h) Would the two gap transit time factor defined in problem set #6, problem #3 calculated for $\beta \simeq$ const be a good approximation for this cavity if the gaps are tuned to be the same length? Why? Qualitative answer only.

10pts

i) Calculate the stored RF energy $\mathcal{U}$ of this cavity for arbitrary p. Show that:

$$\mathcal{U} = \frac{\pi \varepsilon_0}{2} V^2 \ell \ln(r_o/r_i)$$

Hint: parallel steps applied to the pillbox cavity in the class notes.

10pts

j) Calculate the average RF power loss $\langle P_{loss} \rangle_{rf}$ of this cavity for arbitrary p. Show that

$$\langle P_{loss} \rangle_{rf} = \frac{\pi V^2 \varepsilon_0 R_{surf}}{2 \mu_0} \left[\ell \left(\frac{1}{r_p} + \frac{1}{r_o} \right) + 4 \ln(r_o/r_i) \right]$$

R_{surf} = surface Resistance

Hints: parallel steps applied to the pillbox cavity in the class notes. Do not forget end contributions at $z = 0, \ell$.

If we take $\vec{H}$ in complex, harmonic form,

$$\langle P_{loss} \rangle_{rf} = \frac{R_{surf}}{2} \int_{\text{Cavity Surface}} |H_{tangent}|^2 d^2x$$

$$H = H(r, z) e^{i\omega t} \quad \text{here.}$$

5pts

k) Calculate the cavity Q using the results of i) and j) and the definition of Q given in class. Show that

$$Q = \frac{p\pi}{R_{surf}} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\ln(r_o/r_i)}{\ell\left(\frac{1}{r_i} + \frac{1}{r_o}\right) + 4 \ln(r_o/r_i)}$$

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Problem #5 Synchrotron Energy Acceptance. 35 pts

Consider the continuous approx RF focusing/acceleration equations derived in class for a ring:

$$\frac{d\phi}{dn} = \frac{-2\pi h \eta_s}{\gamma_s \beta_s^2} \frac{\Delta W}{mc^2}$$

$$\frac{d}{dn}\Delta W = qV_{rf}\left[\sin\phi - \sin\phi_s\right]$$

$$\eta_s = \frac{1}{\gamma_s^2} - \frac{1}{\gamma_{tr}^2}$$

Assume coefficients approx constant lap by lap.

10 pts

a) Show that for a particle evolving in the bucket that

$$\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 - \frac{2\pi h \eta_s}{\gamma_s \beta_s^2} \frac{qV_{rf}}{mc^2}\left(\cos\phi + \phi\sin\phi_s\right) = \text{const.}$$

Hint: Apply analogous steps used in class for linac eqns using continuous lap variable n.

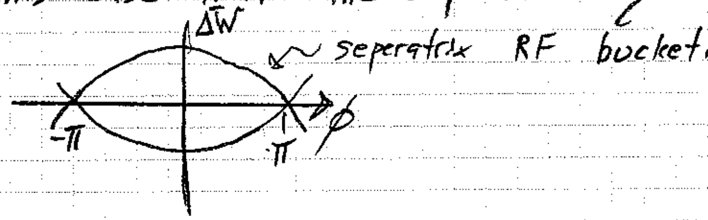
5 pts

b) Show that the result in a) can be expressed as:

$$\Delta W^2 + \frac{\beta_s^2 \gamma_s mc^2}{\pi h \eta_s} qV_{rf}\left(\cos\phi + \phi\sin\phi_s\right) = \text{const}$$

10 pts

c) Choose ϕ_s for max acceptance (corresponding to no acceleration) below transition and show for this case that the seperatrix equation satisfies:



$$\Delta W^2 - \frac{2\gamma_s \beta_s^2 mc^2 qV_{rf}}{\pi h \eta_s}\cos\left(\frac{\phi}{2}\right) = 0$$

Hint: easy to identify const at x-points at $\phi = \pm\pi$.

10 pts

d) Apply the result in part c) to estimate the energy acceptance $\Delta W/W$ of a proton synchrotron with:

$h = 80$ harmonic number
$\eta_s = 0.05$ slip factor.
$V_{rf} = 1\,\text{MV}$
$W = 200\,\text{MeV}$ kinetic energy

Assume incoming particles have correct synchronous phase and machine is operated for max acceptance.

