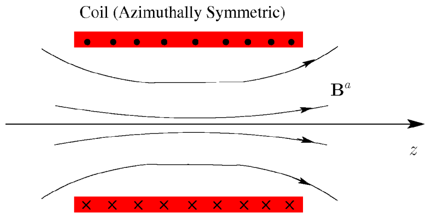
S2E: Solenoidal Focusing

The field of an ideal magnetic solenoid is invariant under transverse rotations about it's axis of symmetry (z) can be expanded in terms of the on-axis field as as:



Vacuum Maxwell equations:

$$\nabla \cdot \mathbf{B}^a = 0$$
$$\nabla \times \mathbf{B}^a = 0$$

Imply $\mathbf{B}^a$ can be expressed in terms of on-axis field $\mathbf{B}^a_z(r=0,z)$

$$\mathbf{E}^a = 0$$

$$\mathbf{B}^a_\perp = \frac{1}{2}\sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!}\frac{\partial^{2\nu-1}B_{z0}(z)}{\partial z^{2\nu-1}}\left(\frac{|\mathbf{x}_\perp|}{2}\right)^{2\nu-2}\mathbf{x}_\perp$$

$$B^a_z = B_{z0}(z) + \sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{(\nu!)^2}\frac{\partial^{2\nu}B_{z0}(z)}{\partial z^{2\nu}}\left(\frac{|\mathbf{x}_\perp|}{2}\right)^{2\nu}$$

$$B_{z0}(z) \equiv B^a_z(\mathbf{x}_\perp = 0, z) = \text{On-Axis Field}$$

See Appendix D or Reiser, *Theory and Design of Charged Particle Beams*, Sec. 3.3.1

Writing out explicitly the terms of this expansion:

$$\mathbf{B}^a(r,z) = \hat{\mathbf{r}}B^a_r(r,z) + \hat{\mathbf{z}}B^a_z(r,z) \qquad r = \sqrt{x^2+y^2}$$
$$= (-\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta)B^a_r(r,z) + \hat{\mathbf{z}}B^a_z(r,z)$$

where

$$B^a_r(r,z) = \sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu!(\nu-1)!}B^{(2\nu-1)}_{z0}(z)\left(\frac{r}{2}\right)^{2\nu-1}$$
$$= \boxed{-\frac{B'_{z0}(z)}{2}r} + \frac{B^{(3)}_{z0}(z)}{16}r^3 - \frac{B^{(5)}_{z0}(z)}{384}r^5 + \frac{B^{(7)}_{z0}(z)}{18432}r^7 - \frac{B^{(9)}_{z0}(z)}{1474560}r^9 + \ldots$$

$$B^a_z(r,z) = \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(\nu!)^2}B^{(2\nu)}_{z0}(z)\left(\frac{r}{2}\right)^{2\nu}$$
$$= \boxed{B_{z0}(z)} - \frac{B''_{z0}(z)}{4}r^2 + \frac{B^{(4)}_{z0}(z)}{64}r^4 - \frac{B^{(6)}_{z0}(z)}{2304}r^6 + \frac{B^{(8)}_{z0}(z)}{147456}r^8 + \ldots$$

$$B_{z0}(z) \equiv B^a_z(r=0,z) = \text{On-axis Field}$$

$\boxed{\cdots}$ Linear Terms

$$B^{(n)}_{z0}(z) \equiv \frac{\partial^n B_{z0}(z)}{\partial z^n} \qquad B'_{z0}(z) \equiv \frac{\partial B_{z0}(z)}{\partial z} \qquad B''_{z0}(z) \equiv \frac{\partial^2 B_{z0}(z)}{\partial z^2}$$

For modeling, we truncate the expansion using only leading-order terms to obtain:

- Corresponds to linear dynamics in the equations of motion

$$B^a_x = -\frac{1}{2}\frac{\partial B_{z0}(z)}{\partial z}x$$
$$B^a_y = -\frac{1}{2}\frac{\partial B_{z0}(z)}{\partial z}y \qquad B_{z0}(z) \equiv B^a_z(\mathbf{x}_\perp = 0, z) = \text{On-Axis Field}$$
$$B^a_z = B_{z0}(z)$$

Note that this truncated expansion is divergence free:

$$\nabla \cdot \mathbf{B}^a = -\frac{1}{2}\frac{\partial B_{z0}}{\partial z}\frac{\partial}{\partial \mathbf{x}_\perp}\cdot\mathbf{x}_\perp + \frac{\partial}{\partial z}B_{z0} = 0$$

but not curl free within the vacuum aperture:

$$\nabla \times \mathbf{B}^a = \frac{1}{2}\frac{\partial^2 B_{z0}(z)}{\partial z^2}(-\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)$$
$$= \frac{1}{2}\frac{\partial^2 B_{z0}(z)}{\partial z^2}r(-\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta) = \frac{1}{2}\frac{\partial^2 B_{z0}(z)}{\partial z^2}r\hat{\theta}$$

- Nonlinear terms needed to satisfy 3D Maxwell equations

Solenoid equations of motion:

- Insert field components into equations of motion and collect terms

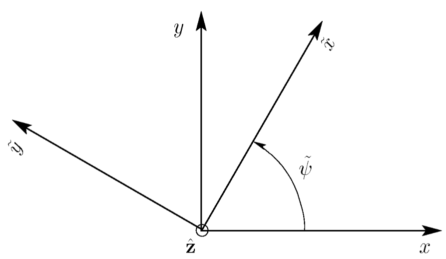
$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' - \frac{B'_{z0}(s)}{2[B\rho]}y - \frac{B_{z0}(s)}{[B\rho]}y' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$
$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \frac{B'_{z0}(s)}{2[B\rho]}x + \frac{B_{z0}(s)}{[B\rho]}x' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$[B\rho] \equiv \frac{\gamma_b\beta_b mc}{q} = \text{Rigidity} \qquad \frac{B_{z0}(s)}{[B\rho]} = \frac{\omega_c(s)}{\gamma_b\beta_b c}$$

$$\omega_c(s) = \frac{qB_{z0}(s)}{m} = \text{Cyclotron Frequency (in applied axial magnetic field)}$$

- Equations are linearly cross-coupled in the applied field terms
  - x equation depends on y, y'
  - y equation depends on x, x'

It can be shown (see: Appendix B) that the linear cross-coupling in the applied field can be removed by an s-varying transformation to a rotating "Larmor" frame:



$\tilde{\ }$ used to denote rotating frame variables

$$\tilde{x} = \quad x\cos\tilde{\psi}(s) + y\sin\tilde{\psi}(s)$$
$$\tilde{y} = -x\sin\tilde{\psi}(s) + y\cos\tilde{\psi}(s)$$

$$\tilde{\psi}(s) = -\int_{s_i}^{s} d\bar{s}\; k_L(\bar{s})$$

$$k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]} = \frac{\omega_c(s)}{2\gamma_b\beta_b c}$$
= Larmor wave number

$s = s_i$ defines initial condition

If the beam space-charge is *axisymmetric*:

$$\frac{\partial\phi}{\partial\mathbf{x}_\perp} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial\mathbf{x}_\perp} = \frac{\partial\phi}{\partial r}\frac{\mathbf{x}_\perp}{r}$$

then the space-charge term also decouples under the Larmor transformation and the equations of motion can be expressed in fully uncoupled form:

$$\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa(s)\tilde{x} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{x}}{r}$$
$$\tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa(s)\tilde{y} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{y}}{r}$$

$$\kappa(s) = k_L^2(s) \equiv \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2$$

Will demonstrate this in problems for the simple case of:
$B_{z0}(s) = \text{const}$

- Because Larmor frame equations are in the same form as continuous and quadrupole focusing with a different κ, for solenoidal focusing we implicitly work in the Larmor frame and simplify notation by dropping the tildes:

$$\tilde{\mathbf{x}}_\perp \rightarrow \mathbf{x}_\perp$$

/// Aside: Notation:

A common theme of this class will be to introduce new effects and generalizations while keeping formulations looking as similar as possible to the the most simple representations given. When doing so, we will often use "tildes" to denote transformed variables to stress that the new coordinates have, in fact, a more complicated form that must be interpreted in the context of the analysis being carried out. Some examples:

- Larmor frame transformations for Solenoidal focusing
  See: Appendix B
- Normalized variables for analysis of accelerating systems
  See: S10
- Coordinates expressed relative to the beam centroid
  See: S.M. Lund, lectures on Transverse Centroid and Envelope Model
- Variables used to analyze Einzel lenses
  See: J.J. Barnard, Introductory Lectures

