S5: Linear Transverse Particle Equations of Motion without Space-Charge, Acceleration, and Momentum Spread

S5A: Hill's Equation

Neglect:

- Space-charge effects: $\partial\phi/\partial\mathbf{x} \simeq 0$
- Nonlinear applied focusing and bends: $\mathbf{E}^a,\ \mathbf{B}^a$ have only linear focus terms
- Acceleration: $\gamma_b\beta_b \simeq \text{const}$
- Momentum spread effects: $v_{zi} \simeq \beta_b c$

Then the transverse particle equations of motion reduce to Hill's Equation:

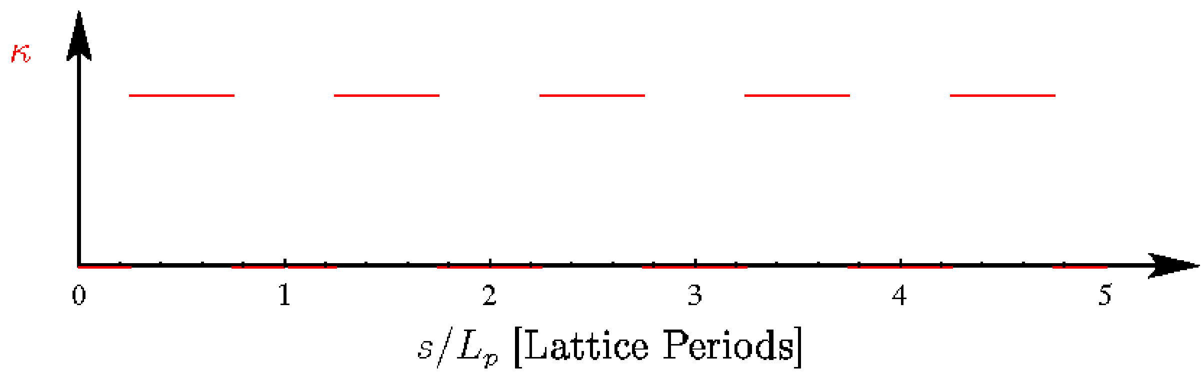
$$\boxed{x''(s) + \kappa(s)x(s) = 0}$$

$$\begin{aligned}
x &= \perp \text{ particle coordinate} \\
&\quad \text{(i.e., } x \text{ or } y \text{ or possibly combinations of coordinates)} \\
s &= \text{Axial coordinate of reference particle} \\
' &= \frac{d}{ds} \\
\kappa(s) &= \text{Lattice focusing function (linear fields)}
\end{aligned}$$

For a periodic lattice:

$$\boxed{\begin{aligned}\kappa(s + L_p) &= \kappa(s) \\ L_p &= \text{Lattice Period}\end{aligned}}$$

/// Example: Hard-Edge Periodic Focusing Function



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For a ring (i.e., circular accelerator), one also has the "superperiod" condition:

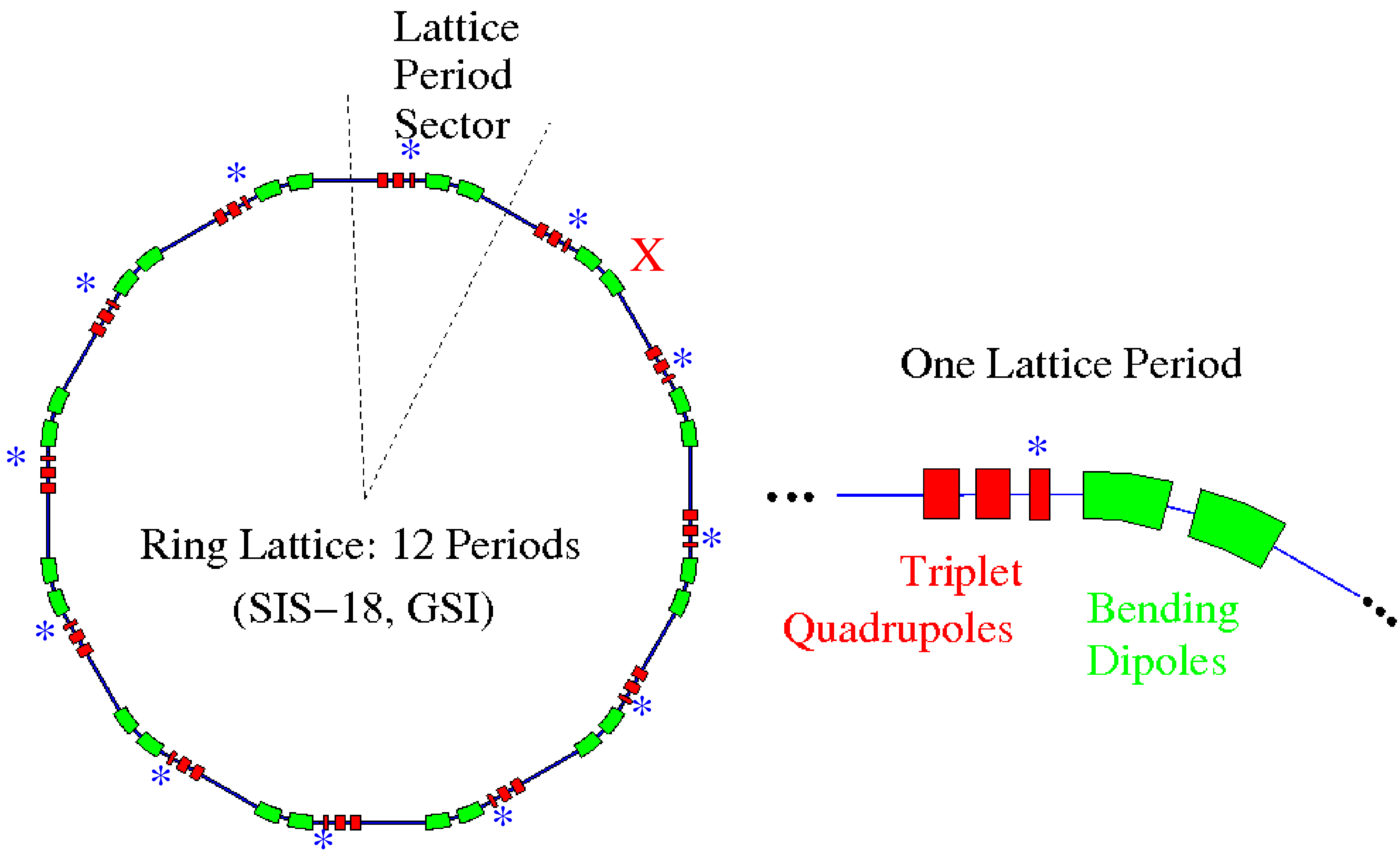
$$\boxed{\begin{aligned}\kappa(s + \mathcal{C}) &= \kappa(s) \\ \mathcal{C} = \mathcal{N} L_p &= \text{Ring Circumfrance} \\ \mathcal{N} &= \text{Superperiod Number}\end{aligned}}$$

- Distinction matters when there are (field) construction errors in the ring
  - Repeat with superperiod but not lattice period
  - See lectures on: Particle Resonances

/// Example: Period and Superperiod distinctions for errors in a ring

* Magnet with systematic defect will be felt every lattice period

X Magnet with random (fabrication) defect felt once per lap



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S5B: Transfer Matrix Form of the Solution to Hill's Equation

Hill's equation is linear. The solution with initial condition:

$$\boxed{\begin{aligned} x(s=s_i) &= x(s_i) \\ x'(s=s_i) &= x'(s_i) \end{aligned}}$$

$s = s_i =$ Axial location of initial condition

can be uniquely expressed in matrix form (**M** is the transfer matrix) as:

$$\boxed{\begin{aligned} \begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} &= \mathbf{M}(s|s_i) \cdot \begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} \\ &= \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \cdot \begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} \end{aligned}}$$

Where $C(s|s_i)$ and $S(s|s_i)$ are "cosine-like" and "sine-like" principal trajectories satisfying:

$$\boxed{\begin{array}{lll} C''(s|s_i) + \kappa(s)C(s|s_i) = 0 & C(s_i|s_i) = 1 & C'(s_i|s_i) = 0 \\ S''(s|s_i) + \kappa(s)S(s|s_i) = 0 & S(s_i|s_i) = 0 & S'(s_i|s_i) = 1 \end{array}}$$

Transfer matrices will be worked out in the problems for a few simple focusing systems discussed in S2 with the additional assumption of piecewise constant $\kappa(s)$

