Particle Equations of Motion
(Magnetic Optics and Bends)

Consider a long magnet where we can approximate the fields as 2D transverse within the magnet:

$$\boxed{\vec{B} = B_x(x,y)\,\hat{x} + B_y(x,y)\,\hat{y}}$$

$$\nabla\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ B_x & B_y & 0 \end{vmatrix} = \hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}}$$

Taylor expand the field about $x = y = 0$ for small x, y:

$$\vec{B} = B_x\,\hat{x} + B_y\,\hat{y}$$

$$\simeq \Big[\underbrace{B_x(0)}_{\substack{\text{Choose symmetry} \\ \text{to } 0 \\ \text{(no y-plane bends)}}} + \frac{\partial B_x}{\partial y}\Big|_0 y + \underbrace{\frac{\partial B_x}{\partial x}\Big|_0}_{\substack{\text{Choose symmetry} \\ \text{to } 0 \\ \text{(skew term)}}} x + \underbrace{\cdots}_{\text{Neglect}}\Big]\hat{x} + \Big[B_y(0) + \frac{\partial B_y}{\partial x}\Big|_0 x + \underbrace{\frac{\partial B_y}{\partial y}\Big|_0}_{\substack{\text{Choose symmetry} \\ \text{to } 0 \\ \text{(skew term)}}} y + \underbrace{\cdots}_{\text{Neglect}}\Big]\hat{y}$$

Denote

$$\boxed{B' = \frac{\partial B_y}{\partial x}\Big|_0 = \frac{\partial B_x}{\partial y} = \text{Quadrupole Gradient}}$$

Then

$$\boxed{\vec{B} \simeq B' y\,\hat{x} + \left[B_y(0) + B' x\right]\hat{y}}$$

$B' \neq 0 \Leftrightarrow$ Quadrupole Magnet Focus / Defocus

$B_y(0) \neq 0 \Leftrightarrow$ x-plane bend of particle trajectory

Lorentz Force Eqn

$$\frac{d\vec{p}}{dt} = q\,\vec{v}\times\vec{B} \qquad \vec{p} = m\gamma\vec{v} = m\gamma\dot{\vec{x}} \qquad \dot{} \equiv \frac{d}{dt}$$

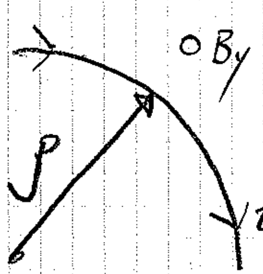
For motion in magnetic field, particle energy does not change

$$\gamma = \text{const} \qquad \frac{d\vec{p}}{dt} = m\gamma\ddot{\vec{x}}$$

$$\boxed{\gamma m\ddot{\vec{x}} = q\,\dot{\vec{x}}\times\vec{B}}$$

Case 1.) If $B_y(0) \neq 0$ and $B' = 0$

Particle will bend in a uniform magnetic field on a circular arc.



$v_s = |\dot{\vec{x}}| = \text{const}$

speed const along arc.

$$\gamma m\ddot{\vec{x}} = q\,\dot{\vec{x}}\times B_y(0)\,\hat{y}$$

$$-\gamma m\frac{v_s^2}{\rho} = -q\,v_s B_y(0)$$

$$\Rightarrow \boxed{\begin{aligned} &\frac{1}{\rho} = \frac{B_y(0)}{(B\rho)} \quad \text{Bend Radius.} \\ &(B\rho) = \frac{p}{q} = \frac{\text{momentum}}{\text{charge}} \equiv \text{Rigidity} \end{aligned}}$$

Dipole Bends used to manipulate "reference" path
- Rings
- Transfer lines

and also, various focusing properties for non-ref path orbit (later).

Rigidity

$$\boxed{(B\rho) = \frac{p}{q} = \frac{\gamma m v}{q} = \frac{mc}{q}\gamma\beta = \frac{\text{momentum}}{\text{charge}} \equiv \text{"Rigidity"}}$$

Read as one symbol "B-rho"!

Is a convenient measure of the coupling strength of particles to applied magnetic fields. Usually measured

$p = \gamma m v$

$p_\mu = \left(\frac{\mathcal{E}}{c}, p\right) \qquad \mathcal{E} = \gamma m c^2 \qquad$ 4-vector $\qquad \mathcal{E} = mc^2 + W \qquad W =$ kinetic Energy

$p_\mu p^\mu = \underbrace{\frac{\mathcal{E}^2}{c^2} - p^2}_{\text{4-vector contraction}} = \underbrace{(mc)^2}_{\text{rest frame evaluation of Lorentz Invariant}}$

$\rightarrow \; cp = \sqrt{\mathcal{E}^2 - (mc^2)^2} = \sqrt{W^2 + 2Wmc^2}$

Alternative derivation

$W = (\gamma-1)/mc^2 \rightarrow \gamma = 1 + \frac{W}{mc^2}$

$\gamma^2 = \frac{1}{1-\beta^2} \rightarrow \beta^2 = 1 - 1/\gamma^2$

$\gamma\beta = \sqrt{\gamma^2 - 1}$

$\gamma\beta = \sqrt{\left(\frac{W}{mc^2}\right)^2 + 2\left(\frac{W}{mc^2}\right)}$

$$\boxed{(B\rho) = \frac{p}{q} = \frac{mc}{q}\gamma\beta = \frac{\sqrt{W^2 + 2Wmc^2}}{qc} = \frac{mc}{q}\sqrt{\left(\frac{W}{mc^2}\right)^2 + 2\left(\frac{W}{mc^2}\right)}}$$