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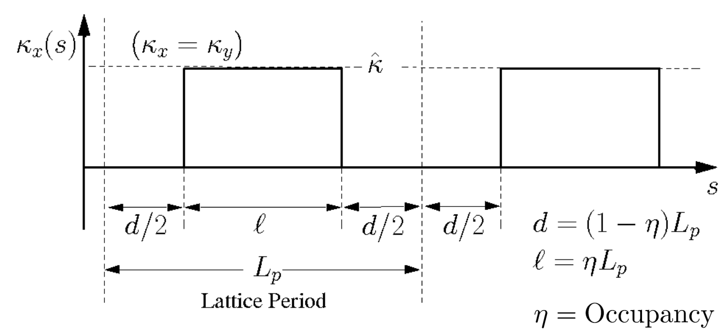
Solenoid periodic lattices can be formed similarly to the quadrupole case
- Drifts placed between solenoids of finite axial length
  - Allows space for diagnostics, pumping, acceleration cells, etc.
- Analogous equivalence cases to quadrupole
  - Piecewise constant κ often used
- Fringe can be more important for solenoids

Simple hard-edge solenoid lattice with piecewise constant κ



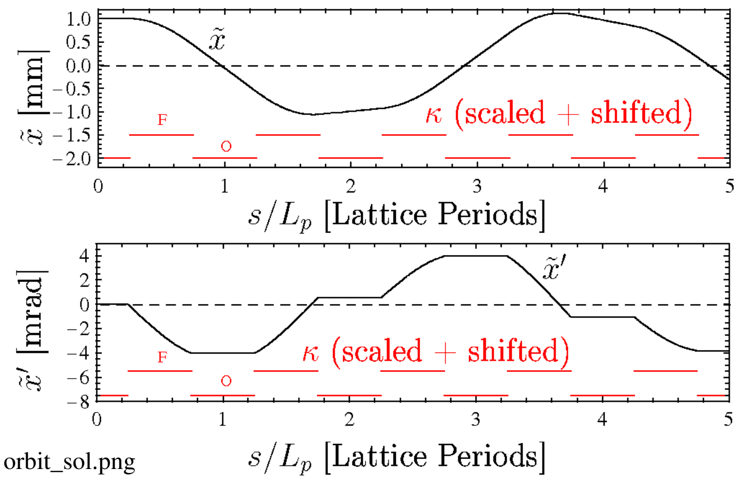
$$d = (1-\eta)L_p$$
$$\ell = \eta L_p$$

$$\eta = \text{Occupancy} \in (0, 1]$$

/// Example: Larmor Frame Particle Orbits in a Periodic Solenoidal Focusing Lattice: $\tilde{x} - \tilde{x}'$ phase-space for hard edge elements and applied fields

$L_p = 0.5$ m $\quad \kappa = 20$ rad/m^2 in Solenoids $\quad \tilde{x}(0) = 1$ mm $\quad \tilde{y}(0) = 0$

$\eta = 0.5 \quad \phi \simeq 0 \quad \gamma_b\beta_b = \text{const} \quad \tilde{x}'(0) = 0 \quad \tilde{y}'(0) = 0$

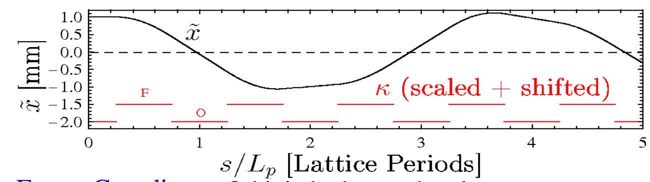


orbit_sol.png

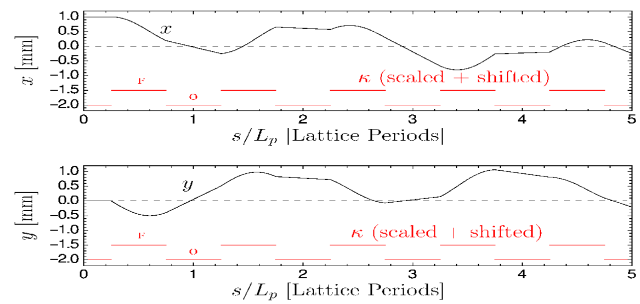
Contrast of Larmor-Frame and Lab-Frame Orbits

- Same initial condition

Larmor-Frame Coordinate Orbit in transformed x-plane only



Lab-Frame Coordinate Orbit in both x- and y-planes

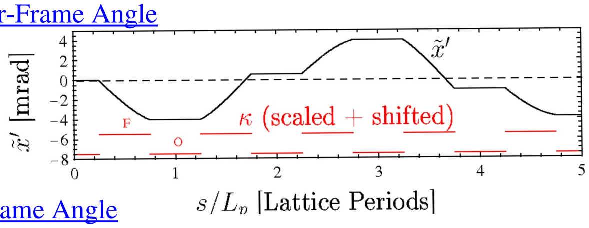


Calculate using transfer matrices in Appendix C

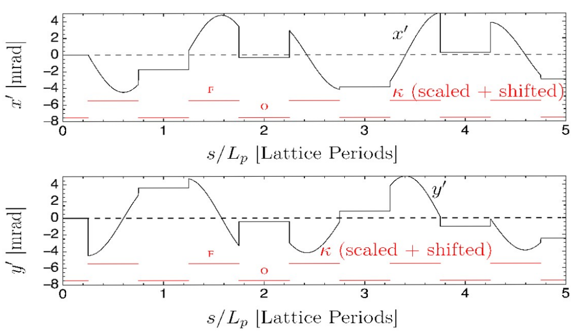
Contrast of Larmor-Frame and Lab-Frame Orbits

- Same initial condition

Larmor-Frame Angle



Lab-Frame Angle

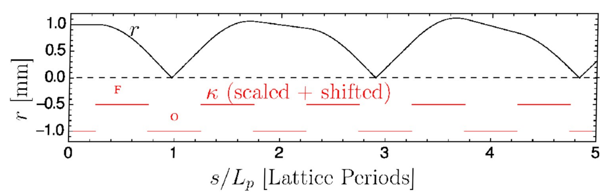


Calculate using transfer matrices in Appendix C

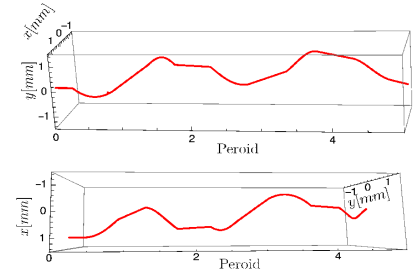
Additional perspectives of particle orbit in solenoid transport channel

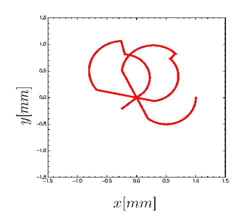
- Same initial condition

Radius evolution (Lab or Larmor Frame: radius same)



Side- (2 view points) and End-View Projections of 3D Lab-Frame Orbit





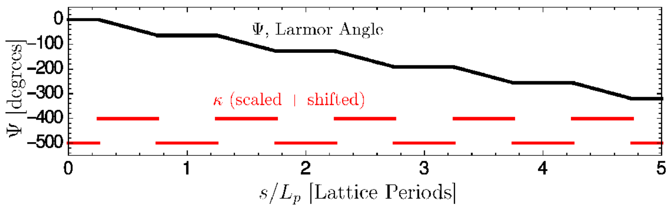
Calculate using transfer matrices in Appendix C

Larmor angle and angular momentum of particle orbit in solenoid transport channel

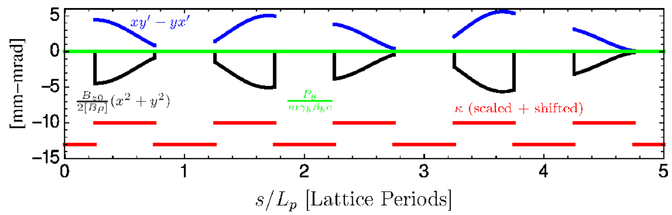
- Same initial condition

Larmor Angle

$$\tilde{\psi}(s) = -\int_{s_i}^{s} d\bar{s}\; k_L(\bar{s}) \qquad k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]}$$



Angular Momentum and Canonical Angular Momentum (see Sec. S2G)



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Comments on Orbits:

- See Appendix C for details on calculation
  - Discontinuous fringe of hard-edge model must be treated carefully if integrating in the laboratory-frame.
- Larmor-frame orbits strongly deviate from simple harmonic form due to periodic focusing
  - Multiple harmonics present
  - Less complicated than quadrupole AG focusing case when interpreted in the Larmor frame due to the optic being focusing in both planes
- Orbits transformed back into the Laboratory frame using Larmor transform (see: Appendix B and Appendix C)
  - Laboratory frame orbit exhibits more complicated *x*-*y* plane coupled oscillatory structure
- Will find later that if the focusing is sufficiently strong, the orbit can become unstable (see: S5)
- Larmor frame *y*-orbits have same properties as the *x*-orbits due to the equations being decoupled and identical in form in each plane
  - In example, Larmor *y*-orbit is zero due to simple initial condition in *x*-plane
  - Lab *y*-orbit is nozero due to *x*-*y* coupling

Comments on Orbits (continued):

- Larmor angle advances continuously even for hard-edge focusing
- Mechanical angular momentum jumps discontinuously going into and out of the solenoid
  - Particle spins up and down going into and out of the solenoid
  - No mechanical angular momentum outside of solenoid due to the choice of intial condition in this example (initial *x*-plane motion)
- Canonical angular momentum P_θ is conserved in the 3D orbit evolution
  - As expected from analysis in S2G
  - Invariance provides a good check on dynamics
  - P_θ in example has zero value due to the specific (*x*-plane) choice of initial condition. Other choices can give nonzero values and finite mechanical angular momentum in drifts.