1) Drift: $\kappa = 0$ $\qquad x'' = 0$

$$\mathbf{M}(s|s_i) = \begin{bmatrix} 1 & s - s_i \\ 0 & 1 \end{bmatrix}$$

2) Continuous Focusing: $\kappa = k_{\beta 0}^2 = \text{const} > 0$ $\qquad x'' + k_{\beta 0}^2 x = 0$

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cos[k_{\beta 0}(s - s_i)] & \frac{1}{k_{\beta 0}} \sin[k_{\beta 0}(s - s_i)] \\ -k_{\beta 0} \sin[k_{\beta 0}(s - s_i)] & \cos[k_{\beta 0}(s - s_i)] \end{bmatrix}$$

3) Solenoidal Focusing: $\kappa = \hat{\kappa} = \text{const} > 0$ $\qquad x'' + \hat{\kappa} x = 0$

Results are expressed within the rotating Larmor Frame
(same as continuous focusing with reinterpretation of variables)

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}(s - s_i)] & \frac{1}{\sqrt{\hat{\kappa}}} \sin[\sqrt{\hat{\kappa}}(s - s_i)] \\ -\sqrt{\hat{\kappa}} \sin[\sqrt{\hat{\kappa}}(s - s_i)] & \cos[\sqrt{\hat{\kappa}}(s - s_i)] \end{bmatrix}$$

4) Quadrupole Focusing-Plane: $\kappa = \hat{\kappa} = \text{const} > 0 \qquad x'' + \hat{\kappa} x = 0$

(Obtain from continuous focusing case)

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] \\ -\sqrt{\hat{\kappa}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] & \cos[\sqrt{\hat{\kappa}}(s-s_i)] \end{bmatrix}$$

5) Quadrupole DeFocusing-Plane: $\kappa = -\hat{\kappa} = \text{const} < 0 \qquad x'' - \hat{\kappa} x = 0$

(Obtain from quadrupole focusing case with $\sqrt{\hat{\kappa}} \rightarrow i\sqrt{\hat{\kappa}} \quad i = \sqrt{-1}$)

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cosh[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] \\ \sqrt{\hat{\kappa}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] & \cosh[\sqrt{\hat{\kappa}}(s-s_i)] \end{bmatrix}$$

6) Thin Lens: $\kappa(s) = \frac{1}{f}\delta(s-s_0) \qquad x'' + \frac{1}{f}\delta(s-s_0)x = 0$

$s_0 = \text{const} = \text{Axial Location Lens}$

$f = \text{const} = \text{Focal Length}$

$\delta(x) = \text{Dirac-Delta Function}$

$$\mathbf{M}(s_0^+|s_0^-) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

S5C: Wronskian Symmetry of Hill's Equation

An important property of this linear motion is a Wronskian invariant/symmetry:

$$\boxed{\begin{aligned} W(s|s_i) &\equiv \det \mathbf{M}(s|s_i) = \det \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \\ &= C(s|s_i)S'(s|s_i) - C'(s|s_i)S(s|s_i) = 1 \end{aligned}}$$

/// Proof: Abbreviate Notation $C \equiv C(s|s_i)$ etc.

Multiply Equations of Motion for C and S by $-S$ and C, respectively:

$$-S(C'' + \kappa C) = 0$$

$$+C(S'' + \kappa S) = 0$$

Add Equations:

$$CS'' - SC'' + \kappa(\overbrace{CS - SC}^{0}) = 0$$

$$\Longrightarrow \frac{dW}{ds} = \frac{}{ds}(CS' - C'S) = CS'' - SC'' = 0$$

$$\Longrightarrow W = \text{const}$$

Apply initial conditions:

$$W(s) = W(s_i) = C_i S_i' - C_i' S_i = 1 \cdot 1 - 0 \cdot 0 = 1 \qquad ///$$

/// Example: Continuous Focusing: Transfer Matrix and Wronskian

$$\kappa(s) = k_{\beta 0}^2 = \text{const} > 0$$

Principal orbit equations are simple harmonic oscillators with solution:

$$C(s|s_i) = \cos[k_{\beta 0}(s - s_i)] \qquad C'(s|s_i) = -k_{\beta 0} \sin[k_{\beta 0}(s - s_i)]$$

$$S(s|s_i) = \frac{\sin[k_{\beta 0}(s - s_i)]}{k_{\beta 0}} \qquad S'(s|s_i) = \cos[k_{\beta 0}(s - s_i)]$$