Electrons: $\quad mc^2 = 511$ keV $\quad q = -e$

$\frac{mc}{q} = -\frac{mc}{e} = -1.7045 \; 10^{-3}$ Tesla-m $\quad \Rightarrow \quad (B\rho) = -1.7045\times 10^{-3}\sqrt{\left(\frac{W}{mc^2}\right)^2 + 2\left(\frac{W}{mc^2}\right)}$ Tesla-m

Ions: $\quad m = A m_u \qquad m_u c^2 = 931.49$ MeV

$q = Qe$

$\frac{mc}{q} = \frac{A}{Q}\frac{m_u c}{e} = 3.107\left(\frac{A}{Q}\right)$ Tesla-m $\quad \Rightarrow \quad (B\rho) = 3.107\left(\frac{A}{Q}\right)\sqrt{\left(\frac{W}{mc^2}\right)^2 + 2\left(\frac{W}{mc^2}\right)}$ Tesla-m

Case 2) If $B_y(0) = 0$ and $B' \neq 0$

Quadrupole magnet with no bend.

$$\gamma m \ddot{\vec{x}} + q \dot{\vec{x}} \times \vec{B}$$

$$\gamma m (\ddot{x}\hat{x} + \ddot{y}\hat{y})_\perp = q \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B'y & B'x & 0 \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}_\perp = -q \dot{z} B' x \hat{x} + q \dot{z} B' y \hat{y}$$

But

$$\dot{} = \frac{d}{dt} = \frac{dz}{dt}\frac{d}{dz} = v_z \frac{d}{dz} \approx v \frac{d}{dz}$$

$v \approx$ const
motion primarily axially directed.

Using this our equations become

$$\gamma m \, v \frac{d}{dz} v \frac{d}{dz} x = -q v B' x$$

$$\gamma m \, v \frac{d}{dz} v \frac{d}{dz} y = q v B' y$$

$$\Rightarrow \boxed{\begin{aligned} \frac{d^2x}{dz^2} + \frac{B'}{(B\rho)} x &= 0 \\ \frac{d^2y}{dz^2} - \frac{B'}{(B\rho)} y &= 0 \end{aligned}}$$

For purposes of later notational uniformity we take

$z = s$ = ref. trajectory (centerline) coordinate

$$\frac{B'}{(B\rho)} = \kappa(s) = \text{Lattice focusing Function}$$

$$[\kappa] = \frac{1}{(\text{length})^2}$$

Allow B' to vary in s

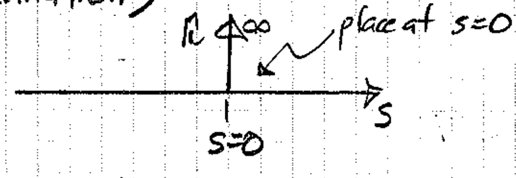
Recall $(B\rho) = \frac{\cancel{\gamma} m v}{q} = \frac{p}{q}$ = Rigidity

Hills' Equation

$$\Rightarrow \boxed{\begin{aligned} \frac{d^2x}{ds^2} + \kappa x &= 0 \\ \frac{d^2y}{ds^2} - \kappa y &= 0 \end{aligned}}$$

Thin Lens Kick

Consider a short quadrupole of axial length $\ell \to 0$
But keep $\mathcal{K}\ell$ finite. (Kick Approximation)



$$\frac{d^2x}{ds^2} + \mathcal{K}x = 0$$

$$\left.\begin{aligned} x(0^-) &= x_0 \\ x'(0^-) &= x_0' \end{aligned}\right\} \text{ initial conditions}$$

$$\int_{0^-}^{0^+} \frac{d^2x}{ds^2}\,ds = -\int_{0^-}^{0^+} \mathcal{K}\,x\,ds$$

$$x'(0^+) - \underbrace{x'(0^-)}_{x_0'} = -\mathcal{K}\ell\, x(0)$$

Integrate again

$$x(0^-) = x(0^+) = x_0$$

Summarize results in matrix form:

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{0^+} = \begin{bmatrix} 1 & 0 \\ -\mathcal{K}\ell & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

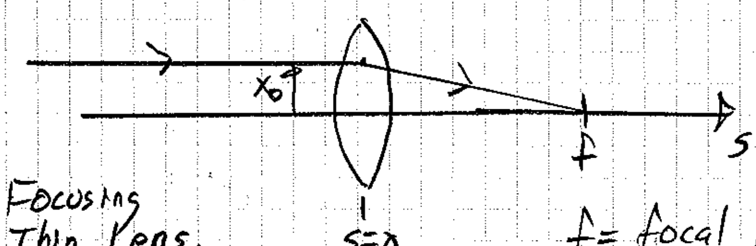
-or-

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{0^+} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{0^-}$$

$$\frac{1}{f} = \mathcal{K}\ell$$

sometimes write as

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{s} = \underbrace{\bar{M}(s|s_i)}_{2\times2 \text{ Transfer Matrix}} \begin{bmatrix} x \\ x' \end{bmatrix}_{s_i}$$



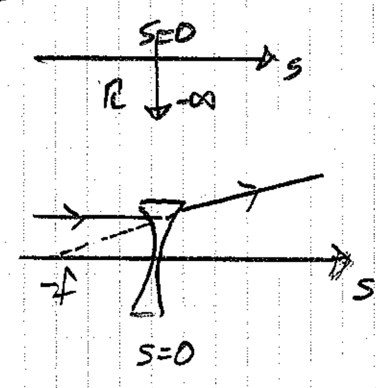
Focusing Thin Lens.

f = focal length

Thin lens kick changes angle but not coordinate.
Interpreted analogously to a thin lens in light optics.