Some properties of particle orbits in solenoids with piecewise $\kappa = \text{const}$ will be analyzed in the problem sets

///

S2F: Summary of Transverse Particle Equations of Motion

In linear applied focusing channels, without momentum spread or radiation, the particle equations of motion in both the *x*- and *y*-planes expressed as:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial x}\phi$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial y}\phi$$

$$\kappa_x(s) = x\text{-focusing function of lattice}$$

$$\kappa_y(s) = y\text{-focusing function of lattice}$$

Common focusing functions:

Continuous: $\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$

Quadrupole (Electric or Magnetic):

$$\kappa_x(s) = -\kappa_y(s) = \kappa(s)$$

Solenoidal (equations must be interpreted in Larmor Frame: see Appendix B):

$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

Although the equations have the same form, the couplings to the fields are different which leads to different regimes of applicability for the various focusing technologies with their associated technology limits:

Focusing:

Continuous:

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

Good qualitative guide (see later material/lecture)
BUT not physically realizable (see S2B)

Quadrupole:

$$\kappa_x(s) = -\kappa_y(s) = \begin{cases} \frac{G(s)}{\beta_b c [B\rho]}, & \text{Electric} \\ \frac{G(s)}{c[B\rho]}, & \text{Magnetic} \end{cases} \qquad [B\rho] = \frac{m\gamma_b\beta_b c}{q}$$

G is the field gradient which for linear applied fields is:

$$G(s) = \begin{cases} -\frac{\partial E_x^a}{\partial x} = \frac{\partial E_y^a}{\partial y} = \frac{2V_q}{r_p^2}, & \text{Electric} \\ \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x} = \frac{B_p}{r_p}, & \text{Magnetic} \end{cases}$$

Solenoid:

$$\kappa_x(s) = \kappa_y(s) = k_L^2(s) = \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2 \qquad \omega_c(s) = \frac{qB_{z0}(s)}{m}$$

It is instructive to review the structure of solutions of the transverse particle equations of motion in the absence of:

Space-charge: $\frac{\partial \phi}{\partial x} \sim \frac{\partial \phi}{\partial y} \sim 0$

Acceleration: $\gamma_b\beta_b \simeq \text{const} \qquad \Longrightarrow \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq 0$

In this simple limit, the *x* and *y*-equations are of the same Hill's Equation form:

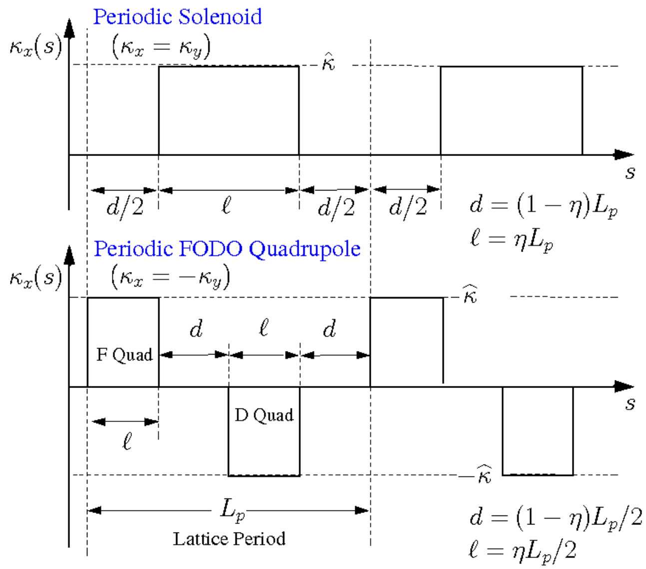
$$\boxed{\begin{aligned} x'' + \kappa_x(s)x &= 0 \\ y'' + \kappa_y(s)y &= 0 \end{aligned}}$$

- These equations are central to transverse dynamics in conventional accelerator physics (weak space-charge and acceleration)
  - Will study how solutions change with space-charge in later lectures

In many cases beam transport lattices are designed where the applied focusing functions are periodic:

$$\boxed{\begin{aligned} \kappa_x(s + L_p) &= \kappa_x(s) \\ \kappa_y(s + L_p) &= \kappa_y(s) \end{aligned} \qquad L_p = \text{Lattice Period}}$$

Common, simple examples of periodic lattices:



However, the focusing functions need not be periodic:

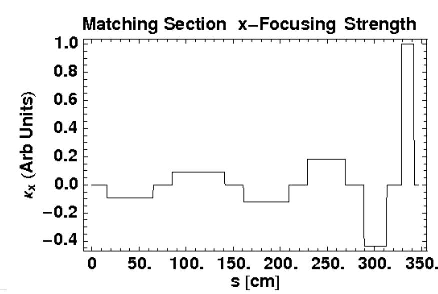
- Often take periodic or continuous in this class for simplicity of interpretation

Focusing functions can vary strongly in many common situations:

- Matching and transition sections
- Strong acceleration
- Significantly different elements can occur within periods of lattices in rings
  - "Panofsky" type (wide aperture along one plane) quadrupoles for beam insertion and extraction in a ring

Example of Non-Periodic Focusing Functions: Beam Matching Section

Maintains alternating-gradient structure but not quasi-periodic



Example corresponds to High Current Experiment Matching Section (hard edge equivalent) at LBNL (2002)

Equations presented in this section apply to a single particle moving in a beam under the action of linear applied focusing forces. In the remaining sections, we will (mostly) neglect space-charge ($\phi \rightarrow 0$) as is conventional in the standard theory of low-intensity accelerators.

- What we learn from treatment will later aid analysis of space-charge effects
  - Appropriate variable substitutions will be made to apply results
- Important to understand basic applied field dynamics since space-charge complicates
  - Results in plasma-like collective response

/// Example: We will see in Transverse Centroid and Envelope Descriptions of Beam Evolution that the linear particle equations of motion can be applied to analyze the evolution of a beam when image charges are neglected

$$x \rightarrow x_c \equiv \langle x \rangle_\perp \quad x - \text{centroid}$$
$$y \rightarrow y_c \equiv \langle y \rangle_\perp \quad y - \text{centroid}$$

///

S2G: Conservation of Angular Momentum in Axisymmetric Focusing Systems

Background:

Goal: find an invariant for axisymmetric focusing systems which can help us further interpret/understand the dynamics.

In Hamiltonian descriptions of beam dynamics one must employ proper canonical conjugate variables such as (x-plane):

$$\boxed{\begin{aligned} x &= && \text{Canonical Coordinate} \\ P_x &= p_x + qA_x = && \text{Canonical Momentum} \end{aligned}}$$

+ analogous y-plane

Here, $\mathbf{A}$ denotes the vector potential of the (static for cases of field models considered here) applied magnetic field with:

$$\boxed{\mathbf{B}^a = \nabla \times \mathbf{A}}$$

For the cases of linear applied magnetic fields in this section, we have:

$$\boxed{\mathbf{A} = \begin{cases} \hat{\mathbf{z}}\frac{G}{2}(y^2 - x^2), & \text{Magnetic Quadrupole Focusing} \\ -\hat{\mathbf{x}}\frac{1}{2}B_{z0}y + \hat{\mathbf{y}}\frac{1}{2}B_{z0}x, & \text{Solenoidal Focusing} \\ 0, & \text{Otherwise} \end{cases}}$$

For continuous, electric or magnetic quadrupole focusing without acceleration ($\gamma_b\beta_b = \text{const}$), it is straightforward to verify that x,x' and y,y' are canonical coordinates and that the correct equations of motion are generated by the Hamiltonian:

$$\boxed{\begin{aligned} H_\perp &= \frac{1}{2}x'^2 + \frac{1}{2}y'^2 + \frac{1}{2}\kappa_x x^2 + \frac{1}{2}\kappa_y y^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^3} \\ \frac{d}{ds}x &= \frac{\partial H_\perp}{\partial x'} \qquad \frac{d}{ds}x = \frac{\partial H_\perp}{\partial y'} \\ \frac{d}{ds}x' &= -\frac{\partial H_\perp}{\partial x} \qquad \frac{d}{ds}y' = -\frac{\partial H_\perp}{\partial y} \end{aligned}}$$

Giving the familiar equations of motion:

$$x'' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial x}$$
$$y'' + \kappa_y y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial y}$$

For solenoidal magnetic focusing without acceleration, it can be verified that we can take (tilde) canonical variables:

- Tildes *do not* denote Larmor transform variables here !

$$\boxed{\begin{aligned} \tilde{x} &= x & \tilde{y} &= y & & \\ \tilde{x}' &= x' - \frac{B_{z0}}{2[B\rho]}y & \tilde{y}' &= y' + \frac{B_{z0}}{2[B\rho]}x & [B\rho] &\equiv \frac{m\gamma_b\beta_b c}{q} \end{aligned}}$$

With Hamiltonian:

$$\boxed{\begin{aligned} \tilde{H}_\perp &= \frac{1}{2}\left[\left(\tilde{x}' + \frac{B_{z0}}{2[B\rho]}\tilde{y}\right)^2 + \left(\tilde{y}' - \frac{B_{z0}}{2[B\rho]}\tilde{x}\right)^2\right] + \frac{q\phi}{m\gamma_b^3\beta_b^2c^3} \\ \frac{d}{ds}\tilde{x} &= \frac{\partial \tilde{H}_\perp}{\partial \tilde{x}'} \qquad \frac{d}{ds}\tilde{y} = \frac{\partial \tilde{H}_\perp}{\partial \tilde{y}'} \\ \frac{d}{ds}\tilde{x}' &= -\frac{\partial \tilde{H}_\perp}{\partial \tilde{x}} \qquad \frac{d}{ds}\tilde{y}' = -\frac{\partial \tilde{H}_\perp}{\partial \tilde{y}} \end{aligned}}$$

Caution:
Primes do not mean *d/ds* in tilde variables here: just notation to distinguish "momentum" variable!

