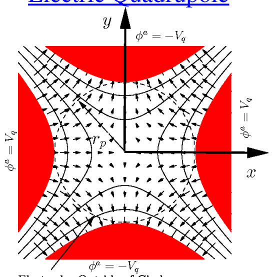
S3: Description of Applied Focusing Fields
S3A: Overview

Applied fields for focusing, bending, and acceleration enter the equations of motion via:

$$\mathbf{E}^a = \text{Applied Electric Field}$$
$$\mathbf{B}^a = \text{Applied Magnetic Field}$$

Generally, these fields are produced by sources (often static or slowly varying in time) located outside an aperture or so-called pipe radius $r = r_p$. For example, the electric and magnetic quadrupoles of S2:

Electric Quadrupole

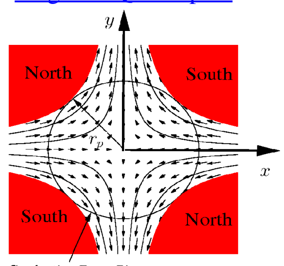


Electrodes Outside of Circle $r = r_p$
Electrodes: $x^2 - y^2 = \mp r_p^2$

Hyperbolic material surfaces outside pipe radius

$$r = r_p$$

Magnetic Quadrupole



Conducting Beam Pipe: $r = r_p$
Poles: $xy = \pm\frac{r_p^2}{2}$

The fields of such classes of magnets obey the vacuum Maxwell Equations within the aperture:

$$\boxed{\begin{array}{ll} \nabla \cdot \mathbf{E}^a = 0 & \nabla \cdot \mathbf{B}^a = 0 \\ \nabla \times \mathbf{E}^a = -\dfrac{\partial}{\partial t}\mathbf{B}^a & \nabla \times \mathbf{B}^a = \dfrac{1}{c^2}\dfrac{\partial}{\partial t}\mathbf{E}^a \end{array}}$$

If the fields are static or sufficiently slowly varying (quasistatic) where the time derivative terms can be neglected, then the fields in the aperture will obey the static vacuum Maxwell equations:

$$\boxed{\begin{array}{ll} \nabla \cdot \mathbf{E}^a = 0 & \nabla \cdot \mathbf{B}^a = 0 \\ \nabla \times \mathbf{E}^a = 0 & \nabla \times \mathbf{B}^a = 0 \end{array}}$$

In general, optical elements are tuned to limit the strength of nonlinear field terms so the beam experiences primarily linear applied fields.

- Linear fields allow better preservation of beam quality

Removal of *all* nonlinear fields cannot be accomplished

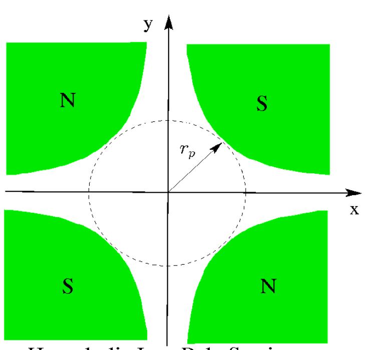
- 3D structure of the Maxwell equations precludes for finite geometry optics
- Even in finite geometries deviations from optimal structures and symmetry will result in nonlinear fields

As an example of this, when an ideal 2D iron magnet with infinite hyperbolic poles is truncated radially for finite 2D geometry, this leads to nonlinear focusing fields even in 2D:

- Truncation necessary along with confinement of return flux in yoke

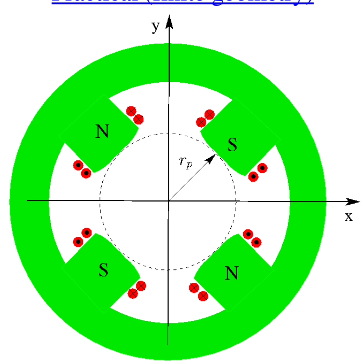
Cross-Sections of Iron Quadrupole Magnets

Ideal (infinite geometry)



Hyperbolic Iron Pole Sections (infinite)

Practical (finite geometry)



Shaped Iron Pole Sections (finite)

The design of optimized electric and magnetic optics for accelerators is a specialized topic with a vast literature. It is not be possible to cover this topic in this brief survey. In the remaining part of this section we will overview a limited subset of material on magnetic optics including:

- (see: S3B) Magnetic field expansions for focusing and bending
- (see: S3C) Hard edge equivalent models
- (see: S3D) 2D multipole models and nonlinear field scalings
- (see: S3E) Good field radius

Much of the material presented can be immediately applied to static Electric Optics since the vacuum Maxwell equations are the same for static Electric $\mathbf{E}^a$ and Magnetic $\mathbf{B}^a$ fields in vacuum.