Similarly, for the defocus plane: $\mathcal{K} \to 0$, $\mathcal{K}l$ finite kick approx:

$$\frac{d^2 y}{ds^2} - \mathcal{K} y = 0$$



Defocusing Thin Lens

leads to

$$\begin{bmatrix} y \\ y' \end{bmatrix}_{0^+} = \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}_{0^-}$$

$$\frac{1}{f} = \mathcal{K}l$$

When the quadrupole polarity is reversed

$$B' \to -B' \qquad \mathcal{K} \to -\mathcal{K}$$

and the focusing (x) and defocusing (y) plane solutions are exchanged.

In a drift; $\mathcal{K} = 0$

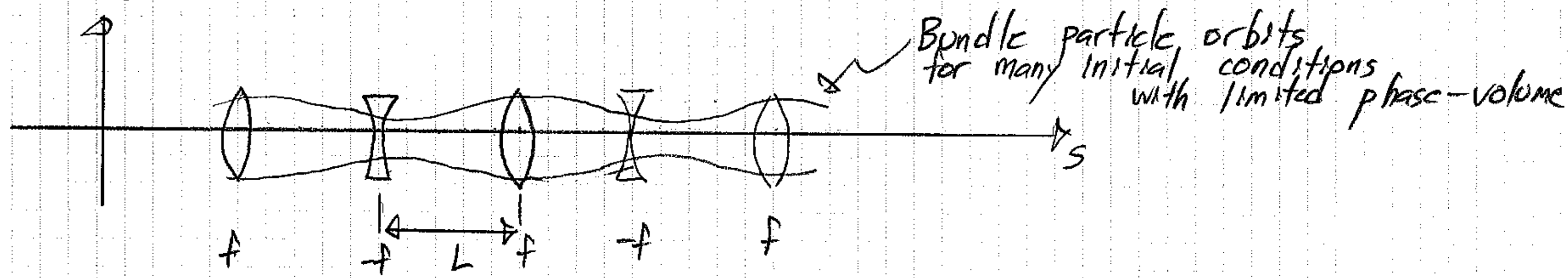
$$\frac{d^2 x}{ds^2} = 0 \quad ; \quad \Rightarrow \quad \begin{aligned} x(s) &= x_0 + x_0'(s - s_i) \\ x'(s) &= x_0' \end{aligned}$$

leads to

$$\begin{bmatrix} x \\ x' \end{bmatrix}_s = \begin{bmatrix} 1 & s - s_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{\text{initial}}$$

$$\begin{bmatrix} y \\ y' \end{bmatrix}_s = \begin{bmatrix} 1 & s - s_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}_{\text{initial}}$$

This gives the tools we need to analyze particles evolving in an alternating-gradient transport lattice.



On average
- Particles farther from axis $x=0$ in focus lens
- " closer to " " in defocus lens

$\Rightarrow$ Net focusing in both planes

Called "Alternating Gradient" Focusing or "Strong Focusing" also.

We are not so far from being able to model a real accelerator now. Will show in homework any linear optic $\overline{M}$ can be replaced by two drifts (d_1, d_2) and between a thin lens kick (f).
So appropriately chosen drifts and thin lens kicks can exactly model a linear beamline.

Now (gulp .....) how do we deal with case of simultaneous focusing ($B' \neq 0$) and bending ($B_y(0) \neq 0$).

- This will also allow us to understand focusing effects from bends when particles do not enter on design (reference) trajectory.

Case 3) $B_y(0) \neq 0$ and $B' \neq 0$

Frenet-Serret Coordinates

Edwards and Syphers

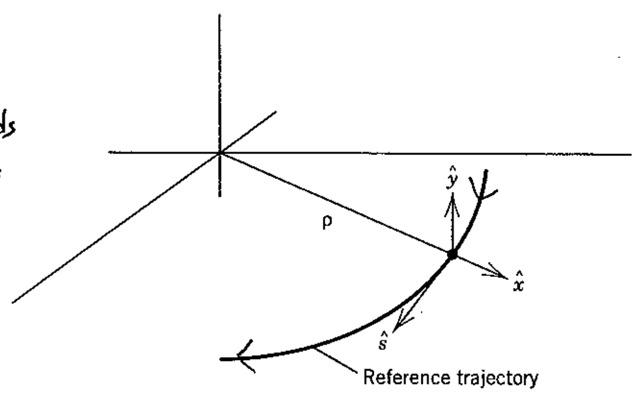


Figure 3.7. Coordinate system for development of equation of motion.

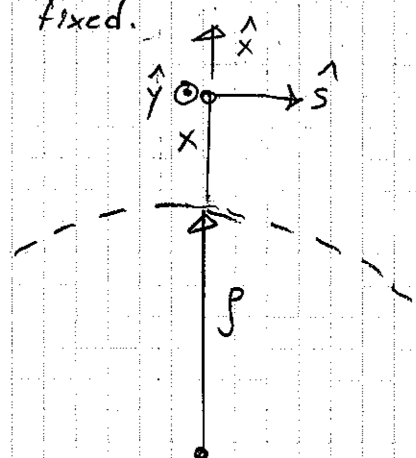
Also called "Reference Orbit"

Design (centerline) trajectory:

- Straight line down center – quadrupoles
- Circular arc segment – dipole. Where $B_y = B_y(0) = \text{const}$.

