Notes on Relativity

D. A. Edwards

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Preface

Many years ago as a junior faculty member at Cornell University I was teaching a course in second-year undergraduate physics for engineering students. I was sure that particle accelerators represented a valuable and developing technology. Particle accelerators run according to Special Relativity, because particle speeds approach that of light. I wanted to express my thoughts on the subject in a way that I could understand and therefore convey with some confidence to class members. Some of my notes from that period appear below.

1 Frames of Reference – Inertial Systems

In discussion of a physical process - for example, the motion of a particle under the influence of known forces - we conventionally express the results made with respect to some set of coordinate axes. Thus, the motion of a particle may be specified by three functions of time, $x(t), y(t), z(t)$ with respect to a particular Cartesian coordinate system. A coordinate system for the description of motion is also called a frame of reference

While certain frames of reference may seem natural for the discussion of a given physical problem, in principle, any coordinate system will do. In recording the motion of a billiard ball on a table, it seems most reasonable to record the positions of the ball with respect to a set of axes fixed to the table. One could, however, record positions with respect to a set of axes moving uniformly with respect to the table, or with respect to a set of axes rotating with respect to the table.

There are frames of reference in which the laws of motion take on a particularly simple form. The statement of Newtonian mechanics - Newton's First Law -that a body initially at rest will remain at rest unless acted upon by a force is not true in all frames of reference. Someone living on a rotating disk will notice that objects set down will tend to slide away; from the perspective of this observer, Newton's First Law is not valid.

The frames of reference for which Newton's First Law holds are called inertial systems or inertial frames of reference. For many purposes, such as the discussion of the motion of a billiard ball on a table, the earth's surface is a very good approximation to an inertial frame, although for others such as missile ballistics the circumstance that the earth rotates must be taken into account. The notion of an inertial frame is in some sense an idealization or abstraction which must be examined in context.

Given one inertial frame, any coordinate system moving uniformly with respect to it will also be an inertial frame. For if an object is at rest in an inertial system, from the point of view of observers in a frame that is moving at constant velocity with respect to the first, the object will be moving at constant velocity with respect to their frame; that is Newton's First Law will also be valid from their point of view. If one inertial frame, there will be many.

2 The Principle of Relativity

In applying Newton's Laws of Motion it is necessary as indicated above that the frame of reference with respect to which coordinates are measured be an inertial one. No particular inertial frame is singled out from the many and our experience confirms that these laws work just as well in a uniformly moving train or airplane as they do for measurements made with respect to the surface of the earth. That this is so is consistent with the transformations which we normally apply to relate measurements made in one frame of reference to measurements made in a second frame of reference which is moving at constant velocity with respect to the first.

In Fig. 1 are two Cartesian coordinate systems. The system with axes x, y which we will refer to

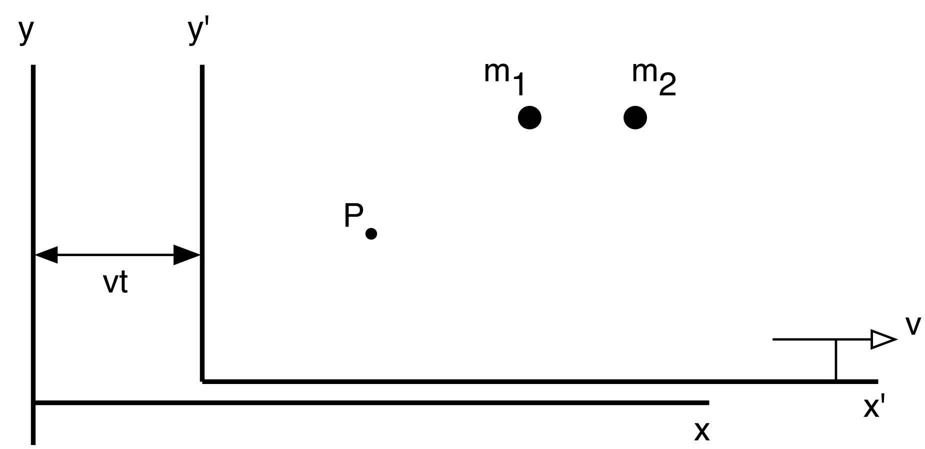

Figure 1: Cartesian Coordinate Systems.