S3B: Magnetic Field Expansions for Focusing and Bending

Forces from transverse $(B_z^a = 0)$ magnetic fields enter the transverse equations of motion (see: S1, S2) via:

Force: $\mathbf{F}_\perp^a \simeq q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}_\perp^a$

Field: $\mathbf{B}_\perp^a = \hat{\mathbf{x}}B_x^a + \hat{\mathbf{y}}B_y^a$

Combined these give:

$$\boxed{\begin{aligned} F_x^a &\simeq -q\beta_b c B_y^a \\ F_y^a &\simeq \;\; q\beta_b c B_x^a \end{aligned}}$$

Field components entering these expressions can be expanded about $\mathbf{x}_\perp = 0$

- Element center and design orbit taken to be at $\mathbf{x}_\perp = 0$

$$B_x^a = \overset{1}{B_x^a(0)} + \overset{2}{\frac{\partial B_x^a}{\partial y}(0)y} + \overset{3}{\frac{\partial B_x^a}{\partial x}(0)x}$$

Nonlinear Focus

$$+\frac{1}{2}\frac{\partial^2 B_x^a}{\partial x^2}(0)x^2 + \frac{\partial^2 B_x^a}{\partial x \partial y}(0)xy + \frac{1}{2}\frac{\partial B_x^a}{\partial y^2}(0)y^2 + \cdots$$

$$B_y^a = \overset{1}{B_y^a(0)} + \overset{2}{\frac{\partial B_y^a}{\partial x}(0)x} + \overset{3}{\frac{\partial B_y^a}{\partial y}(0)y}$$

Nonlinear Focus

$$+\frac{1}{2}\frac{\partial^2 B_y^a}{\partial x^2}(0)x^2 + \frac{\partial^2 B_y^a}{\partial x \partial y}(0)xy + \frac{1}{2}\frac{\partial B_y^a}{\partial y^2}(0)y^2 + \cdots$$

Terms:
1: Dipole Bend
2: Normal Quad Focus
3: Skew Quad Focus

Sources of undesired nonlinear applied field components include:

- Intrinsic finite 3D geometry and the structure of the Maxwell equations
- Systematic errors or sub-optimal geometry associated with practical trade-offs in fabricating the optic
- Random construction errors in individual optical elements
- Alignment errors of magnets in the lattice giving field projections in unwanted directions
- Excitation errors effecting the field strength
  - Currents in coils not correct and/or unbalanced

More advanced treatments exploit less simple power-series expansions to express symmetries more clearly:

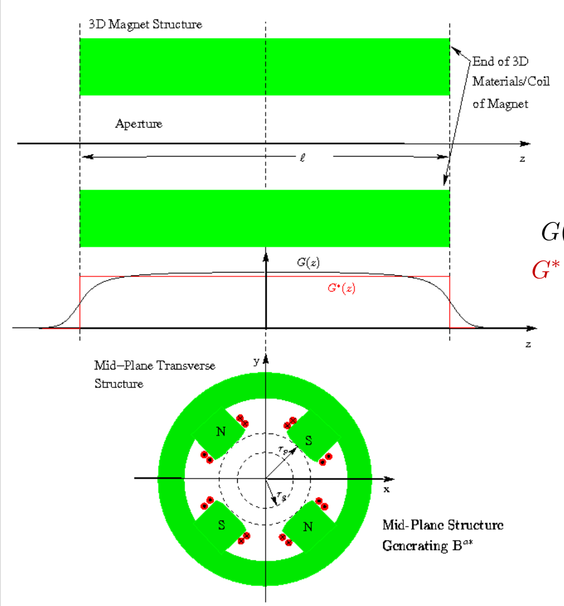
- Maxwell equations constrain structure of solutions
  - Expansion coefficients are NOT all independent
- Forms appropriate for bent coordinate systems in dipole bends can become complicated