Giving (after some algebra) the familiar equations of motion:

$$x'' - \frac{B'_{z0}(s)}{2[B\rho]}y - \frac{B_{z0}(s)}{[B\rho]}y' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial x}$$
$$y'' + \frac{B'_{z0}(s)}{2[B\rho]}x + \frac{B_{z0}(s)}{[B\rho]}x' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial y}$$

Canonical angular momentum

One expects from general considerations (Noether's Theorem in dynamics) that systems with a symmetry have a conservation constraint associated with the generator of the symmetry. So for systems with azimuthal symmetry $(\partial/\partial\theta = 0)$, one expects there to be a conserved canonical angular momentum (generator of rotations). Based on the Hamiltonian dynamics structure, examine:

$$\boxed{P_\theta \equiv [\mathbf{x}\times\mathbf{P}]\cdot\hat{\mathbf{z}} = [\mathbf{x}\times(\mathbf{p}+q\mathbf{A})]\cdot\hat{\mathbf{z}}}$$

This is exactly equivalent to

- Here γ factor is exact (*not* paraxial)

$$\boxed{\begin{aligned} P_\theta &= (xp_y - yp_x) + q(xA_y - yA_x) \\ &= r(p_\theta + qA_\theta) = m\gamma r^2\dot{\theta} + qrA_\theta \end{aligned}}$$

Or employing the usual paraxial approximation steps:

$$\boxed{\begin{aligned} P_\theta &\simeq m\gamma_b\beta_b c(xy' - yx') + q(xA_y - yA_x) \\ &= m\gamma_b\beta_b c r^2\theta' + qrA_\theta \end{aligned}}$$

Inserting the vector potential components consistent with linear approximation solenoid focusing in the paraxial expression gives:

- Applies to (superimposed or separately) to continuous, magnetic or electric quadrupole, or solenoidal focusing since $A_\theta \neq 0$ only for solenoidal focusing

$$\boxed{\begin{aligned} P_\theta &\simeq m\gamma_b\beta_b c(xy' - yx') + \frac{qB_{z0}}{2}(x^2+y^2) \\ &= m\gamma_b\beta_b c r^2\theta' + \frac{qB_{z0}}{2}r^2 \end{aligned}}$$

For a coasting beam $(\gamma_b\beta_b = \text{const})$, it is often convenient to analyze:

- Later we will find this is analogous to use of "unnormalized" variables used in calculation of ordinary emittance rather than normalized emittance

$$\begin{aligned} \frac{P_\theta}{m\gamma_b\beta_b c} &= xy' - yx' + \frac{B_{z0}}{2[B\rho]}(x^2+y^2) \qquad\qquad [B\rho] \equiv \frac{m\gamma_b\beta_b c}{q} \\ &= r^2\theta' + \frac{B_{z0}}{2[B\rho]}r^2 \end{aligned}$$

Conservation of canonical angular momentum

To investigate situations where the canonical angular momentum is a constant of the motion for a beam evolving in linear applied fields, we differentiate P_θ with respect to *s* and apply equations of motion

Equations of Motion:

Including acceleration effects again, we summarize the equations of motion as:

- Applies to continuous, quadrupole (electric + magnetic), and solenoid focusing as expressed
- Several types of focusing can also be superimposed
  - Show for superimposed solenoid

$$\boxed{\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x - \frac{B'_{z0}(s)}{2[B\rho]}y - \frac{B_{z0}(s)}{[B\rho]}y' &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y y + \frac{B'_{z0}(s)}{2[B\rho]}x + \frac{B_{z0}(s)}{[B\rho]}x' &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y} \end{aligned}}$$

$$[B\rho] = \frac{m\gamma_b\beta_b c}{q} \qquad \kappa_x(s) = \begin{cases} k_{\beta 0}^2 = \text{const}, & \text{Continuous Focus } (\kappa_y = \kappa_x) \\ \frac{G(s)}{\beta_b c[B\rho]}, & \text{Electric Quadrupole Focus } (\kappa_y = -\kappa_x) \\ \frac{G(s)}{c[B\rho]}, & \text{Magnetic Quadrupole Focus } (\kappa_y = -\kappa_x) \end{cases}$$

Employ the paraxial form of P_θ consistent with the possible existence of a solenoid magnetic field:

- Formula also applies as expressed to continuous and quadrupole focusing

$$\boxed{\mathrm{P}_\theta = m\gamma_b\beta_b c(xy' - yx') + \tfrac{qB_{z0}}{2}(x^2+y^2)}$$

Differentiate and apply equations of motion:

- Intermediate algebraic steps not shown

$$\begin{aligned} \frac{d}{ds}P_\theta &= mc(\gamma_b\beta_b)'(xy' - yx') + mc(\gamma_b\beta_b)(xy'' - yx'') \\ &\quad + \frac{qB'_{z0}}{2}(x^2+y^2) + qB_{z0}(xx' + yy') \\ &= mc(\gamma_b\beta_b)[\kappa_x - \kappa_y]xy - \frac{q}{\gamma_b^2\beta_b c}\left(x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x}\right) \end{aligned}$$

So *IF*:

1) $\kappa_x = \kappa_y$
  - Valid continuous or solenoid focusing
  - Invalid for quadrupole focusing

2) $x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\theta} = 0$
  - Axisymmetric beam

$$\boxed{\frac{d}{ds}P_\theta = 0 \qquad \Longrightarrow \qquad P_\theta = \text{const}}$$

For:
- Continuous focusing
- Linear optics solenoid magnetic focusing
- Other axisymmetric electric optics not covered such as Einzel lenses …

$$\boxed{\mathrm{P}_\theta = m\gamma_b\beta_b c(xy' - yx') + \tfrac{qB_{z0}}{2}(x^2+y^2) = \text{const}}$$

$$m\gamma_b\beta_b c(xy' - yx') = \text{Mechanical Angular Momentum Term}$$

$$\frac{qB_{z0}}{2}(x^2+y^2) = \text{Vector Potential Angular Momentum Term}$$

In S2E we plot for solenoidal focusing :
- Mechanical angular momentum $\propto xy' - yx'$
- Larmor rotation angle $\tilde{\psi}$
- Canonical angular momentum (constant) P_θ

Comments:
- Where valid, $P_\theta = \text{const}$ provides a powerful constraint to check dynamics
- If $P_\theta = \text{const}$ for all particles, then $\langle P_\theta \rangle = \text{const}$ for the beam as a whole and it is found in envelope models that canonical angular momentum can act effectively act phase-space area (emittance-like term) defocusing the beam
- Valid for acceleration: similar to a "normalized emittance": see S10

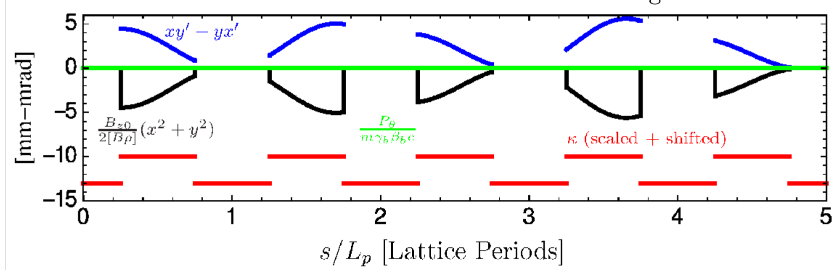
Example: solenoidal focusing channel

Employ the solenoid focusing channel example in S2E and plot:
- Mechanical angular momentum $\propto xy' - yx'$
- Vector potential contribution to canonical angular momentum $\propto B_{z0}(x^2+y^2)$
- Canonical angular momentum (constant) P_θ

$$\frac{P_\theta}{m\gamma_b\beta_b c} = xy' - yx' + \frac{B_{z0}}{2[B\rho]}(x^2+y^2) = \text{const}$$

$$xy' - yx' = r^2\theta' = \text{Mechanical Angular Momentum}$$

$$\frac{B_{z0}}{2[B\rho]}(x^2+y^2) = \sqrt{\kappa}(x^2+y^2) = \text{Vector Potential Component Canonical Angular Momentum}$$