as S is at rest with respect to the paper. The system with axes x', y', which we will refer to as S', is moving to the right with constant speed v. The x and x' axes coincide – for schematic purposes, they are shown with a small vertical separation. If the y and y' axes crossed at the zero of time, then we would expect the coordinates of a point in S to be related to its coordinates in S' to be

$$
\begin{aligned}
x' &= x - vt && (1)\\
y' &= y && (2)\\
t' &= t && (3)
\end{aligned}
$$

The third relation, which would not have ordinarily been written down in the days of Newton, asserts that the time as recorded by clocks everywhere in S is the same as the time of clocks everywhere in S'. These relations are often called the Galilean transformations.

Now in Fig. 1 suppose that m_1 and m_2 are the masses of two particles located at the points indicated. Using Newton's Law of Gravitation, the equation of motion for the particle of mass m_1 would be

$$m_1 \frac{d^2 x_1}{dt^2} = G \frac{m_1 m_2}{(x_1 - x_2)^2} \tag{4}$$

where we have written this equation with respect to S. Using the Galilean transformations, we can express Eq. 4 with respect to S'. Since

$$
\begin{aligned}
x'_1 - x'_2 &= x_1 - x_2 && (5)\\
\frac{d^2 x'_1}{dt^2} &= \frac{d^2 x_1}{dt^2} && (6)
\end{aligned}
$$

that gives

$$m_1 \frac{d^2 x'_1}{dt^2} = G \frac{m_1 m_2}{(x'_1 - x'_2)^2}. \tag{7}$$

But Eq. 7 is exactly what we would have written down as the equation of motion with respect to S' in the first place (the masses and the gravitational constant are assumed, in Newtonian mechanics, to be independent of the state of motion). That is, the Galilean transformations assure the validity of Newton's Law of Gravity in all inertial frames.

The statement that any inertial frame is just as good as any other inertial frame for the validity of the laws of physics is the principle of relativity. In the next section, we shall attempt to widen the range of applicability.

3 The Problem of the Velocity of Light

Earlier in this course, we have seen as a consequence of Maxwell's equations that electromagnetic radiation propagates in vacuum with a velocity given by

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 3 \times 10^8 \text{ m/sec.} \tag{8}$$

It is natural to ask with respect to what frame of reference is this velocity to be measured, for the Galilean transformations will not permit the velocity to be the same with respect to all frames. Referring again to Fig. 1, if a light signal is traveling to the right with velocity c as measured in S, the velocity as measured in S' will be $c - v$. This of course implies the rejection of the principle of relativity for electromagnetic phenomena, since some one frame of reference, presumably an inertial system, is singled out with respect to which Maxwell's equations are to be applied.

One of the most celebrated experiments in the history of physics, the Michelson-Morley experiment, was carried out in order to verify the existence of the preferred frame of reference. Their approach was as follows. Since the earth moves in an orbit around the sun, it is unreasonable to suppose that the earth is at rest in this preferred frame. The orbital velocity of the earth is some 30 kilometers per second; therefore if one measures the speed of light along some fixed path on the earth's surface one should obtain a different result throughout the year. During the year, the speed of light should exhibit excursions of 30 kilometers per second about the average value. (The average value need not be as stated in Eq. 8 since the sun need not be at rest in the preferred frame.) Actually, the speed of light along a single path could not be measured to the required accuracy; Michelson and Morley compared the speed of light along two paths at right angles to one another by means of an interferometric technique, and expected to find a difference of 30 kilometers per second at some time during a six month period. Their result, however, was negative; the speed of light did not exhibit the expected variation.

As an explanation of this negative result, it was suggested that the preferred frame was somehow swept along by the earth. Another observation, the "aberration of starlight" denies this excuse. In the course of a year, distant stars change their position by some 41 seconds of arc. This angular deviation is consistent with v/c, where v is the speed of the earth in its orbit.