Transfer matrix gives the familiar solution:

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} \cos[k_{\beta 0}(s - s_i)] & \frac{\sin[k_{\beta 0}(s - s_i)]}{k_{\beta 0}} \\ -k_{\beta 0} \sin[k_{\beta 0}(s - s_i)] & \cos[k_{\beta 0}(s - s_i)] \end{bmatrix} \cdot \begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix}$$

Wronskian invariant is elementary:

$$W = \cos^2[k_{\beta 0}(s - s_i)] + \sin^2[k_{\beta 0}(s - s_i)] = 1$$

///

S5D: Stability of Solutions to Hill's Equation in a Periodic Lattice

The transfer matrix must be the same in any period of the lattice:

$$\mathbf{M}(s + L_p | s_i + L_p) = \mathbf{M}(s|s_i)$$

For a propagation distance $s - s_i$ satisfying

$$N L_p \leq s - s_i \leq (N+1) L_p \qquad N = 0, 1, 2, \cdots$$

the transfer matrix can be resolved as

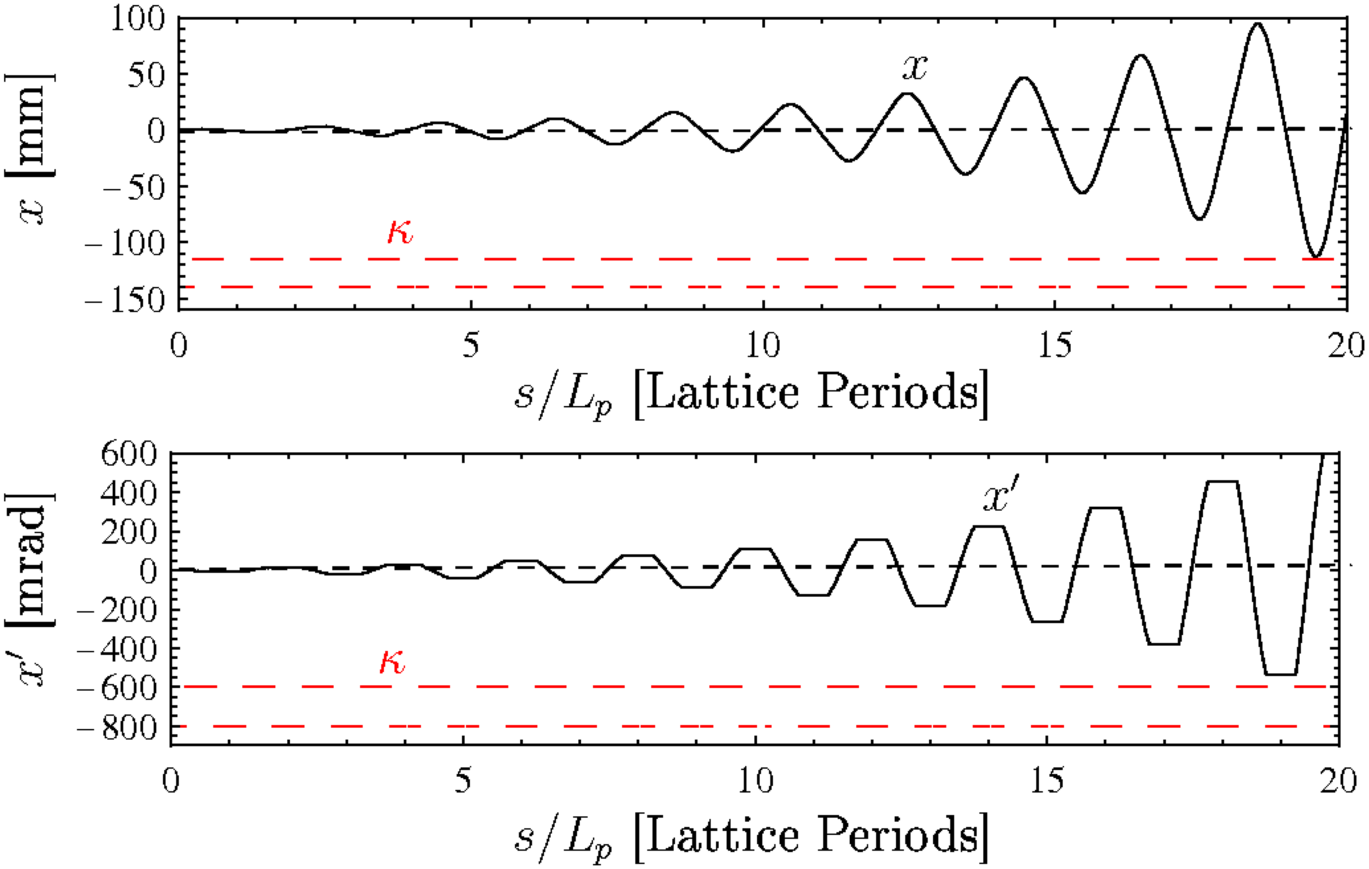
$$\boxed{\begin{aligned} \mathbf{M}(s|s_i) &= \mathbf{M}(s - NL_p|s_i) \cdot \mathbf{M}(s_i + NL_p|s_i) \\ &= \underbrace{\mathbf{M}(s - NL_p|s_i)}_{\text{Residual}} \cdot \underbrace{[\mathbf{M}(s_i + L_p|s_i)]^N}_{N \text{ Full Periods}} \end{aligned}}$$