S3C: Hard Edge Equivalent Models

Real 3D magnets can often be modeled with sufficient accuracy by 2D hard-edge "equivalent" magnets that give the same approximate focusing impulse to the particle as the full 3D magnet

- Objective is to provide same approximate applied focusing "kick" to particles with different gradient focusing gradient functions $G(s)$

See Figure Next Slide



$G(z) =$ 3D Field Gradient

$G^*(z) =$ Hard-Edge Equivalent Field Gradient

Many prescriptions exist for calculating the effective axial length and strength of hard-edge equivalent models

- See Review: Lund and Bukh, PRSTAB 7 204801 (2004), Appendix C

Here we overview a simple equivalence method that has been shown to work well:

For a relatively long, but finite axial length magnet with 3D gradient function:

$$G(z) \equiv \left. \frac{\partial B_x^a}{\partial y} \right|_{x=y=0}$$

Take hard-edge equivalent parameters:

- Take $z = 0$ at the axial magnet mid-plane

Gradient: $$G^* \equiv G(z=0)$$

Axial Length: $$\ell \equiv \frac{1}{G(z=0)} \int_{-\infty}^{\infty} dz\; G(z)$$

- More advanced equivalences can be made based more on particle optics
  - Disadvantage of such methods is "equivalence" changes with particle energy and must be revisited as optics are tuned

S3D: 2D Transverse Multipole Magnetic Fields

In many cases, it is sufficient to characterize the field errors in 2D hard-edge equivalent as:

$$\overline{B}_x(x,y) = \frac{1}{\ell}\int_{-\infty}^{\infty} dz\; B_x^a(x,y,z)$$

$$\overline{B}_y(x,y) = \frac{1}{\ell}\int_{-\infty}^{\infty} dz\; B_y^a(x,y,z)$$

2D Effective Fields 3D Fields

Operating on the vacuum Maxwell equations with: $\displaystyle\int_{-\infty}^{\infty} \frac{dz}{\ell} \cdots$

yields the (exact) 2D Transverse Maxwell equations :

$$\frac{\partial \overline{B}_x(x,y)}{\partial y} = \frac{\partial \overline{B}_y(x,y)}{\partial x} \quad \Leftarrow \text{From} \quad \nabla \times \mathbf{B} = 0$$

$$\frac{\partial \overline{B}_x(x,y)}{\partial x} = -\frac{\partial \overline{B}_y(x,y)}{\partial y} \quad \Leftarrow \text{From} \quad \nabla \cdot \mathbf{B} = 0$$

These equations are recognized as the Cauchy-Riemann conditions for a complex field variable:

$$\underline{B}^* \equiv \overline{B}_x - i\overline{B}_y \qquad i \equiv \sqrt{-1}$$

to be an analytical function of the complex variable:

$$\underline{z} \equiv x + iy \qquad i \equiv \sqrt{-1}$$

Notation: Underlines denote complex variables where confusion may arise

Cauchy-Riemann Conditions		2D Magnetic Field
$\underline{F} = u(x,y) + iv(x,y)$		$u = \overline{B}_x \quad v = -\overline{B}_y$
$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$	$\longrightarrow$	$\dfrac{\partial \overline{B}_x(x,y)}{\partial x} = -\dfrac{\partial \overline{B}_y(x,y)}{\partial y}$
$\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$	$\longrightarrow$	$\dfrac{\partial \overline{B}_x(x,y)}{\partial y} = \dfrac{\partial \overline{B}_y(x,y)}{\partial x}$
$\underline{F} = u + iv$ analytic func of $\underline{z} = x + iy$		$\underline{F} = \overline{B}_x - i\overline{B}_y$ analytic func of $\underline{z} = x + iy$

Note the complex field which is an analytic function of $\underline{z} = x + iy$ is $\underline{B}^* = \overline{B}_x - i\overline{B}_y$ NOT $\underline{B} = \overline{B}_x + i\overline{B}_y$. This is *not* a typo and is necessary for $\underline{B}^*$ to satisfy the Cauchy-Riemann conditions.

- See problem sets for illustration

It follows that $\underline{B}^*(\underline{z})$ can be analyzed using the full power of the highly developed theory of analytical functions of a complex variable.