These considerations and others of a similar nature led Einstein, in a paper published in 1905, to investigate the consequences of asserting that the velocity of light is the same in all inertial frames, which had as an immediate consequence the abandonment of the Galilean transformations and modification to Newtonian mechanics. These consequences, while extensive, are not quite as destructive as they may sound, for we know very well that the Galilean transformations and Newton's laws work very well for phenomena involving the moderate velocities with which we are familiar from everyday experience. As we shall see, Einstein's assumptions lead to modifications of these ideas for processes where one encounters velocities which are a significant fraction of the speed of light, and the older ideas appear as approximations – usually very good approximations – to the more complete notions of space, time, and mechanics to which these assumptions lead us.

4 The Relative Character of Simultaneity

It is clear from the arguments of the preceding section that Einstein's assumption, the consequences of which are usually called the Special Theory of Relativity, lead to a modification of the law of addition of velocities. A simple example will indicate, however, that much more than this must be revised. Consider again, in Fig. 2, the two frames S and S'. At the instant the origins are superimposed, let a light flash be emitted from the common origin. Since the velocity of light is independent of motion of the source, it does not matter which frame we assume the source to be attached to. We have drawn the sketch at a time somewhat after the emission of the light flash, and the position of the wave front is drawn as a dotted line. But the word "time" presents a problem.

If x is the distance that the light flash has moved along the x-axis in the time interval t after the

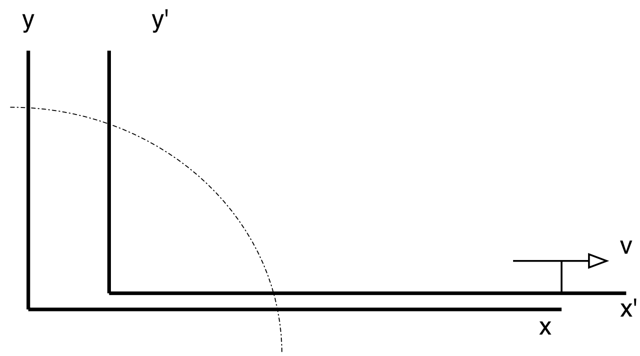

Figure 2: Light Flash.

crossing of the origins, then $x/t = c$. From the point of view of an observer in S', the light flash will have moved some distance x' along the x' axis, where x' is less than x. The time interval recorded by an observer moving with S' one might suppose would be t, the same as the time interval recorded by observers in S. But if $x/t = c$ then x'/t cannot be c, in contradiction to the assumption that the velocity of light must be the same in all such frames. Therefore, the time interval recorded by observers in S' cannot be the same as t, but something less. We must abandon the notion of a uniform "public" time which is the same for all observers, regardless of their state of motion.

A further conclusion can be drawn from Fig. 2. We have drawn the wave front as a circle with center at the origin of S, as it must be from the point of view of observers in S. But since observers in S' have an equal right to the assertion that the speed of light is c in all directions, they must also see the wave front as a sphere with its center at the origin of S', which it certainly is not in the sketch. This tells us that although the various points on the dotted line representing the wave front are all reached at the same time from the point of view of observers in S, these positions must be reached at different times from the point of view of observers in S' – that is, events which are simultaneous in one frame need not be simultaneous in another frame moving relatively to the first.

An example due to Einstein indicates rather clearly the relative character of simultaneity and certain of its consequences. Suppose that we wish to measure the length of a moving train. We can do this by stationing a number of observers by the side of the track with the instructions that at some specified time, say exactly 2:00 PM, the two observers who find an end of the train opposite their positions are to record their position; we can assume that we have previously marked off a length scale along the track for this purpose. From the preceding example, we are aware that there may be difficulties in the specification of times for various observers; we should therefore specify exactly how the clocks of the observers situated along the track are to be synchronized. Fortunately, we have a standard velocity which can be used for this purpose. The observers are instructed that when the clock located at the zero position of the coordinate axis reads exactly 1:00 PM, a light signal will be sent out from this position. Since the speed of light is exactly c, the time at which the light signal reaches a distance x from the origin will be x/c later. So the observer at x sets his clock accordingly when he detects the light flash and is confident that his or her clock is synchronized with the one at the origin.