Comments on Orbits (see also info in S2E on 3D orbit):
- Mechanical angular momentum jumps discontinuously going into and out of the solenoid
  - Particle spins up (θ' jumps) and down going into and out of the solenoid
  - No mechanical angular momentum outside of solenoid due to the choice of intial condition in this example (initial *x*-plane motion)
- Canonical angular momentum P_θ is conserved in the 3D orbit evolution
  - Invariance provides a strong check on dynamics
  - P_θ in example has zero value due to the specific (*x*-plane) choice of initial condition of the particle. Other choices can give nonzero values and finite mechanical angular momentum in drifts.
- Solenoid provides focusing due to radial kicks associated with the "fringe" field entering the solenoid
  - Kick is abrupt for hard-edge solenoids
  - Details on radial kick/rotation structure can be found in Appendix C

Alternative expressions of canonical angular momentum

It is insightful to express the canonical angular momentum in (denoted tilde here) in the solenoid focusing canonical variables used earlier in this section and rotating Larmor frame variables:
- See Appendix B for Larmor frame transform
- Might expect simpler form of expressions given the relative simplicity of the formulation in canonical and Larmor frame variables

Canonical Variables:

$$\tilde{x} = x \qquad \tilde{y} = y$$

$$\tilde{x}' = x' - \frac{B_{z0}}{2[B\rho]}y \qquad \tilde{y}' = y' + \frac{B_{z0}}{2[B\rho]}x$$

$$\Longrightarrow \boxed{\begin{aligned}\frac{P_\theta}{m\gamma_b\beta_b c} &\equiv xy' - yx' + \frac{B_{z0}}{2[B\rho]}(x^2+y^2) \\ &= \tilde{x}\tilde{y}' - \tilde{x}\tilde{y}'\end{aligned}}$$

- Applies to acceleration also since just employing transform as a definition here

Larmor (Rotating) Frame Variables:

Larmor transform following formulation in Appendix B:

- Here tildes denote Larmor frame variables

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix} = \begin{bmatrix} \cos\tilde{\psi} & 0 & -\sin\tilde{\psi} & 0 \\ k_L \sin\tilde{\psi} & \cos\tilde{\psi} & k_L\cos\tilde{\psi} & -\sin\tilde{\psi} \\ \sin\tilde{\psi} & 0 & \cos\tilde{\psi} & 0 \\ -k_L\cos\tilde{\psi} & \sin\tilde{\psi} & k_L \sin\tilde{\psi} & \cos\tilde{\psi} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{bmatrix} \qquad \tilde{\psi}(s) = -\int_{s_i}^{s} d\bar{s}\, k_L(\bar{s}) \qquad k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]}$$

gives after some algebra:

$$\Longrightarrow \qquad x^2 + y^2 = \tilde{x}^2 + \tilde{y}^2$$

$$xy' - yx' = \tilde{x}\tilde{y}' - \tilde{y}\tilde{x}' - \frac{B_{z0}}{2[B\rho]}(\tilde{x}^2 + \tilde{y}^2)$$

Showing that:

$$\boxed{\begin{aligned} \frac{P_\theta}{m\gamma_b\beta_b c} &\equiv xy' - yx' + \frac{B_{z0}}{2[B\rho]}(x^2+y^2) \\ &= \tilde{x}\tilde{y}' - \tilde{x}\tilde{y}' \end{aligned}}$$

- Same form as previous canonical variable case due to notation choices. However, steps/variables and implications different in this case !

Bush's Theorem expression of canonical angular momentum conservation

Take:

$$\boxed{\mathbf{B}^a = \nabla \times \mathbf{A}}$$

and apply Stokes Theorem to calculate the magnetic flux Ψ through a circle of radius *r*:

$$\boxed{\Psi = \int_r d^2x\, \mathbf{B}^a \cdot \hat{\mathbf{z}} \; = \int_r d^2x\, (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{z}} = \oint_r \mathbf{A} \cdot d\vec{\ell}}$$

For a nonlinear, but axisymmetric solenoid, one can always take:

- Also applies to linear field component case

$$\mathbf{A} = \hat{\theta} A_\theta(r,z)$$

$$\Longrightarrow \qquad \mathbf{B}^a = -\hat{\mathbf{r}}\frac{\partial A_\theta}{\partial z} + \hat{\mathbf{z}}\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)$$

Thus:

$$\boxed{\Psi = 2\pi r A_\theta}$$

// Aside: Nonlinear Application of Vector Potential

Given the magnetic field components

$$B_r^a(r,z) \qquad B_z^a(r,z)$$

the equations

$$B_r^a(r,z) = -\frac{\partial}{\partial z} A_\theta(r,z)$$

$$B_z^a(r,z) = \frac{1}{r}\frac{\partial}{\partial r}[rA_\theta(r,z)]$$

can be integrated for a single isolated magnet to obtain *equivalent* expressions for A_θ

$$A_\theta(r,z) = -\int_{-\infty}^{z} d\tilde{z}\, B_r^a(r,\tilde{z})$$

$$A_\theta(r,z) = \frac{1}{r}\int_0^r d\tilde{r}\tilde{r}\, B_z^a(\tilde{r},z)$$

- Resulting A_θ contains consistent nonlinear terms with magnetic field

//

Then the exact form of the canonical angular momentum for for solenoid focusing can be expressed as:

- Here γ factor is exact (not paraxial)

$$\boxed{\begin{aligned} P_\theta &= m\gamma r^2\dot{\theta} + qrA_\theta \\ &= m\gamma r^2\dot{\theta} + \frac{q\Psi}{2\pi} \end{aligned}}$$

This form is often applied in solenoidal focusing and is known as “Bush's Theorem” with

$$\boxed{P_\theta = m\gamma r^2\dot{\theta} + \tfrac{q\Psi}{2\pi} = \text{const}}$$

- In a static applied magnetic field, $\gamma = \text{const}$ further simplifying use of eqn
- Exact as expressed, but easily modified using familiar steps for paraxial form and/or linear field components
- Expresses how a particle “spins up” when entering a solenoidal magnetic field

Appendix B: The Larmor Transform to Express Solenoidal Focused Particle Equations of Motion in Uncoupled Form

Solenoid equations of motion:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' - \frac{B'_{z0}(s)}{2[B\rho]}y - \frac{B_{z0}(s)}{[B\rho]}y' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \frac{B'_{z0}(s)}{2[B\rho]}x + \frac{B_{z0}(s)}{[B\rho]}x' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$B_{z0}(s) = B_z^a(r=0, z=s) = \text{On-Axis Field}$$

$$[B\rho] = \frac{\gamma_b\beta_b mc}{q} = \text{Rigidity}$$

To simplify algebra, introduce the complex coordinate

$$\underline{z} \equiv x + iy \qquad\qquad i \equiv \sqrt{-1}$$

Note* context clarifies use of i (particle index, initial cond, complex i)

Then the two equations can be expressed as a single complex equation

$$\underline{z}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\underline{z}' + i\frac{B'_{z0}(s)}{2[B\rho]}\underline{z} + i\frac{B_{z0}(s)}{[B\rho]}\underline{z}' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right)$$

B1

If the potential is axisymmetric with $\phi = \phi(r)$:

$$\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial r}\frac{\underline{z}}{r} \qquad\qquad r \equiv \sqrt{x^2+y^2}$$

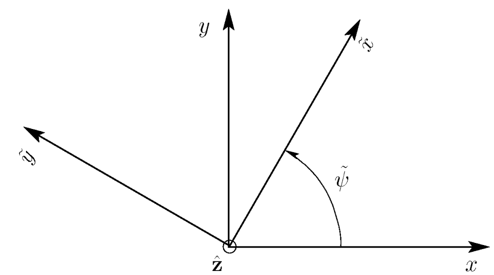
then the complex form equation of motion reduces to:

$$\underline{z}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\underline{z}' + i\frac{B'_{z0}(s)}{2[B\rho]}\underline{z} + i\frac{B_{z0}(s)}{[B\rho]}\underline{z}' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\underline{z}}{r}$$