For a lattice to have stable orbits, both $x(s)$ and $x'(s)$ should remain bounded on propagation through an arbitrary number N of lattice periods. This is equivalent to requiring that the elements of **M** remain bounded on propagation through any number of lattice periods:

$$\mathbf{M}^N \equiv [\mathbf{M}^N{}_{ij}]$$

$$\boxed{\lim_{N \to \infty} \left| \mathbf{M}^N{}_{ij} \right| < \infty \quad \Longrightarrow \text{Stable Motion}}$$

Clarification of stability notion: Unstable Orbit



$$L_p = 0.5 \text{ m}$$
$$\eta = 0.5$$

$$\kappa = \begin{cases} 48/\text{m}^2 & \text{where } \kappa \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$x(0) = 1 \text{ mm}$$
$$x'(0) = 0$$

For energetic particle:

$$H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 \quad \sim \text{Large, but } \neq \text{const}$$

$$\text{where } |x'| \text{ small, } |x| \text{ large}$$
$$\text{where } |x| \text{ small, } |x'| \text{ large}$$

The matrix criterion corresponds to our intuitive notion of stability: as the particle advances there are no large oscillation excursions in position and angle.

To analyze the stability condition, examine the eigenvectors/eigenvalues of **M** for transport through one lattice period:

$$\boxed{\begin{aligned} &\mathbf{M}(s_i + L_p|s_i) \cdot \mathbf{E} \equiv \lambda \mathbf{E} \\ &\mathbf{E} = \text{Eigenvector} \\ &\lambda = \text{Eigenvalue} \end{aligned}}$$

- Eigenvectors and Eigenvalues are generally complex
- Eigenvectors and Eigenvalues generally vary with s_i
- Two independent Eigenvalues and Eigenvectors
  - Degeneracies special case

Derive the two independent eigenvectors/eigenvalues through analysis of the characteristic equation: Abbreviate Notation

$$\mathbf{M}(s_i + L_p|s_i) = \left[\begin{array}{cc} C(s_i + L_p|s_i) & S(s_i + L_p|s_i) \\ C'(s_i + L_p|s_i) & S'(s_i + L_p|s_i) \end{array} \right] \equiv \left[\begin{array}{cc} C & S \\ C' & S' \end{array} \right]$$

Nontrivial solutions exist when:

$$\det \left[\begin{array}{cc} C - \lambda & S \\ C' & S' - \lambda \end{array} \right] = \lambda^2 - (C + S')\lambda + (CS' - SC') = 0$$

But we can apply the Wronskian condition:

$$CS' - SC' = 1$$

and we make the notational definition

$$C + S' = \mathrm{Tr}\,\mathbf{M} \equiv 2\cos\sigma_0$$

The characteristic equation then reduces to:

$$\boxed{\lambda^2 - 2\lambda\cos\sigma_0 + 1 = 0 \qquad \cos\sigma_0 \equiv \frac{1}{2}\mathrm{Tr}\,\mathbf{M}(s_i + L_p|s_i)}$$

The use of $2\cos\sigma_0$ to denote Tr **M** is in anticipation of later results (see S6) where σ_0 is identified as the phase-advance of a stable orbit