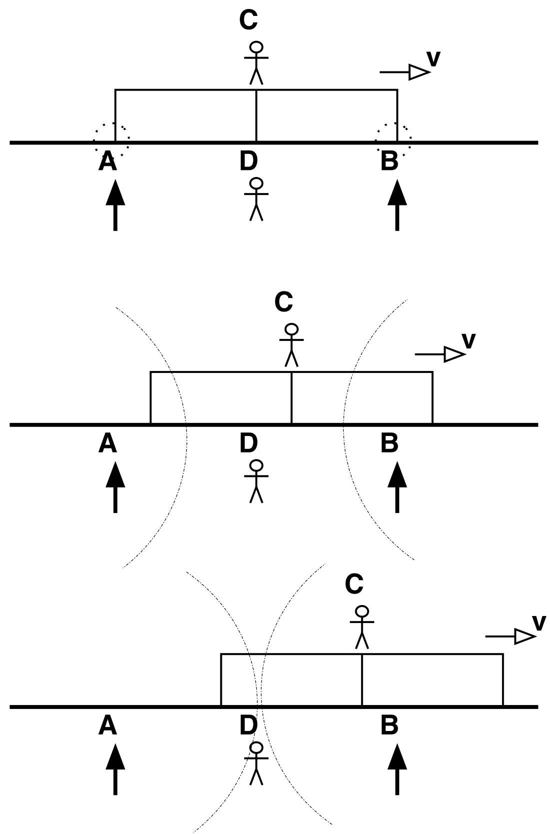

Figure 3: Moving Train.

Now suppose that, in addition to recording the time at their positions, the two observers who find themselves at opposite ends of the train at 2:00 PM also send out light flashes (or radio signals) at that time. The observer who at 2:00 PM had been opposite the midpoint of the train in S would receive these signals simultaneously. An observer standing on the train at its midpoint would not detect the two light flashes simultaneously as we may see from Fig. 3. The leftmost of the sketches is drawn at 2:00 PM – 2:00 PM so far as the observers standing by the tracks are concerned. The observers C and D are confronting each other; C is standing on the train at its midpoint, and D is the observer standing by the track who is opposite the midpoint at 2:00 PM. The middle sketch shows the situation a short time later, with the light flashes moving away from their sources. In the rightmost sketch, still later, the two wave fronts are reaching D simultaneously. This implies that the flash from the front end of the train has already passed the observer C, while the flash from the back end of the train has not yet reached him or her. The observer on the train will of necessity conclude that the light flashes were not emitted at the same time. Thus, events that are simultaneous from observers in one inertial frame of reference need not be simultaneous for measurements conducted in another inertial frame.

5 Comparison of Lengths and Clock Rates

We must now give quantitative expression to the ideas raised above, and lay the groundwork for the coordinate transformations that will replace the Galilean transformations.

5.1 Distances at right angles to the direction of motion

Referring back to Fig. 1, if the distance from the point P from the x' axis is y' as measured in S', then the Galilean transformation assures us that the corresponding distance as measured in S will be the same. That this relation will remain true we can see from the following argument.

Suppose that we have marked off distances on the y and y' axes with rulers which were identical when at rest with respect to each other. Now we place observers in the S frame at the origin and at the positions $\pm h$ on the y axis, These observers are given instructions to note the position on the y' axis which corresponds to theirs when the y' axis passes their position, and also, when this happens, to send out a light signal. If the observer at $+h$ sees $+h'$ on the y' axis, then the observer at $-h$ must certainly see $-h'$ passing, otherwise that would imply some inherent difference in space above and below the x-axis. The two light signals will reach the observer and the origin of S at the same time, for the assumption that we orient the y' axis parallel to to the y axis implies that the crossing of the axes will be simultaneous events in S at $\pm h$.

The two light flashes will also reach an observer in S' at the origin of S' simultaneously, for, referring to Fig. 4, according to observers in S, the two light flashes emitted simultaneously at points A and

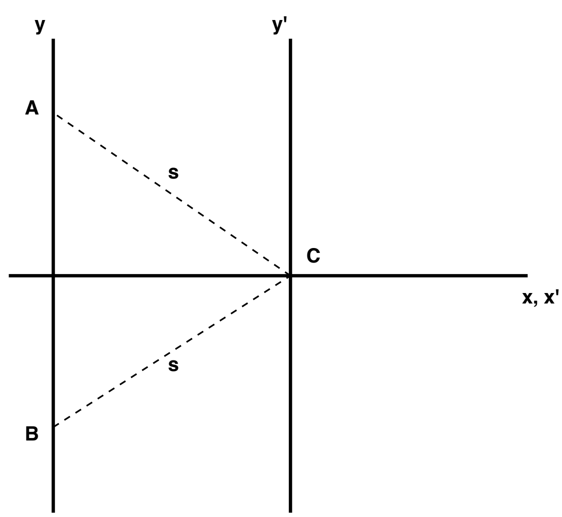

Figure 4: Transverse Coordinate.