Following Wiedemann, Vol II, pg 82, introduce a transformed complex variable that is a local (s-varying) rotation:

$$\underline{\tilde{z}} \equiv \underline{z}e^{-i\tilde{\psi}(s)} = \tilde{x} + i\tilde{y}$$

$$\tilde{\psi}(s) = \text{phase-function (real-valued)}$$



B2

Then: $\underline{z} = \underline{\tilde{z}}e^{i\tilde{\psi}}$

$$\underline{z}' = \left(\underline{\tilde{z}}' + i\tilde{\psi}'\underline{\tilde{z}}\right)e^{i\tilde{\psi}}$$

$$\underline{z}'' = \left(\underline{\tilde{z}}'' + 2i\tilde{\psi}'\underline{\tilde{z}}' + i\tilde{\psi}''\underline{\tilde{z}} - \tilde{\psi}'^2\underline{\tilde{z}}\right)e^{i\tilde{\psi}}$$

and the complex form equations of motion become:

$$\underline{\tilde{z}}'' + \left[i\left(2\tilde{\psi}' + \frac{B_{z0}}{[B\rho]}\right) + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\right]\underline{\tilde{z}}' + \left[-\tilde{\psi}'^2 - \frac{B_{z0}}{[B\rho]}\tilde{\psi}' + i\left(\tilde{\psi}'' + \frac{B'_{z0}}{2[B\rho]} + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{\psi}'\right)\right]\underline{\tilde{z}} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\underline{\tilde{z}}}{r}$$

Free to choose the form of $\tilde{\psi}$ Can choose to eliminate imaginary terms in i(....) in equation by taking:

$$\tilde{\psi}' \equiv -\frac{B_{z0}}{2[B\rho]} \qquad \Longrightarrow \qquad \tilde{\psi}'' = -\frac{B'_{z0}}{2[B\rho]} + \frac{B_{z0}}{2[B\rho]}\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}$$

B3

Using these results, the complex form equations of motion reduce to: B4

$$\underline{\tilde{z}}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\underline{\tilde{z}}' + \left(\frac{B_{z0}}{2[B\rho]}\right)^2\underline{\tilde{z}} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\underline{\tilde{z}}}{r}$$

Or using $\underline{\tilde{z}} = \tilde{x} + i\tilde{y}$, the equations can be expressed in decoupled $\tilde{x},\ \tilde{y}$ variables in the Larmor Frame as:

$$\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa(s)\tilde{x} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{x}}{r}$$

$$\tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa(s)\tilde{y} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{y}}{r}$$

$$\kappa(s) \equiv k_L^2(s) \qquad k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]} = \frac{\omega_c(s)}{2\gamma_b\beta_b c} \qquad [B\rho] = \frac{\gamma_b\beta_b mc}{q}$$

$$= \text{Larmor Wave-Number}$$

Equations of motion are uncoupled but must be interpreted in the rotating Larmor frame

- Same form as quadrupoles but with focusing function same sign in each plane

The rotational transformation to the Larmor Frame can be effected by integrating the equation for $\tilde{\psi}' = -\frac{B_{z0}}{2[B\rho]}$

$$\tilde{\psi}(s) = -\int_{s_i}^{s} d\tilde{s}\, \frac{B_{z0}(\tilde{s})}{2[B\rho]} = -\int_{s_i}^{s} d\tilde{s}\, k_L(\tilde{s})$$

Here, s_i is some value of *s* where the initial conditions are taken.

- Take $s = s_i$ where axial field is zero for simplest interpretation (see: pg B6)

Because

$$\tilde{\psi}' = -\frac{B_{z0}}{2[B\rho]} = \frac{\omega_c}{2\gamma_b\beta_b c}$$

the local $\tilde{x} - \tilde{y}$ Larmor frame is rotating at ½ of the local *s*-varying cyclotron frequency

- If $B_{z0} = \text{const}$, then the Larmor frame is uniformly rotating as is well known from elementary textbooks (see problem sets)

B5

The complex form phase-space transformation and inverse transformations are:

$$\underline{z} = \tilde{\underline{z}} e^{i\tilde{\psi}} \qquad \tilde{\underline{z}} = \underline{z} e^{-i\tilde{\psi}}$$

$$\underline{z}' = \left(\tilde{\underline{z}}' + i\tilde{\psi}'\tilde{\underline{z}}\right) e^{i\tilde{\psi}} \qquad \tilde{\underline{z}}' = \left(\underline{z}' - i\tilde{\psi}'\underline{z}\right) e^{-i\tilde{\psi}}$$

$$\underline{z} = x + iy \qquad \tilde{\underline{z}} = \tilde{x} + i\tilde{y} \qquad \tilde{\psi}' = -k_L$$

$$\underline{z}' = x' + iy' \qquad \tilde{\underline{z}}' = \tilde{x}' + i\tilde{y}'$$

Apply to:

- Project initial conditions from lab-frame when integrating equations
- Project integrated solution back to lab-frame to interpret solution

If the initial condition $s = s_i$ is taken outside of the magnetic field where $B_{z0}(s_i) = 0$, then:

$$\tilde{x}(s = s_i) = x(s = s_i) \qquad \tilde{x}'(s = s_i) = x'(s = s_i)$$

$$\tilde{y}(s = s_i) = y(s = s_i) \qquad \tilde{y}'(s = s_i) = y'(s = s_i)$$

$$\tilde{\underline{z}}(s = s_i) = \underline{z}(s = s_i) \qquad \tilde{\underline{z}}'(s = s_i) = \underline{z}'(s = s_i)$$

B6

The transform and inverse transform between the laboratory and rotating frames can then be applied to project initial conditions into the rotating frame for integration and then the rotating frame solution back into the laboratory frame.

Using the real and imaginary parts of the complex-valued transformations:

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix} = \tilde{\mathbf{M}}_r(s|s_i) \cdot \begin{bmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{bmatrix} \qquad \begin{bmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{bmatrix} = \tilde{\mathbf{M}}_r^{-1}(s|s_i) \cdot \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}$$

$$\tilde{\mathbf{M}}_r(s|s_i) = \begin{bmatrix} \cos\tilde{\psi} & 0 & -\sin\tilde{\psi} & 0 \\ k_L\sin\tilde{\psi} & \cos\tilde{\psi} & k_L\cos\tilde{\psi} & -\sin\tilde{\psi} \\ \sin\tilde{\psi} & 0 & \cos\tilde{\psi} & 0 \\ -k_L\cos\tilde{\psi} & \sin\tilde{\psi} & k_L\sin\tilde{\psi} & \cos\tilde{\psi} \end{bmatrix}$$

$$\tilde{\mathbf{M}}_r^{-1}(s|s_i) = \begin{bmatrix} \cos\tilde{\psi} & 0 & \sin\tilde{\psi} & 0 \\ k_L\sin\tilde{\psi} & \cos\tilde{\psi} & -k_L\cos\tilde{\psi} & \sin\tilde{\psi} \\ -\sin\tilde{\psi} & 0 & \cos\tilde{\psi} & 0 \\ k_L\cos\tilde{\psi} & -\sin\tilde{\psi} & k_L\sin\tilde{\psi} & \cos\tilde{\psi} \end{bmatrix}$$

Here we used:

$$\tilde{\psi}' = -k_L$$

and it can be verified that:

$$\tilde{\mathbf{M}}_r^{-1} = \text{Inverse}[\tilde{\mathbf{M}}_r]$$

B7

Appendix C: Transfer Matrices for Hard-Edge Solenoidal Focusing

Using results and notation from Appendix B, derive transfer matrix for single particle orbit with:

- No space-charge
- No momentum spread

- Details of decompositions can be found in: Conte and Mackay, "An Introduction to the Physics of Particle Accelerators" (2nd edition; 2008)

First, the solution to the Larmor-frame equations of motion:

$$\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa(s)\tilde{x} = 0$$

$$\tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa(s)\tilde{y} = 0$$

$$\kappa = k_L^2 = \left(\frac{B_{z0}}{2[B\rho]}\right)^2$$

Can be expressed as:

$$\begin{bmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{bmatrix}_z = \tilde{\mathbf{M}}_L(z|z_i) \cdot \begin{bmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{bmatrix}_{z=z_i}$$

- In this appendix we use *z* rather than *s* for the axial coordinate since there are not usually bends in a solenoid

C1

Transforming the solution back to the laboratory frame:

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_z = \tilde{\mathbf{M}}_r(z|z_i)\cdot\tilde{\mathbf{M}}_L(z|z_i)\cdot\underbrace{\overbrace{\tilde{\mathbf{M}}_r^{-1}(z_i|z_i)}^{\text{From project of initial conditions to Larmor Frame}}}_{= I \text{ Identity Matrix}}\cdot\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_{z=z_i}$$

- Here we assume the initial condition is outside the magnetic field so that there is no adjustment to the Larmor frame angles, i.e., $\tilde{\mathbf{M}}_r^{-1}(z_i|z_i) = \mathbf{I}$

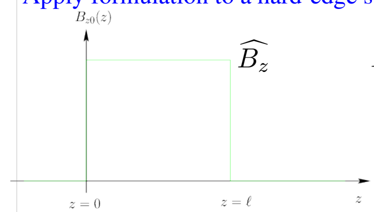
$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_z \equiv \mathbf{M}(z|z_i)\cdot\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_{z=z_i} = \tilde{\mathbf{M}}_r(z|z_i)\cdot\tilde{\mathbf{M}}_L(z|z_i)\cdot\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_{z=z_i}$$

$$\mathbf{M}(z|z_i) = \tilde{\mathbf{M}}_r(z|z_i)\cdot\tilde{\mathbf{M}}_L(z|z_i)$$