$\hat{x}, \hat{y}, \hat{s}$ unit vectors form right-hand system

- $\hat{x}, \hat{s}$ change orientation depending on location s on ref. trajectory
- $\hat{y}$ fixed.



Transverse Particle Coordinate

$$\vec{R} = r\hat{x} + y\hat{y}$$

$$r = \rho + x$$

Lorentz Force:

$$\frac{d\vec{p}}{dt} = q\,\vec{v}\times\vec{B} \qquad \vec{p} = m\gamma\,\dot{\vec{R}} \qquad \bullet = \frac{d}{dt}$$

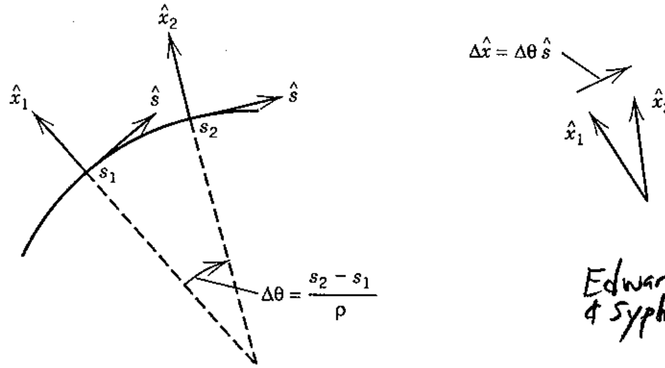
$\gamma = \text{const}$ Magnetic field only bends.

$$\boxed{m\gamma\,\ddot{\vec{R}}_\perp = q\begin{vmatrix}\hat{x} & \hat{y} & \hat{s}\\ v_x & \cancel{v_y} & v_s\\ B_x & \cancel{B_y} & 0\end{vmatrix}_\perp = -q v_s B_y\,\hat{x} + q v_s B_x\,\hat{y}}$$

Must evaluate $\ddot{\vec{R}}$ in coordinates:

$$\vec{R} = r\hat{x} + \cancel{y}\hat{y}$$
$$\dot{\vec{R}} = \dot{r}\hat{x} + r\dot{\hat{x}} + \cancel{\dot{y}}\hat{y}$$

unit vector changes orientation



Edwards & Syphers

Figure 3.8. Time rate of change of unit vector $\hat{x}$.

$$\boxed{\begin{aligned}\dot{\hat{x}} &= \dot{\theta}\hat{s} = \frac{v_s}{r}\hat{s}\\ &\text{and for later}\\ \dot{\hat{s}} &= -\dot{\theta}\hat{x}\end{aligned}} \qquad \dot{\theta} = \frac{v_s}{r}$$

Giving

$$\boxed{\dot{\vec{R}} = \dot{r}\hat{x} + \cancel{\dot{y}\hat{y}} + r\dot{\theta}\hat{s}}$$

Differentiate again

$$\ddot{\vec{R}} = \ddot{r}\hat{x} + \dot{r}\dot{\hat{x}} + \cancel{\ddot{y}\hat{y}} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s} + r\dot{\theta}\dot{\hat{s}}$$

$$\boxed{\ddot{\vec{R}} = (\ddot{r} - r\dot{\theta}^2)\hat{x} + \cancel{\ddot{y}\hat{y}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{s}}$$

Put these results in Lorentz force equation:

$$m\gamma \ddot{\vec{R}}_\perp = -q v_s B_y \hat{x} + q v_s B_x \hat{y}$$

$$\boxed{\begin{aligned} \hat{x}: \quad & \ddot{r} - r\dot{\theta}^2 = \frac{-q v_s B_y}{\gamma m} = \frac{-q v_s}{\gamma m}\left[B_y(0) + B'x\right] \\ \hat{y}: \quad & \ddot{y} = \frac{q v_s B_x}{\gamma m} = \frac{-q v_s}{\gamma m} B' y \end{aligned}}$$

Take: $|v_x| \ll v_s$, $|v_y| \ll v_s$ $\Rightarrow$ Motion primarily directed axially in beam
Paraxial Approximation

Then

$$p \simeq \gamma m v_s$$

Change from t (time) to s (ref. trajectory position) as independent variable.

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds}$$

But

$$ds = \rho\, d\theta = v_s\, dt \frac{\rho}{r} \qquad \dot{\theta} = \frac{v_s}{r}$$

$$\Rightarrow \frac{ds}{dt} = v_s \frac{\rho}{r} = \dot{\theta}\rho$$

$$\boxed{\begin{aligned} \frac{d}{dt} &= \left(\frac{ds}{dt}\right)\frac{d}{ds} = \left(\frac{v_s \rho}{r}\right)\frac{d}{ds} \\ \frac{d^2}{dt^2} &= \cancel{\left(\frac{d^2 s}{dt^2}\right)}^{\,0 \text{ neglect}}\frac{d}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2}{ds^2} = \left(v_s \frac{\rho}{r}\right)^2 \frac{d^2}{ds^2} \end{aligned}}$$

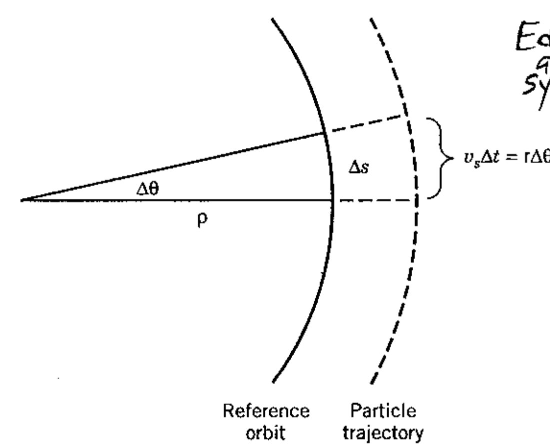


Figure 3.9. Comparison of reference orbit path length ds and particle path length $v_s\, dt$.