B, after traveling the same distance s, will arrive simultaneously at an observer C, at rest in S, at the position on the x-axis corresponding to the origin of the S' frame. Since the coincidence of the two wave fronts at a position in space is an event which must be independent of the state of motion of observers, the observer on S' at its origin will also the detect the arrival of the two light flashes simultaneously, and will conclude that the observers at $\pm h$ in S have made a valid measurement of the distance $2h'$ on the y' axis not only from their own point of view but also from the point of view of observers in S'. To put it another way, observers in S' stationed at $\pm h'$ would agree with observers in S that the positions $\pm h$ on the y-axis correspond simultaneously with their own. In that case, the only result consistent with the principle of relativity is that $h = h'$. For if observers in S found that h was, say, greater than h', the observers in S' would agree with them, which would imply some fundamental distinction between the two frames. Hence, the relation $y' = y$ will remain valid for the new transformations.

It is important to realize the difference between the situation described above and the measurement of the length of the train described above. In the case of the train, the simultaneous recording of the positions of the two ends of the train by the observers situated by the track was not simultaneous for observers traveling with the train; it was therefore possible for the two sets of observers to arrive at different conclusions concerning the length.

5.2 Comparison of clock rates

Note that in comparing the rates of clocks which are attached to coordinate frames in relative motion, we cannot compare the rate of a single clock in one frame directly with a single clock in the other, since the two clocks will not stay at the same place. The best we can do is compare the time interval elapsed for a single clock in one frame with the difference in time between two separate clocks in the other frame which have been synchronized with light signals.

We can determine the relation between these time intervals in the following fashion. In Fig. 5 representing the usual two frames, a mirror is shown attached to the y' axis at a height h above the

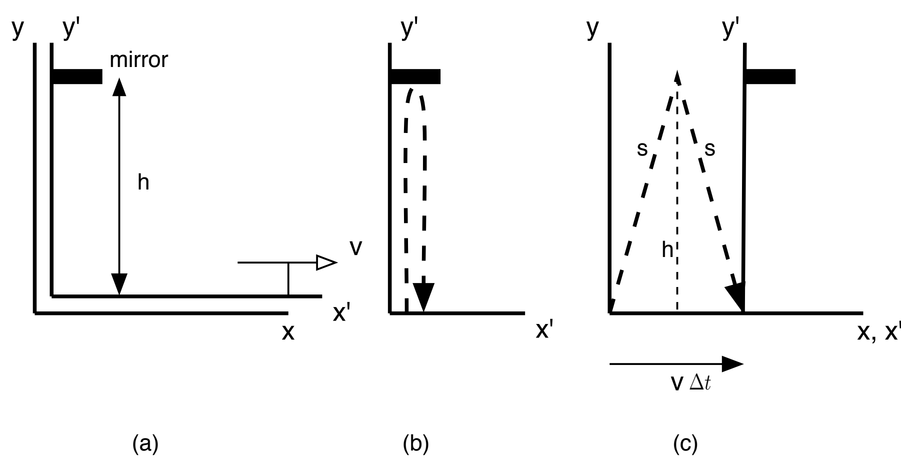

Figure 5: Light Clock.

origin of S'. According to the argument above concerning distances at right angles to the direction of motion, the distance h will be the same in the two frames. Suppose that a light flash is emitted from the common origin of the two systems at the instant they cross, and observers in the two systems measure the time interval for the signal to return to the x and x' axes. In S', the light signal will be detected at the origin after it has moved up the y' axis, been reflected, and returned. The time interval elapsed for the clock located at the origin of S' will be $\Delta t' = 2h/c$.