- Care must be taken when applying to discontinuous (hard-edge) field models of solenoids to correctly calculate transfer matrices
  - Fringe field influences beam “spin-up” and “spin-down” entering and exiting the magnet

C2

Apply formulation to a hard-edge solenoid with no acceleration [$(\gamma_b\beta_b)' = 0$]:



$$B_{z0}(z) = \widehat{B_z}\,[\Theta(z) - \Theta(z-\ell)]$$

$\widehat{B_z} = \text{const} = \text{Hard-Edge Field}$

$\ell = \text{const} = \text{Hard-Edge Magnet Length}$

Note coordinate choice: z=0 is start of magnet

Calculate the Larmor-frame transfer matrix in $0 \leq z \leq \ell$:

$$\tilde{x}'' + k_L^2\tilde{x} = 0 \qquad \tilde{y}'' + k_L^2\tilde{y} = 0$$

$$k_L = \frac{qB_{z0}}{2\gamma_b\beta_b mc} = \frac{B_{z0}}{2[B\rho]} = \frac{\widehat{B_z}}{2[B\rho]} = \text{const}$$

$0^- \leq z \leq \ell^+$

$$\tilde{\mathbf{M}}_L(z|0^-) = \begin{bmatrix} C & S/k_L & 0 & 0 \\ -k_L S & C & 0 & 0 \\ 0 & 0 & C & S/k_L \\ 0 & 0 & -k_L S & C \end{bmatrix}$$

$$C \equiv \cos(k_L z) \qquad S \equiv \sin(k_L z)$$

Subtle Point:
Larmor frame transfer matrix is valid both sides of discontinuity in focusing entering and exiting solenoid.

C3

The Larmor-frame transfer matrix can be decomposed as:
- Useful for later constructs

$$\tilde{\mathbf{M}}_L(z|0^-) = \begin{bmatrix} C & S/k_L & 0 & 0 \\ -k_L S & C & 0 & 0 \\ 0 & 0 & C & S/k_L \\ 0 & 0 & -k_L S & C \end{bmatrix} = \begin{bmatrix} \mathbf{F}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{F}(z) \end{bmatrix}$$

with

$$\tilde{\mathbf{F}}(z) \equiv \begin{bmatrix} C(z) & S(z)/k_L \\ -k_L S(z) & C(z) \end{bmatrix} \qquad \mathbf{0} \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using results from Appendix E, ***F*** can be further decomposed as:

$$\begin{aligned}\tilde{\mathbf{F}}(z) &= \begin{bmatrix} C(z) & S(z)/k_L \\ -k_L S(z) & C(z) \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{1}{k_L}\tan\left(\frac{k_L z}{2}\right) \\ 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ -k_L\sin(k_L z) & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & \frac{1}{k_L}\tan\left(\frac{k_L z}{2}\right) \\ 0 & 1 \end{bmatrix} \\ &= \mathbf{M}_{\text{drift}}(z)\cdot\mathbf{M}_{\text{thin-lens}}(z)\cdot\mathbf{M}_{\text{drift}}(z)\end{aligned}$$

C4

Applying these results and the formulation of Appendix B, we obtain the rotation matrix within the magnet $0 < z < \ell$:
- Here we apply $\tilde{\mathbf{M}}_r$ formula with $\tilde{\psi} = -k_L z$ for the hard-edge solenoid

$$\tilde{\mathbf{M}}_r(z|0^-) = \begin{bmatrix} C & 0 & S & 0 \\ -k_L S & C & k_L C & S \\ -S & 0 & C & 0 \\ -k_L C & -S & -k_L S & C \end{bmatrix}$$

With special magnet end-forms:
- Here we exploit continuity of $\tilde{\mathbf{M}}_r$ in Larmor frame

Entering solenoid

$$\tilde{\mathbf{M}}_r(0^+|0^-) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_L & 0 \\ 0 & 0 & 1 & 0 \\ -k_L & 0 & 0 & 1 \end{bmatrix}$$

- Direct plug-in from formula for $\tilde{\mathbf{M}}_r$

Exiting solenoid

$$\tilde{\mathbf{M}}_r(\ell^+|\ell^-) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -k_L & 0 \\ 0 & 0 & 1 & 0 \\ k_L & 0 & 0 & 1 \end{bmatrix}$$

- Slope of fringe field is reversed so replace in entrance formula: $k_L \rightarrow -k_L$

C5

The rotation matrix through the full solenoid is:

$$\tilde{\mathbf{M}}_r(\ell^+|0^-) = \begin{bmatrix} \cos\Phi & 0 & \sin\Phi & 0 \\ 0 & \cos\Phi & 0 & \sin\Phi \\ -\sin\Phi & 0 & \cos\Phi & 0 \\ 0 & -\sin\Phi & 0 & \cos\Phi \end{bmatrix} = \begin{bmatrix} \mathbf{I}\cos\Phi & \mathbf{I}\sin\Phi \\ -\mathbf{I}\sin\Phi & \mathbf{I}\cos\Phi \end{bmatrix}$$

$$\Phi \equiv k_L \ell \qquad \mathbf{I} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the rotation matrix within the solenoid is:

$$\tilde{\mathbf{M}}_r(z|0^-) = \begin{bmatrix} C(z) & 0 & S(z) & 0 \\ 0 & C(z) & 0 & S(z) \\ -S(z) & 0 & C(z) & 0 \\ 0 & -S(z) & 0 & C(z) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_L & 0 \\ 0 & 0 & 1 & 0 \\ -k_L & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C(z)\mathbf{I} & S(z)\mathbf{I} \\ -S(z)\mathbf{I} & C(z)\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I} & \mathbf{K} \\ -\mathbf{K} & \mathbf{I} \end{bmatrix} \qquad \mathbf{K} \equiv \begin{bmatrix} 0 & 0 \\ k_L & 0 \end{bmatrix}$$

$$= \tilde{\mathbf{M}}_r(z|0^+) \cdot \tilde{\mathbf{M}}_r(0^+|0^-) \qquad 0 < z < \ell$$

Note that the rotation matrix kick entering the solenoid is expressible as

$$\tilde{\mathbf{M}}_r(0^+|0^-) = \begin{bmatrix} \mathbf{I} & \mathbf{K} \\ -\mathbf{K} & \mathbf{I} \end{bmatrix}$$

C6

The lab-frame advance matrices are then (after expanding matrix products):

Inside Solenoid $0^+ \leq z \leq \ell^-$

$$\mathbf{M}(z|0^-) = \tilde{\mathbf{M}}_r(z|0^-)\tilde{\mathbf{M}}_L(z|0^-)$$

$$= \begin{bmatrix} \cos^2\phi & \frac{1}{2k_L}\sin(2\phi) & \frac{1}{2}\sin(2\phi) & \frac{1}{k_L}\sin^2\phi \\ -k_L\sin(2\phi) & \cos(2\phi) & k_L\cos(2\phi) & \sin(2\phi) \\ -\frac{1}{2}\sin(2\phi) & -\frac{1}{k_L}\sin^2\phi & \cos^2\phi & \frac{1}{2k_L}\sin(2\phi) \\ -k_L\cos(2\phi) & -\sin(2\phi) & -k_L\sin(2\phi) & \cos(2\phi) \end{bmatrix}$$

$$\phi \equiv k_L z$$

$$= \begin{bmatrix} C(z)\mathbf{I} & S(z)\mathbf{I} \\ -S(z)\mathbf{I} & C(z)\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I} & \mathbf{K} \\ -\mathbf{K} & \mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{F}(z) \end{bmatrix}$$

$$= \begin{bmatrix} C(z)\mathbf{I} - S(z)\mathbf{K} & C(z)\mathbf{K} + S(z)\mathbf{I} \\ -C(z)\mathbf{K} - S(z)\mathbf{I} & C(z)\mathbf{I} - S(z)\mathbf{K} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{F}(z) \end{bmatrix}$$

$$= \begin{bmatrix} C(z)\mathbf{F}(z) - S(z)\mathbf{K}\cdot\mathbf{F}(z) & C(z)\mathbf{K}\cdot\mathbf{F}(z) + S(z)\mathbf{F}(z) \\ -C(z)\mathbf{K}\cdot\mathbf{F}(z) - S(z)\mathbf{F}(z) & C(z)\mathbf{F}(z) - S(z)\mathbf{K}\cdot\mathbf{F}(z) \end{bmatrix}$$