Edwards and Syphers

Use these results to simplify

$$\hat{x}: \quad \ddot{r} - r\dot{\theta}^2 = \frac{-q v_s}{\gamma m}\left[B_y(0) + B'x\right]$$

$$\hat{y}: \quad \ddot{y} = \frac{q v_s}{\gamma m} B' y$$

$$r = \rho + x \qquad \ddot{r} = \ddot{x} \qquad \dot{\theta} = \frac{v_s}{r}$$

$$\frac{d^2}{dt^2} = \left(\frac{v_s \rho}{r}\right)^2 \frac{d^2}{ds^2}$$

to obtain

$$\hat{x}: \quad \left(\frac{v_s \rho}{r}\right)^2 \frac{d^2 x}{ds^2} - r\left(\frac{v_s}{r}\right)^2 = \frac{-q v_s}{\gamma m}\left[B_y(0) + B'x\right]$$

$$\hat{y}: \quad \left(\frac{v_s \rho}{r}\right)^2 \frac{d^2 y}{ds^2} = \frac{q v_s}{\gamma m} B' y$$

or

$$\boxed{\begin{aligned} \hat{x}: \quad & \frac{d^2 x}{ds^2} - \frac{\rho + x}{\rho^2} = \frac{-1}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2\left[B_y(0) + B'x\right] \\ \hat{y}: \quad & \frac{d^2 y}{ds^2} = \frac{1}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2 B' y \end{aligned}}$$

Rigidity:

$$(B\rho) \equiv \frac{m\gamma v_s}{q} = \frac{p}{q}$$

Expand results to leading linear order, neglecting terms $\propto x^2$ and xy

$$\hat{x}: \quad \frac{d^2 x}{ds^2} + \left[\frac{-1}{\rho^2} + \frac{2 B_y(0)}{\rho (B\rho)} + \frac{B'}{(B\rho)}\right] x = \frac{1}{\rho} - \frac{B_y(0)}{(B\rho)}$$

$$\hat{y} \quad \frac{d^2 y}{ds^2} - \frac{B'}{(B\rho)} y = 0$$

Employ bend constraint: $\quad \dfrac{1}{\rho} = \dfrac{B_y(0)}{(B\rho)}$

Final result: Hills' equation in both planes:

$$\boxed{\begin{aligned} \hat{x}: &\quad \frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2} + \frac{B'}{(B\rho)}\right] x = 0 \\ \hat{y}: &\quad \frac{d^2y}{ds^2} - \frac{B'}{(B\rho)} y = 0 \end{aligned}}$$

often write as:

$$\frac{d^2x}{ds^2} + K_x(s)\, x = 0$$

$$\frac{d^2y}{ds^2} + K_y(s)\, y = 0$$

Lattice focusing functions:

$$K_x = \frac{1}{\rho^2} + \frac{B'}{(B\rho)}$$

$$K_y = \frac{-B'}{(B\rho)}$$

$$B' = \left.\frac{\partial B_y}{\partial x}\right|_0 = \left.\frac{\partial B_x}{\partial y}\right|_0$$

$$B' = B'(s)$$

Comment:

- x- and y-equations same mathematical form (Hill's equation) to allow general conclusions on orbit properties
- Relativity incorporated in K lattice functions but don't need to think much about relativity when solving.
- Solve for initial conditions

  $x(0) \quad x'(0)$ $\qquad s=0$ "initial" ref.

  $y(0) \quad y'(0)$

  to understand evolution of bundle of orbits.

Forms of Lattice Functions

Drift

$\mathcal{K}_x = 0$
$\mathcal{K}_y = 0$

Free expansion, straight-line trajectories

Quadrupole Magnet (no superimposed bend $\rho \to \infty$)

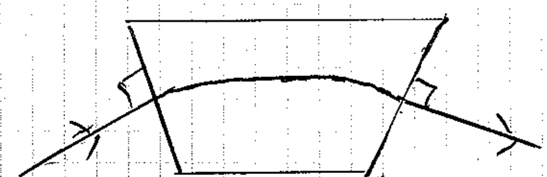
$\mathcal{K}_x = \frac{B'}{(B\rho)}$ x-plane focus

$\mathcal{K}_y = -\frac{B'}{(B\rho)}$ y-plane defocus

Sector Dipole Bend

$\mathcal{K}_x = \frac{1}{\rho^2}$ Focusing: • strong for tight bend (ρ small) • weak for small bend (ρ large)

$\mathcal{K}_y = 0$ Free expansion

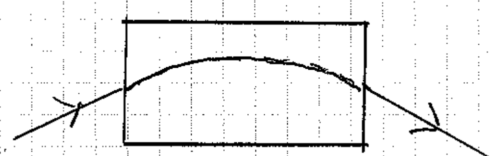


Normal incidence ref trajectory

$\mathcal{K}_x$ independent of x entering/exiting.