For observers in S, the light signal will follow the dotted path in the figure; the time at which the signal returns to the x-axis as recorded by the clock at A will be

$$\Delta t = \frac{2s}{c} = \frac{2}{c}\left[h^2 + \frac{v^2 \Delta t^2}{4}\right]^{1/2} \tag{9}$$

and so

$$\Delta t' = \frac{\Delta t}{\gamma}, \qquad \gamma \equiv \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} \ . \tag{10}$$

The time interval recorded by the single clock in S' is therefore shorter than this time difference between the two clocks in S'; this result is a direct consequence of the assertion that the velocity of light is the same in the two frames. The quantity γ is called the Lorentz factor.

We may obtain from Eq. 10 the time interval recorded by a clock attached to a body which is moving in any manner with respect to an inertial frame S. Even though the velocity of the body may not be uniform with respect to S, for a sufficiently short time interval dt, as measured in S, we may consider the body to be at rest in some inertial frame which is in motion with respect to the body at that instant. The time interval dt' for the clock attached to the body corresponding to dt will be

$$dt' = \frac{dt}{\gamma(v)}. \tag{11}$$

If we conceptually perform this operation throughout the motion of the body, the total time elapsed for the moving clock will be

$$\Delta t' = \int_{t_1}^{t_2} \frac{dt}{\gamma(v)} \tag{12}$$

there $t_2 - t_1 = \Delta t$ is the time interval recorded by clocks in S. The time recorded by a clock attached to a body is called the proper time. From Eq. 10 we see that proper time intervals are always shorter than time intervals recorded by clocks in frames with respect to which one is in motion.

A striking confirmation of the predictions above is provided by the rapidly moving radioactive particles such as pi mesons. The lifetime of such mesons which are moving with velocities at a significant fraction of the speed of light are found to be greater than the same variety of particle at rest by just the factor predicted above.

5.3 Distances parallel to the direction of motion; the Lorentz contraction

Suppose we lay out a distance L on the x-axis of S. An observer located at the origin of S' can measure this interval by noting the time $\Delta t'$ required for this piece of the x-axis to go by the positions; since the relative velocity of the frames is v, the observer would conclude that the interval is of length $L' = v\Delta t'$. In S, the time for the origin of S' to travel the distance L would be L/v. From the preceding discussion of clock rates, however, the time interval $\Delta t'$ recorded by the single clock at the origin of S' must be related to the time interval Δt by the Lorentz factor. So $L' = L/\gamma$.

This astonishing result was put forth by Lorentz and Fitzgerald in order to account for the negative result of the Michelson-Morley experiment, and preceded Einstein's conclusive 1905 paper. So the effect is often referred to as the Lorentz-Fitzgerald contraction.

6 The Lorentz Transformation

6.1 Transformation equations

We may make use of the results of the preceding section to find the new coordinate transformations. Suppose that the point P in Fig. 6 has the coordinates x', y' at time t', all as measured in S'. The

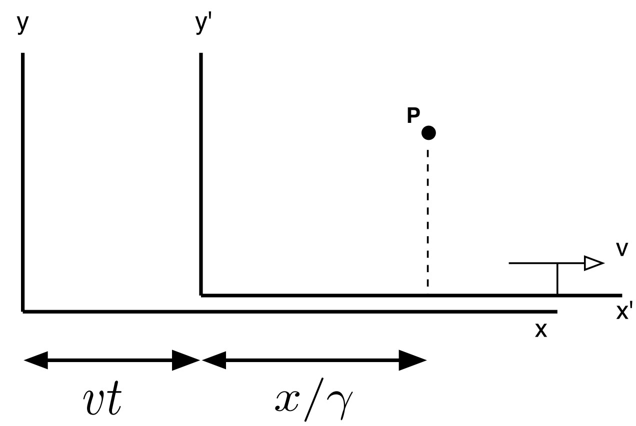

Figure 6: Lorentz Transformation.

figure is drawn at the time t for observers in S. As measured in S, x' will be shortened by the factor γ; the x coordinate of P will therefore be $x = vt + x'/\gamma$. After rearrangement, $x' = \gamma(x - vt)$. This result differs from the Galilean transformation by the factor of γ and reduces to the Galilean transformation for speeds much less than c as one would expect.