Expand $\underline{B}^*(\underline{z})$ as a Laurent Series within the vacuum aperture as:

$$\underline{B}^*(\underline{z}) = \overline{B}_x(x,y) - i\overline{B}_y(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \underline{z}^{n-1}$$

$$\underline{b}_n = \text{const (complex)}$$

$$n = \text{Multipole Index}$$

The $\underline{b}_n$ are called "multipole coefficients" and give the structure of the field.
The multipole coefficients can be resolved into real and imaginary parts as:

$$\underline{b}_n = \mathcal{A}_n - i\mathcal{B}_n$$
$$\mathcal{B}_n \Longrightarrow \text{"Normal" Multipoles}$$
$$\mathcal{A}_n \Longrightarrow \text{"Skew" Multipoles}$$

Some algebra identifies the polynomial symmetries of low-order terms as:

Cartesian projections: $\overline{B_x} - i\overline{B_y} = (\mathcal{A}_n - i\mathcal{B}_n)(x+iy)^{n-1}$

Index	Name	Normal ($\mathcal{A}_n = 0$)		Skew ($\mathcal{B}_n = 0$)	
n		$\overline{B_x}/\mathcal{B}_n$	$\overline{B_y}/\mathcal{B}_n$	$\overline{B_x}/\mathcal{A}_n$	$\overline{B_y}/\mathcal{A}_n$
1	Dipole	0	1	1	
2	Quadrupole	y	x	x	$-y$
3	Sextupole	$2xy$	x^2-y^2	x^2-y^2	$-2xy$
4	Octupole	$3x^2y-y^3$	x^3-3xy^2	x^3-3xy^2	$-3x^2y+y^3$
5	Decapole	$4x^3y-4xy^3$	$x^4-6x^2y^2+y^4$	$x^4-6x^2y^2+y^4$	$-4x^3y+4xy^3$

Comments:

- Reason for pole names most apparent from polar representation (see following pages) and sketches of the magnetic pole structure
- Caution: In so-called "US notation", poles are labeled with index n -> n -1
  - Arbitrary in 2D but US choice *not* good notation in 3D generalizations

Comments continued:

- Normal and Skew symmetries can be taken as a symmetry *definition.* But this choice makes sense for n = 2 quadrupole focusing terms:

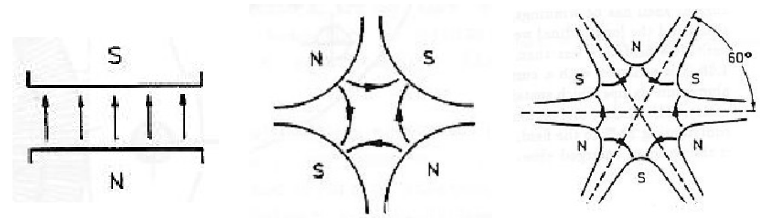
$$\overline{F^a_x} = -q\beta_b c\overline{B_y} \quad = -q\beta_b c(\mathcal{B}_2 x - \mathcal{A}_2 y)$$
$$\overline{F^a_y} = \quad q\beta_b c\overline{B_x} \quad = \quad q\beta_b c(\mathcal{B}_2 y + \mathcal{A}_2 x)$$

In equations of motion:

$$\begin{array}{llll} \text{Normal} \Rightarrow & \mathcal{B}_2: & x\text{-eqn, } x\text{-focus} & y\text{-eqn, } y\text{-defocus} \\ \text{Skew} \Rightarrow & \mathcal{A}_2: & x\text{-eqn, } y\text{-defocus} & y\text{-eqn, } x\text{-defocus} \end{array}$$

Magnetic Pole Symmetries (normal orientation):

Dipole (n=1) Quadrupole (n=2) Sextupole (n=3)



- Actively rotate normal field structures clockwise through an angle of $\pi/(2n)$ for skew field component symmetries

Multipole scale/units

Frequently, in the multipole expansion:

$$\underline{B}^*(\underline{z}) = \overline{B_x}(x,y) - i\overline{B_y}(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \underline{z}^{n-1}$$

the multipole coefficients $\underline{b}_n$ are rescaled as

$$\underline{b}_n \longrightarrow \underline{b}_n r_p^{n-1} \qquad r_p = \text{Aperture "Pipe" Radius}$$

Closest radius of approach of magnetic sources and/or aperture materials

so that the expansions becomes

$$\underline{B}^*(\underline{z}) = \overline{B_x}(x,y) - i\overline{B_y}(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1}$$

Advantages of alternative notaiton:

- Multipoles $\underline{b}_n$ given directly in field units regardless of index n
- Scaling of field amplitudes with radius within the magnet bore becomes clear

Scaling of Fields produced by multipole term:

Higher order multipole coefficients (larger n values) leading to nonlinear focusing forces decrease rapidly within the aperture. To see this use a polar representation for $\underline{z}$, $\underline{b}_n$

$$\underline{z} = x + iy = re^{i\theta} \qquad r = \sqrt{x^2+y^2}, \quad \theta = \arctan[y,x]$$
$$\underline{b}_n = |\underline{b}_n| e^{i\psi_n} \qquad \psi_n = \text{Real Const}$$

Thus, the nth order multipole terms scale as

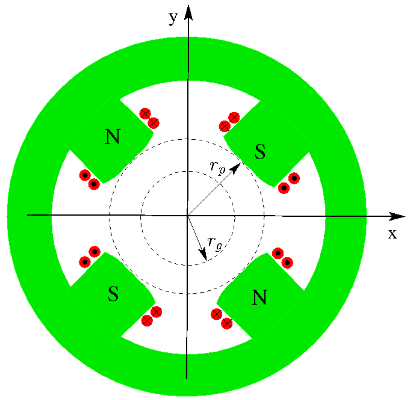
$$\underline{b}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1} = |\underline{b}_n| \left(\frac{r}{r_p}\right)^{n-1} e^{i[(n-1)\theta+\psi_n]}$$

- Unless the coefficient $|\underline{b}_n|$ is very large, high order terms in n will become small rapidly as r_p decreases
- Better field quality can be obtained for a given magnet design by simply making the clear bore r_p larger, or alternatively using smaller bundles (more tight focus) of particles
  - Larger bore machines/magnets cost more. So designs become trade-off between cost and performance.
  - Stronger focusing to keep beam from aperture can be unstable (see: S5)

S3E: Good Field Radius

Often a magnet design will have a so-called "good-field" radius $r = r_g$ that the maximum field errors are specified on.

- In superior designs the good field radius can be around ~70% or more of the clear bore aperture to the beginning of material structures of the magnet.
- Beam particles should evolve with radial excursions with $r < r_g$



$r_p =$ Clear Bore Radius
$\sim$ Pole Radius Typical

$r_g =$ Good Field Radius
$\sim 70\%\ r_p$ Typical

Comments:

- Particle orbits are designed to remain within radius r_g
- Field error statements are readily generalized to 3D since:

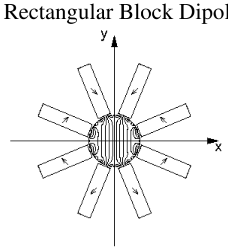
$$\begin{aligned} \nabla \cdot \mathbf{B}^a &= 0 \\ \nabla \times \mathbf{B}^a &= 0 \end{aligned} \quad \Longrightarrow \quad \nabla^2 \mathbf{B}^a = 0$$

and therefore each component of $\mathbf{B}^a$ satisfies a Laplace equation within the vacuum aperture. Therefore, field errors decrease when moving more deeply within a source-free region.

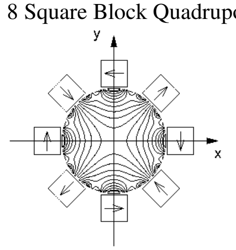
S3F: Example Permanent Magnet Assemblies

A few examples of practical permanent magnet assemblies with field contours are provided to illustrate error field structures in practical devices

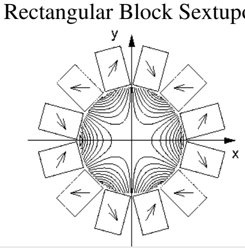
8 Rectangular Block Dipole



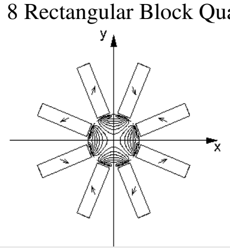
8 Square Block Quadrupole



12 Rectangular Block Sextupole



8 Rectangular Block Quadrupole



For more info on permanent magnet design see: Lund and Halbach, Fusion Engineering Design, **32-33**, 401-415 (1996)

S4: Transverse Particle Equations of Motion with Nonlinear Applied Fields S4A: Overview