Other Dipoles

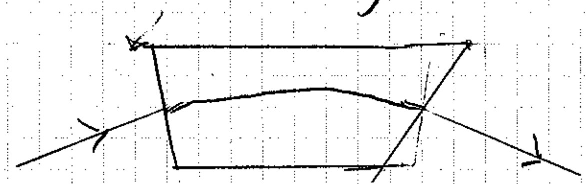
Rectangular box



Particles with $\pm x$ incident see less/more bending

Augment formulation

Slanted Edges



See Syphers notes for these cases.

See 05.lecture.pdf.

<u>More</u> Problems can de mapped to Hill's Equation with κ_x, κ_y functions

<u>Electric Quadropoles</u>
<u>Electric Bends</u>
} Similar results
see Lund: USPAS notes for
Beam Physics with Intense Space Charge
Transverse Particle Dynamics

<u>Solenoids</u> — Apply rotating frame "Larmor" transform — Lund USPAS

<u>Accelerating Beam</u> — Apply normalized coordinates — Lund USPAS

Try to map new problems to what you know .... linear Oscillator equation.

For more info on these topics see supplemental notes on course web site:

Lund USPAS Notes
- 04. supplement.pdf — 1 slide / page
- 04. supplement_ho.pdf — 4 slides / page

Transfer Matrix

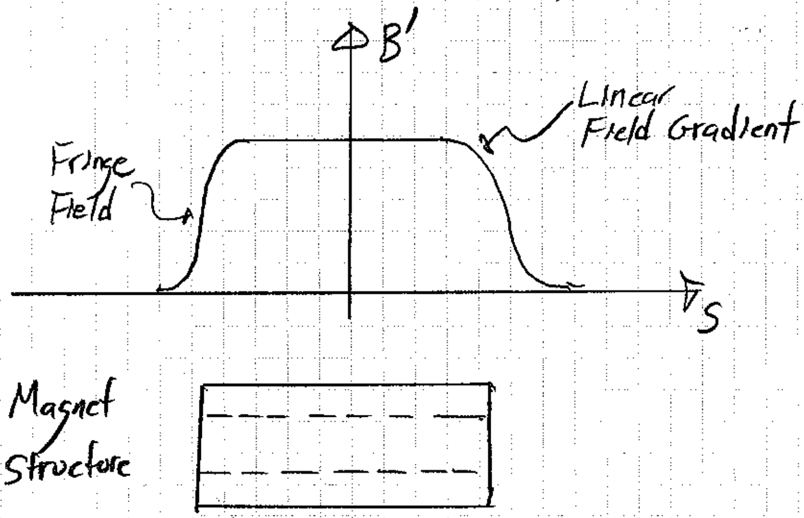
Solutions to the Hill eqns are often exposed in matrix form

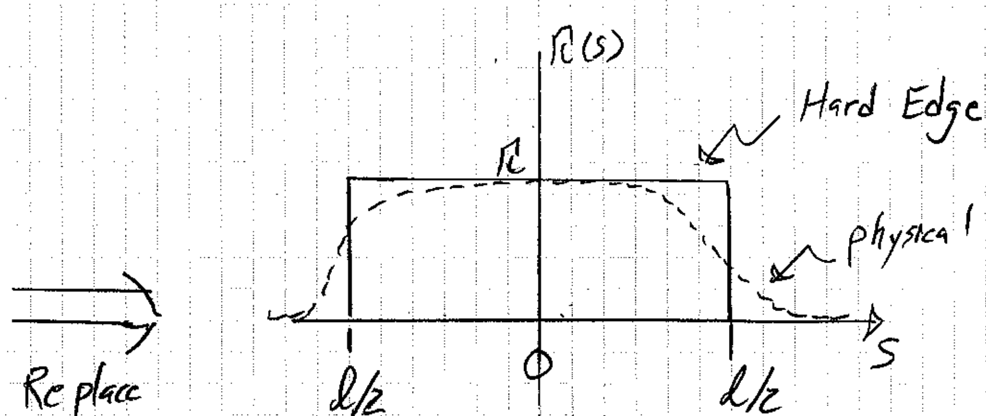
$$\boxed{\begin{bmatrix} x \\ x' \end{bmatrix}_s = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{s_i} \equiv \underbrace{\bar{M}(s|s_i)}_{2\times 2 \text{ matrix}} \begin{bmatrix} x \\ x' \end{bmatrix}_{s_i}}$$

$s = s_i$ = initial condition

Piecewise Method

Solutions can be constructed using magnet codes to get actual s-variation in $K(s)$ lattice function. This enables more realistic modeling. However, can approximate $K(s)$ = const in a regions to model the focusing properties of a lattice.





l = Effective Length

K = const Effective Strength

In applying this method need to relate $\mathcal{K}$ value / length to actual magnet properties. Many ways to do this:

Quadrupole

ℓ = Physical length magnet

$\mathcal{K}$ = Set from peak (middle) gradient B'

Dipole

ℓ = Length ref trajectory in magnet

ρ Set from center (middle) value of $B_y(0)$ from $\frac{1}{\rho} = \frac{B_y(0)}{(B\rho)}$

Augment for non-sector dipoles with thin lens focus corrections for slanted edges.