- 2^{nd} forms useful to see structure of transfer matrix

C7

Through entire Solenoid $z = \ell^+$

$$\mathbf{M}(\ell^+|0^-) = \tilde{\mathbf{M}}_r(\ell^+|0^-)\tilde{\mathbf{M}}_L(\ell^+|0^-)$$

$$= \begin{bmatrix} \cos^2\Phi & \frac{1}{2k_L}\sin(2\Phi) & \frac{1}{2}\sin(2\Phi) & \frac{1}{k_L}\sin^2\Phi \\ -\frac{k_L}{2}\sin(2\Phi) & \cos^2\Phi & -k_L\sin^2\Phi & \frac{1}{2}\sin(2\Phi) \\ -\frac{1}{2}\sin(2\Phi) & -\frac{1}{k_L}\sin^2\Phi & \cos^2\Phi & \frac{1}{2k_L}\sin(2\Phi) \\ k_L\sin^2\Phi & -\frac{1}{2}\sin(2\Phi) & -\frac{k_L}{2}\sin(2\Phi) & \cos^2\Phi \end{bmatrix}$$

$$\Phi \equiv k_L \ell$$

$$= \begin{bmatrix} \cos\Phi\mathbf{I} & \sin\Phi\mathbf{I} \\ -\sin\Phi\mathbf{I} & \cos\Phi\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F}(\ell) & \mathbf{0} \\ \mathbf{0} & \mathbf{F}(\ell) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Phi\mathbf{F}(\ell) & \sin\Phi\mathbf{F}(\ell) \\ -\sin\Phi\mathbf{F}(\ell) & \cos\Phi\mathbf{F}(\ell) \end{bmatrix}$$

- 2^{nd} forms useful to see structure of transfer matrix

Note that due to discontinuous fringe field:

$$\mathbf{M}(0^+|0^-) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_L & 0 \\ 0 & 0 & 1 & 0 \\ -k_L & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{K} \\ -\mathbf{K} & \mathbf{I} \end{bmatrix} \neq I$$

Fringe going in kicks angles of beam

C8

$$\mathbf{M}(\ell^-|0^-) \neq \mathbf{M}(\ell^+|0^-)$$

Due to fringe exiting kicking angles of beam

In more realistic model with a continuously varying fringe to zero, all transfer matrix components will vary continuously across boundaries

- Still important to get this right in idealized designs often taken as a first step!

Focusing kicks on particles entering/exiting the solenoid can be calculated as:

Entering:

$$x(0^+) = x(0^-) \qquad x'(0^+) = x'(0^-) + k_L y(0^-)$$
$$y(0^+) = y(0^-) \qquad y'(0^+) = y'(0^-) - k_L x(0^-)$$

Exiting:

$$x(\ell^+) = x(\ell^-) \qquad x'(\ell^+) = x'(\ell^-) - k_L y(\ell^-)$$
$$y(\ell^+) = y(\ell^-) \qquad y'(\ell^+) = y'(\ell^-) + k_L x(\ell^-)$$

- Beam spins up/down on entering/exiting the (abrupt) magnetic fringe field
- Sense of rotation changes with entry/exit of hard-edge field.

C9

The transfer matrix for a hard-edge solenoid can be resolved into thin-lens kicks entering and exiting the optic and an rotation in the central region of the optic as:

$$\mathbf{M}(\ell^+|0^-) = \tilde{\mathbf{M}}_r(\ell^+|0^-)\tilde{\mathbf{M}}_L(\ell^+|0^-)$$

$$= \begin{bmatrix} \cos^2\Phi & \frac{1}{2k_L}\sin(2\Phi) & \frac{1}{2}\sin(2\Phi) & \frac{1}{k_L}\sin^2\Phi \\ -\frac{k_L}{2}\sin(2\Phi) & \cos^2\Phi & -k_L\sin^2\Phi & \frac{1}{2}\sin(2\Phi) \\ -\frac{1}{2}\sin(2\Phi) & -\frac{1}{k_L}\sin^2\Phi & \cos^2\Phi & \frac{1}{2k_L}\sin(2\Phi) \\ k_L\sin^2\Phi & -\frac{1}{2}\sin(2\Phi) & -\frac{k_L}{2}\sin(2\Phi) & \cos^2\Phi \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -k_L & 0 \\ 0 & 0 & 1 & 0 \\ k_L & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2k_L}\sin(2\Phi) & 0 & \frac{1}{k_L}\sin^2\Phi \\ 0 & \cos(2\Phi) & 0 & \sin(2\Phi) \\ 0 & \frac{1}{k_L}\sin^2\Phi & 1 & \frac{1}{2k_L}\sin(2\Phi) \\ 1 & -\sin(2\Phi) & 0 & \cos(2\Phi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_L & 0 \\ 0 & 0 & 1 & 0 \\ -k_L & 0 & 0 & 1 \end{bmatrix}$$

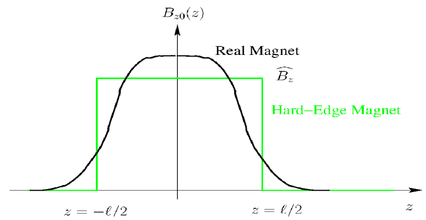
$$= \mathbf{M}(\ell^+|\ell^-)\cdot\mathbf{M}(\ell^-|0^+)\cdot\mathbf{M}(0^+|0^-)$$

where $\Phi \equiv k_L\ell$

- Focusing effect effectively from thin lens kicks at entrance/exit of solenoid as particle traverses the (abrupt here) fringe field

C10

The transfer matrix for the hard-edge solenoid is exact within the context of linear optics. However, real solenoid magnets have an axial fringe field. An obvious need is how to best set the hard-edge parameters B_z, ℓ from the real fringe field.



Hard-Edge and Real Magnets axially centered to compare

Simple physical motivated prescription by requiring:

1) Equivalent Linear Focus Impulse $\propto \int dz\, k_L^2 \propto \int dz B_{z0}^2$

$$\Longrightarrow \boxed{\int_{-\infty}^{\infty} dz\ B_{z0}^2(z) = \ell\widehat{B}_z^{\,2}}$$

2) Equivalent Net Larmor Rotation Angle $\propto \int dz\ k_L \propto \int dz\ B_{z0}$

$$\Longrightarrow \boxed{\int_{-\infty}^{\infty} dz\ B_{z0}(z) = \ell\widehat{B}_z}$$

C11

Solve 1) and 2) for harde edge parameters $\widehat{B}_z$, ℓ

$$\widehat{B}_z = \frac{\int_{-\infty}^{\infty} dz\ B_{z0}^2(z)}{\int_{-\infty}^{\infty} dz\ B_{z0}(z)}$$

$$\ell = \frac{\left[\int_{-\infty}^{\infty} dz\ B_{z0}(z)\right]^2}{\int_{-\infty}^{\infty} dz\ B_{z0}^2(z)}$$

C12

Appendix D: Axisymmetric Applied Magnetic or Electric Field Expansion

Static, rationally symmetric static applied fields $\mathbf{E}^a$, $\mathbf{B}^a$ satisfy the vacuum Maxwell equations in the beam aperture:

$$\nabla\cdot\mathbf{E}^a = 0 \qquad \nabla\times\mathbf{E}^a = 0 \qquad \nabla\cdot\mathbf{B}^a = 0 \qquad \nabla\times\mathbf{B}^a = 0$$

This implies we can take for some electric potential $\phi^e$and magnetic potential ϕ^m:

$$\mathbf{E}^a = -\nabla\phi^e \qquad \mathbf{B}^a = -\nabla\phi^m$$

which in the vacuum aperture satisfies the Laplace equations:

$$\nabla^2\phi^e = 0 \qquad \nabla^2\phi^m = 0$$

We will analyze the magnetic case and the electric case is analogous. In axisymmetric $(\partial/\partial\theta = 0)$ geometry we express Laplace's equation as:

$$\boxed{\nabla^2\phi^m(r,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi^m}{\partial r}\right) + \frac{\partial^2\phi^m}{\partial z^2} = 0}$$

Due to symmetry, $\phi^m(r,z)$ is an even function of *r* and can be expanded as

$$\phi^m(r,z) = \sum_{\nu=0}^{\infty} f_{2\nu}(z)r^{2\nu} = f_0 + f_2r^2 + f_4r^4 + \ldots$$

where $f_0 = \phi^m(r=0,z)$ is the on-axis potential

D1

Plugging ϕ^m into Laplace's equation yields the recursion relation for $f_{2\nu}$

$$(2\nu+2)^2 f_{2\nu+2} + f''_{2\nu} = 0$$

Iteration then shows that

$$\phi^m(r,z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} f(0,z)}{\partial z^{2\nu}} \left(\frac{r}{2}\right)^{2\nu}$$

Using $B_z^a(r=0,z) \equiv B_{z0}(z) = -\frac{\partial \phi_m(0,z)}{\partial z}$ and diffrentiating yields:

$$\boxed{\begin{aligned} B_r^a(r,z) &= -\frac{\partial \phi_m}{\partial r} = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{(\nu!)(\nu-1)!} \frac{\partial^{2\nu-1} B_{z0}(z)}{\partial z^{2\nu-1}} \left(\frac{r}{2}\right)^{2\nu-1} \\ B_z^a(r,z) &= -\frac{\partial \phi_m}{\partial z} = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} B_{z0}(z)}{\partial z^{2\nu}} \left(\frac{r}{2}\right)^{2\nu} \end{aligned}}$$

- Electric case immediately analogous and can arise in electrostatic Einzel lens focusing systems often employed near injectors

D2