In S1 we showed that the particle equations of motion can be expressed as:

$$\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}^a_\perp + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}}\times\mathbf{B}^a_\perp + \frac{qB^a_z}{m\gamma_b\beta_b c}\mathbf{x}'_\perp\times\hat{\mathbf{z}} - \frac{q}{\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}_\perp}\phi$$

When momentum spread is neglected and results are interpreted in a Cartesian coordinate system (no bends). In S2, we showed that these equations can be further reduced when the applied focusing fields are linear to:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial x}\phi$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial y}\phi$$

where

$\kappa_x(s) = x$-focusing function of lattice

$\kappa_y(s) = y$-focusing function of lattice describe the linear applied focusing forces and the equations are implicitly analyzed in the rotating Larmor frame when $B_z^a \neq 0$.

Lattice designs attempt to minimize nonlinear applied fields. However, the 3D Maxwell equations show that there will *always* be some finite nonlinear applied fields for an applied focusing element with finite extent. Applied field nonlinearities also result from:

- Design idealizations
- Fabrication and material errors

The largest source of nonlinear terms will depend on the case analyzed.

Nonlinear applied fields must be added back in the idealized model when it is appropriate to analyze their effects

- Common problem to address when carrying out large-scale numerical simulations to design/analyze systems

There are two basic approaches to carry this out:

Approach 1: Explicit 3D Formulation

Approach 2: Perturbations About Linear Applied Field Model

We will now discuss each of these in turn

S4B: Approach 1: Explicit 3D Formulation

This is the simplest. Just employ the full 3D equations of motion expressed in terms of the applied field components $\mathbf{E}^a$, $\mathbf{B}^a$ and avoid using the focusing functions κ_x, κ_y

Comments:

- Most easy to apply in computer simulations where many effects are simultaneously included
  - Simplifies comparison to experiments when many details matter for high level agreement
- Simplifies simultaneous inclusion of transverse and longitudinal effects
  - Accelerating field E_z^a can be included to calculate changes in β_b, γ_b
  - Transverse and longitudinal dynamics cannot be fully decoupled in high level modeling – especially try when acceleration is strong in systems like injectors
- Can be applied with time based equations of motion (see: S1)
  - Helps avoid unit confusion and continuously adjusting complicated equations of motion to identify the axial coordinate *s* appropriately

S4C: Approach 2: Perturbations About Linear Applied Field Model

Exploit the linearity of the Maxwell equations to take:

$$\boxed{\begin{aligned}\mathbf{E}_\perp^a &= \mathbf{E}_\perp^a|_L + \delta\mathbf{E}_\perp^a \\ \mathbf{B}^a &= \mathbf{B}^a|_L + \delta\mathbf{B}^a\end{aligned}}$$

where

$\mathbf{E}_\perp^a|_L$, $\mathbf{B}^a|_L$ are the linear field components incorporated in κ_x, κ_y

to express the equations of motion as:

$$\boxed{\begin{aligned}x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x &= \frac{q}{m\gamma_b\beta_b^2c^2}\delta E_x^a - \frac{q}{m\gamma_b\beta_b c}\delta B_y^a + \frac{q}{m\gamma_b\beta_b c}\delta B_z^a y' \\ &\quad - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y y &= \frac{q}{m\gamma_b\beta_b^2c^2}\delta E_y^a + \frac{q}{m\gamma_b\beta_b c}\delta B_x^a - \frac{q}{m\gamma_b\beta_b c}\delta B_z^a x' \\ &\quad - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}\end{aligned}}$$

This formulation can be most useful to understand the effect of deviations from the usual linear model where intuition is developed

Comments:

- Best suited to non-solenoidal focusing
  - Simplified Larmor frame analysis for solenoidal focusing is only valid for axisymmetric potentials $\phi = \phi(r)$ which may not hold in the presence of non-ideal perturbations.
  - Applied field perturbations $\delta\mathbf{E}_\perp^a$, $\delta\mathbf{B}^a$ would also need to be projected into the Larmor frame
- Applied field perturbations $\delta\mathbf{E}_\perp^a$, $\delta\mathbf{B}^a$ will not necessarily satisfy the 3D Maxwell Equations by themselves
  - Follows because the linear field components $\mathbf{E}_\perp^a|_L$, $\mathbf{B}^a|_L$ will not, in general, satisfy the 3D Maxwell equations by themselves