Problem #6 RMS Envelope Equation with Space Charge and Acceleration 50 pts

Consider a long unbunched beam evolving according to

$$x'' + \frac{(\gamma\beta)'}{(\gamma\beta)} x' + \kappa_x x = -\frac{q}{m\gamma^3\beta^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma\beta)'}{(\gamma\beta)} y' + \kappa_y y = -\frac{q}{m\gamma^3\beta^2c^2}\frac{\partial\phi}{\partial y}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0} \qquad \frac{\partial}{\partial z} = 0 \quad \text{unbunched}$$

ρ = beam charge density, continuous approximation

κ_x, κ_y = x and y-plane lattice focusing functions

15 pts

a) Define a statistical edge measure of the beam x-width

$$r_x = 2\langle x^2\rangle^{1/2} \qquad \langle \cdots \rangle = \frac{\int\int \cdots f(\vec{x}_\perp, \vec{x}_\perp', s)\, d^2x_\perp' \, d^2x_\perp}{\int\int f(\vec{x}_\perp, \vec{x}_\perp', s)\, d^2x_\perp' \, d^2x_\perp}$$

$$\rho = q\int f(\vec{x}_\perp, \vec{x}_\perp', s)\, d^2x_\perp'$$

Show that

$$r_x'' + \frac{(\gamma\beta)'}{(\gamma\beta)} r_x' + \kappa_x r_x + \frac{4q\left\langle x\frac{\partial\phi}{\partial x}\right\rangle}{m\gamma^3\beta^2c^2 r_x} - \frac{16\varepsilon_{x,rms}^2}{r_x^3} = 0$$

$$\varepsilon_{x,rms} = 4\left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right]^{1/2}$$

Hint: parallel class notes on space charge effects.

No need to work, but by analogy also have

$$r_y = 2\langle y^2\rangle^{1/2}$$

$$r_y'' + \frac{(\gamma\beta)'}{(\gamma\beta)} r_y' + \kappa_y r_y + \frac{4q\left\langle y\frac{\partial\phi}{\partial y}\right\rangle}{4m\gamma^3\beta^2c^2 r_y} - \frac{16\varepsilon_{y,rms}^2}{r_y^3} = 0$$

$$\varepsilon_{y,rms} = 4\left[\langle y^2\rangle\langle y'^2\rangle - \langle yy'\rangle^2\right]^{1/2}$$

20 pts

b) Consider an axisymmetric beam focused with

$$\kappa_x = \kappa_y = \kappa(s)$$

$$\phi = \phi(r) \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = -\frac{\rho(r)}{\varepsilon_0}$$

Show that for <u>any</u> axisymmetric beam that

$$\left\langle x\frac{\partial \phi}{\partial x}\right\rangle = \frac{-\lambda}{8\pi\varepsilon_0} \qquad \text{where} \quad \lambda = 2\pi\int_0^\infty \rho(r)\, r\, dr$$
= beam line charge.

Hint:

$$\left\langle x\frac{\partial \phi}{\partial x}\right\rangle = \frac{1}{2}\left\langle r\frac{\partial \phi}{\partial r}\right\rangle \quad \text{for axisymmetric beam}$$

$$\langle f(r)\rangle = \frac{2\pi}{\lambda}\int_0^\infty f(r)\,\rho(r)\, r\, dr \quad \text{for any function } f(r)$$

Use these and Poisson's equation.

5 pts

c) Use the results in a) and b) to obtain for an axisymmetric beam

$$r_b = 2\langle x^2\rangle^{1/2} = 2\langle y^2\rangle^{1/2}$$

evolving with

$$\kappa_x = \kappa_y = \kappa(s)$$

that

$$r_b'' + \frac{(\gamma\beta)'}{(\gamma\beta)} r_b' + \kappa\, r_b - \frac{Q}{r_b} - \frac{16\,\varepsilon_{x,rms}^2}{r_b^3} = 0$$

$$Q \equiv \frac{q\lambda}{2\pi\varepsilon_0 m\gamma^3\beta^2 c^2} = \text{Dimensionless perveance.}$$

This result shows that the envelope equation can apply, as expressed, to any self-consistently evolving nonuniform beam if the self-consistent emittance evolution contained in $\varepsilon_{x,rms}$ is known.

Should we expect $Q = \text{const}$ and $\varepsilon_{x,rms} = \text{const}$ for an accelerating beam with <u>linear</u> (uniform density beam) space-charge forces? Why? Qualitative only needed. You may wish to review Problem Set #6, Problem #1.

5 pts

d) For a uniform density beam show that

$$-\frac{\partial \phi}{\partial x} = \frac{\lambda}{2\pi\varepsilon_0}\frac{x}{r_b^2} \qquad 0 \le r \le r_b \qquad \text{(inside beam)}$$

$$-\frac{\partial \phi}{\partial y} = \frac{\lambda}{2\pi\varepsilon_0}\frac{y}{r_b^2}$$

corresponding to linear forces.

5 pts

e) Show that for a uniform density nonrelativistic beam, that:

$$Q = \frac{q\lambda}{2\pi\varepsilon_0 m\gamma^3\beta^2c^2} \simeq \frac{q\Delta\phi}{\mathcal{E}}$$

$$\Delta\phi = \phi(r=0) - \phi(r=r_b)$$

$$\mathcal{E} = \text{Axial kinetic energy}$$

$\Rightarrow q\Delta\phi$ = Potential energy accross beam.