There are two solutions to the characteristic equation that we denote $\lambda_\pm$

$$\boxed{\begin{aligned} &\lambda_\pm = \cos\sigma_0 \pm \sqrt{\cos^2\sigma_0 - 1} = \cos\sigma_0 \pm i\sin\sigma_0 = e^{\pm i\sigma_0} \\ &\mathbf{E}_\pm = \text{Corresponding Eigenvectors} \qquad i \equiv \sqrt{-1} \end{aligned}}$$

Note that: $\lambda_+\lambda_- = 1$

$\lambda_+ = 1/\lambda_-$

Consider a vector of initial conditions:

$$\begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} = \begin{bmatrix} x_i \\ x'_i \end{bmatrix}$$

The eigenvectors $\mathbf{E}_\pm$ span two-dimensional space. So any initial condition vector can be expanded as:

$$\boxed{\begin{bmatrix} x_i \\ x'_i \end{bmatrix} = \alpha_+ \mathbf{E}_+ \; + \; \alpha_- \mathbf{E}_- \qquad \alpha_\pm = \text{Complex Constants}}$$

Then using $\mathbf{M}\mathbf{E}_\pm = \lambda_\pm \mathbf{E}_\pm$

$$\boxed{\mathbf{M}^N(s_i + L_p|s_i) \begin{bmatrix} x_i \\ x'_i \end{bmatrix} = \alpha_+ \lambda_+^N \mathbf{E}_+ \; + \; \alpha_- \lambda_-^N \mathbf{E}_-}$$

Therefore, if $\lim_{N \to \infty} \lambda^N$ is bounded, then the motion is stable. This will always be the case if $|\lambda_\pm| = |e^{\pm i\sigma_0}| \leq 1$, corresponding to σ_0 real with $|\cos\sigma_0| \leq 1$

This implies for stability or the orbit that we must have:

$$\boxed{\begin{aligned}\frac{1}{2}\,|\mathrm{Trace}\,\mathbf{M}(s_i+L_p|s_i)| &= \frac{1}{2}|C(s_i+L_p|s_i)+S'(s_i+L_p|s_i)| \\ &= |\cos\sigma_0| \leq 1\end{aligned}}$$

In a periodic focusing lattice, this important stability condition places restrictions on the lattice structure (focusing strength) that are generally interpreted in terms of phase advance limits (see: S6).

- Accelerator lattices almost always tuned for single particle stability to maintain beam control
  - Even for intense beams, beam centroid approximately obeys single particle equations of motion when image charges are negligible
- Space-charge and nonlinear applied fields can further limit particle stability
  - Resonances: see: Particle Resonances ....
  - Envelope Instability: see: Transverse Centroid and Envelope ....
  - Higher Order Instability: see: Transverse Kinetic Stability
- We will show (see: S6) that for stable orbits σ_0 can be interpreted as the phase-advance of single particle oscillations

/// Example: Continuous Focusing Stability

$$\kappa(s) = k_{\beta 0}^2 = \text{const} > 0$$

Principal orbit equations are simple harmonic oscillators with solution:

$$C(s|s_i) = \cos[k_{\beta 0}(s - s_i)] \qquad C'(s|s_i) = -k_{\beta 0}\sin[k_{\beta 0}(s - s_i)]$$

$$S(s|s_i) = \frac{\sin[k_{\beta 0}(s - s_i)]}{k_{\beta 0}} \qquad S'(s|s_i) = \cos[k_{\beta 0}(s - s_i)]$$

Stability bound then gives:

$$\begin{aligned} \frac{1}{2}\,|\text{Trace}\,\mathbf{M}(s_i + L_p|s_i)| &= \frac{1}{2}|C(s_i + L_p|s_i) + S'(s_i + L_p|s_i)| \\ &= |\cos[k_{\beta 0}(s - s_i)]| \leq 1 \end{aligned}$$

- Always satisfied for real $k_{\beta 0}$
- Confirms known result using formalism: continuous focusing stable
  - Energy not pumped into or out of particle orbit

///

The simplest example of the stability criterion applied to periodic lattices will be given in the problem sets: Stability of a periodic thin lens lattice