Similarly, the inverse relation which for any x' and t' yields x will be $x = \gamma(x' + vt')$. Finally, the time t' recorded by a clock in S' at the position corresponding to the x, t in S may be found by elimination of x' from these last two equations, yielding $t' = \gamma(t - xv/c^2)$. The replacements to the

Galilean transformations are

$$x' = \gamma(x - vt) \tag{13}$$
$$y' = y \tag{14}$$
$$t' = \gamma\left[t - \frac{xv}{c^2}\right]. \tag{15}$$

These relations were obtained by H. A. Lorentz in 1904, although their full significance for the relativity of motion was not realized at that time. Einstein, unaware of the work of Lorentz, derived them independently in his 1905 treatment of relativity.

6.2 Transformation of velocity

If no velocity can exceed c, then the Galiean addition of velocities needs to be replaced. Suppose that a particle is traveling to the right in S' parallel to the x' axis with speed u'_x as measured in S'. Without loss of generality, we can assume that the particle started from $x' = 0$ at $t' = 0$. Then $x' = u'_x t'$ gives the position of the particle in S' as a function of time. Applying the Lorentz transformations yields

$$x = \frac{u'_x + v}{1 + u'_x v/c^2} t. \tag{16}$$

so for observers in S, the particle is moving with speed

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}. \tag{17}$$

As u'_x approaches c then so does u_x also, as consistent with the limit of the speed of light.

If there is a vertical velocity component, $y' = u'_y t'$, then application of the Lorentz transformation yields

$$u_y = \frac{u'_y}{1 + \frac{u'_x v}{c^2}}. \tag{18}$$

6.3 The limit of the speed of light

The continually occurring γ becomes infinite for $v = c$, and imaginary for speed above c. We know from particle accelerators that we can accelerate particles close to the speed of light, so we conceive of a frame S' containing the usual observers, clocks, and coordinate axes moving with a speed very close to that of light. In S, the position of a fast signal is given by $x = ut$. For observers in S', the signal left the origin of S' at $t' = 0$. The time t' at which the signal arrives at a point x' coincident with a point on the x-axis will be

$$t' = \gamma\left(t - \frac{xv}{c^2}\right) = \gamma t\left(1 - \frac{uv}{c^2}\right). \tag{19}$$

If we set v very close to c, we have

$$t' = \gamma t\left(1 - \frac{u}{c}\right). \tag{20}$$

If u were greater than c, a signal will be received on the x' axis at times earlier than zero; that is, observers in S' would receive the signal before it was sent, in violation of the normal cause and effect relation of events.

7 Newton's Second Law and $E = mc^2$

Here, I have to change my language from that of a half-century ago, because that language is quite properly out of date. Back then, there were words about the variation of mass with velocity which introduced confusion into issues of invariance. Today, particle attributes such as charge and mass are regarded as invariants not subject to modification by the Lorentz transformation. Respect for Maxwell's equations supports this point of view regarding charge invariance; that could be the subject for another set of notes.

Newton's Second Law which states that a force leads to a time change of momentum is preserved by a new definition of momentum. The old definition of $p = mv$ becomes $p = \gamma mv$ and Newton's Second Law remains valid, though with totally different consequences. Consistent with observation, a particle undergoing persistent acceleration will not exceed the speed of light regardless of the force applied. The Second Law is still $F = dp/dt$. But the definition of force is now consistent with $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$, as it must be given the primacy of Maxwell's equations. Newton's Law of Gravitation remains intact as a force law at speeds low compared with the speed of light.

If a particle is accelerated from rest to a velocity v, the kinetic energy of the particle remains defined as

$$T = \int F \cdot ds = \int \frac{dp}{dt} ds \tag{21}$$

For a particle of mass m, the result is

$$T = (\gamma - 1)mc^2 \tag{22}$$

which reduces in the limit of low speed to the familiar $(1/2)mv^2$.

Then in what was a wonderfully imaginative leap, Einstein endowed a particle with a "rest energy" $E = mc^2$ in recognition of the prospects of radioactivity only then underway and other processes even now awaiting exploration. The "total energy" becomes γmc^2 which with the replacement of γ with the momentum leads to the more general expression

$$E = \sqrt{p^2c^2 + m^2c^4}. \tag{23}$$