Orbits within elements in piecewise approximation:

$$\begin{bmatrix} x_0 \\ x_0' \end{bmatrix} \equiv \begin{bmatrix} x \\ x' \end{bmatrix}_{s=s_i}$$

Drift

$\mathcal{K}=0$

$$\frac{d^2x}{ds^2} = 0$$

solution:

$$x(s) = x_0 + x_0'(s-s_i)$$

$$x'(s) = x_0'$$

$$\overline{M}(s|s_i) = \begin{bmatrix} 1 & s-s_i \\ 0 & 1 \end{bmatrix}$$

Through drift element:

$$\overline{M}(s_i+\ell, s_i) = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

Focusing Element

$K > 0$

$$\frac{d^2x}{ds^2} + Kx = 0$$

orbit solution

$$x(s) = x_0 \cos\left[\sqrt{K}(s-s_i)\right] + \frac{x_0'}{\sqrt{K}} \sin\left[\sqrt{K}(s-s_i)\right]$$

$$x'(s) = -x_0\sqrt{K} \sin\left[\sqrt{K}(s-s_i)\right] + x_0' \cos\left[\sqrt{K}(s-s_i)\right]$$

- Quadrupole
  $K = \frac{B'}{(B\rho)} > 0$ focal
- Dipole Bend (Sector)
  $K = \frac{1}{\rho^2} > 0$ focal

$$\overline{M}(s|s_i) = \begin{bmatrix} \cos\left[\sqrt{K}(s-s_i)\right] & \frac{1}{\sqrt{K}} \sin\left[\sqrt{K}(s-s_i)\right] \\ -\sqrt{K} \sin\left[\sqrt{K}(s-s_i)\right] & \cos\left[\sqrt{K}(s-s_i)\right] \end{bmatrix}$$

Through element:

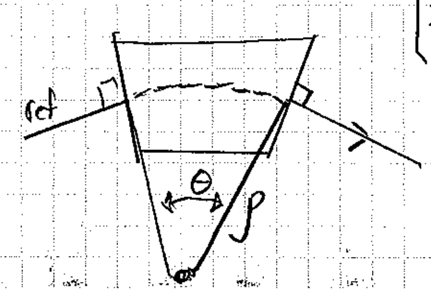
$$\overline{M}(s_i+l|s_i) = \begin{bmatrix} \cos(\sqrt{K}l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}l) \\ -\sqrt{K} \sin(\sqrt{K}l) & \cos(\sqrt{K}l) \end{bmatrix}$$

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} \cos(\sqrt{K}l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}l) \\ -\sqrt{K} \sin(\sqrt{K}l) & \cos(\sqrt{K}l) \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

For a dipole bend, note that

$K = \frac{1}{\rho^2}$ $\qquad \sqrt{K}\,l = \frac{l}{\rho} = \Theta_{bend}$

Θ_{bend} = Bend Angle



Bend

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} \cos(l/\rho) & \rho \sin(l/\rho) \\ -\frac{\sin(l/\rho)}{\rho} & \cos(l/\rho) \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

Defocusing Element

$\mathcal{K} < 0$

$$\frac{d^2}{ds^2}x - \mathcal{K}x = 0$$

- Quadrupole

$$\mathcal{K} = -\frac{B'}{(B\rho)} < 0 \quad \text{defocus}$$

Orbit solution: $\quad K \equiv -\mathcal{K} > 0$

$$x(s) = x_0 \cosh[\sqrt{K}(s-s_i)] + \frac{x_0'}{\sqrt{K}} \sinh[\sqrt{K}(s-s_i)]$$

$$x'(s) = x_0 \sinh[\sqrt{K}(s-s_i)] + x_0' \cosh[\sqrt{K}(s-s_i)]$$

$$\bar{M}(s|s_i) = \begin{pmatrix} \cosh[\sqrt{K}(s-s_i)] & \frac{1}{\sqrt{K}}\sinh[\sqrt{K}(s-s_i)] \\ \sqrt{K}\sinh[\sqrt{K}(s-s_i)] & \cosh[\sqrt{K}(s-s_i)] \end{pmatrix}$$

Through an element:

$$\bar{M}(s_i+\ell|s_i) = \begin{pmatrix} \cosh[\sqrt{K}\ell] & \frac{1}{\sqrt{K}}\sinh[\sqrt{K}\ell] \\ \sqrt{K}\sinh[\sqrt{K}\ell] & \cosh[\sqrt{K}\ell] \end{pmatrix} \qquad \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh[\sqrt{K}\ell] & \frac{1}{\sqrt{K}}\sinh[\sqrt{K}\ell] \\ \sqrt{K}\sinh[\sqrt{K}\ell] & \cosh[\sqrt{K}\ell] \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Thin Lens Limit

For focusing / defocusing elements take

$$\ell \to 0 \qquad \mathcal{K}\ell \text{ finite.}$$

Find

Focusing

$$\overline{M} = \begin{bmatrix} \cos(\sqrt{\mathcal{K}}\ell) & \frac{1}{\sqrt{\mathcal{K}}}\sin(\sqrt{\mathcal{K}}\ell) \\ -\sqrt{\mathcal{K}}\sin(\sqrt{\mathcal{K}}\ell) & \cos(\sqrt{\mathcal{K}}\ell) \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ -\mathcal{K}\ell & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

$\frac{1}{f} = \mathcal{K}\ell$

f = focal length

Defocusing $\quad K = -\mathcal{K} > 0$

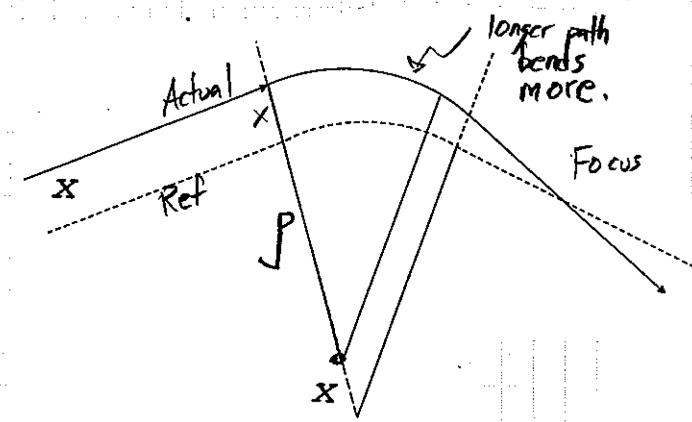
$$\overline{M} = \begin{pmatrix} \cosh(\sqrt{K}\ell) & \frac{1}{\sqrt{K}}\sinh(\sqrt{K}\ell) \\ \sqrt{K}\sinh(\sqrt{K}\ell) & \cosh(\sqrt{K}\ell) \end{pmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ |\mathcal{K}|\ell & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix}$$

$\frac{1}{f} = |\mathcal{K}|\ell$

f = virtual focal length.

For the focusing case with $\mathcal{K} = \frac{1}{\rho^2}$ for a dipole bend

$$\Rightarrow \quad \frac{1}{f} = \frac{\ell}{\rho^2} = \left(\frac{\ell}{\rho}\right)\frac{1}{\rho} = \frac{\theta_{bend}}{\rho}$$



Consistent with expectation that sector dipole focuses;
See Syphers' Notes/Slides

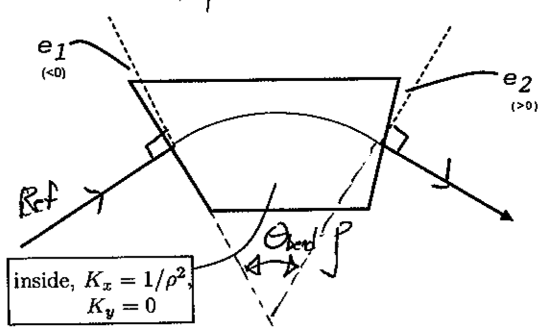
More on Bending Dipoles

See Sypher's notes for details, but in a bend with arbitrary edge angles:

- x-plane: More / less path length in dipole field leads to focusing / defocusing effect when entering and exiting magnet.
- y-plane: Fringe field structure gives impulse in vertical plane on etering and exiting magnet.

Model these edge effects as thin lens kicks:

Dipole Bend



$$\overline{M}_{Total} = \underbrace{\overline{M}_{e2}}_{\text{Thin lens exit}} \cdot \underbrace{\overline{M}_{body}}_{\text{Thick lens body (sector)}} \cdot \underbrace{\overline{M}_{e1}}_{\text{Thin lens enter}}$$

$$\overline{M}_x = \begin{bmatrix} 1 & 0 \\ \frac{\tan e_2}{\rho} & 1 \end{bmatrix} \begin{bmatrix} \cos(l/\rho) & \rho\sin(l/\rho) \\ -\frac{1}{\rho}\sin(l/\rho) & \cos(l/\rho) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\tan e_1}{\rho} & 1 \end{bmatrix}$$

$$\overline{M}_y = \begin{bmatrix} 1 & 0 \\ \frac{-\tan e_2}{\rho} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}}_{\text{Drift}} \begin{bmatrix} 1 & 0 \\ \frac{-\tan e_1}{\rho} & 1 \end{bmatrix}$$

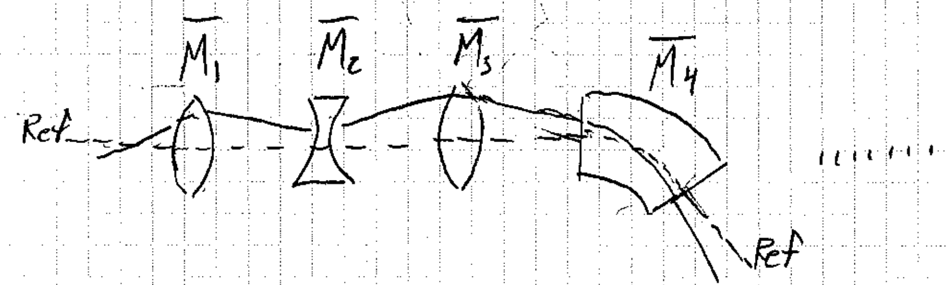
Note: ∘ Sector Dipole: $e_1 = e_2 = 0$
$\Rightarrow$ No entry/exit corrections
$\overline{M}_{e1} = \overline{M}_{e2} = \overline{I}$

∘ Box Dipole: $e_1 = e_2 = \theta_{bend}/2$
Relevant case: easier to fabricate.

For ρ large corrections are small
But beam may enter/exit many magnets to accumulate.

Summary

We now have the basic parts needed to analyze transverse focusing optics in beamlines:



$$\begin{pmatrix} x \\ x' \end{pmatrix}_N = \bar{M}_N \cdot \bar{M}_{N-1} \cdots\cdots \bar{M}_4 \cdot \bar{M}_3 \cdot \bar{M}_2 \cdot \bar{M}_1 \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Much more to do though!

- Solenoid focusing
- Energy gain from RF cavities
- Longitudinal physics
- Effect of "off" (not design) momentum
- Statistical properties of beams
- Control / steering
- Space-Charge effects
- Application examples

Also we will first develop more machinery to analyze orbits / Hill's eqn more efficiently: Phase-Amplitude methods + Courant-Snyder invariant.