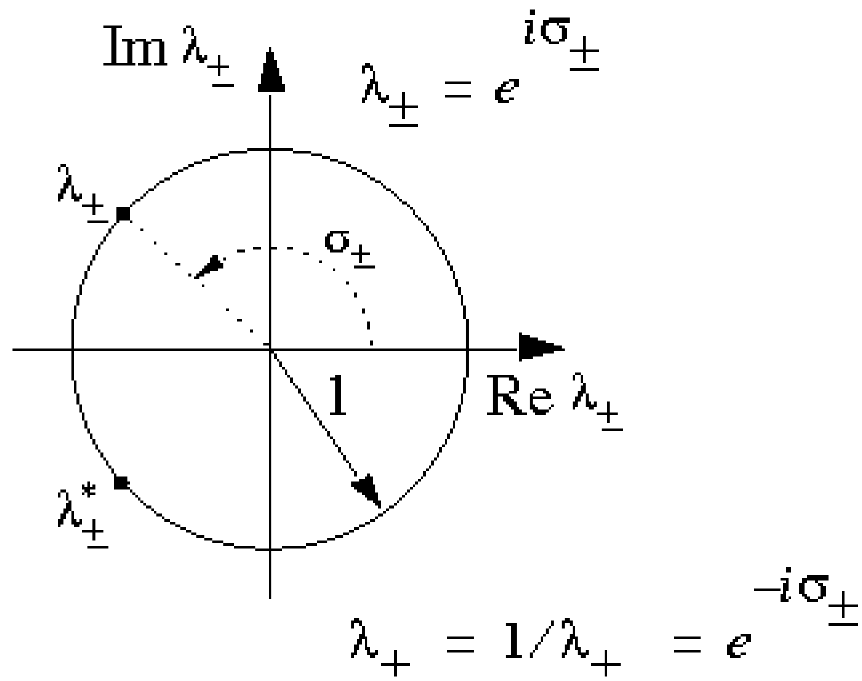
- Analytically find that lattice unstable when focusing kicks sufficiently strong

More advanced treatments

- See: Dragt, *Lectures on Nonlinear Orbit Dynamics*, AIP Conf Proc 87 (1982)

show that symplectic 2x2 transfer matrices associated with Hill's Equation have only two possible classes of eigenvalue symmetries:

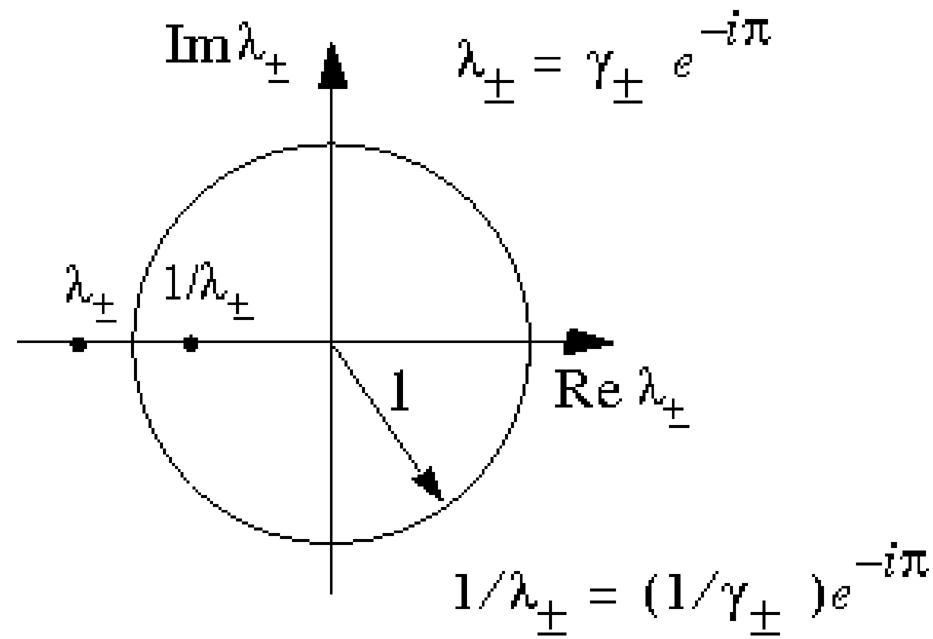
1) Stable



Occurs for:

$$0 \leq \sigma_0 \leq 180^\circ/\text{period}$$

2) Unstable, Lattice Resonance



Occurs in bands when focusing strength is increased beyond

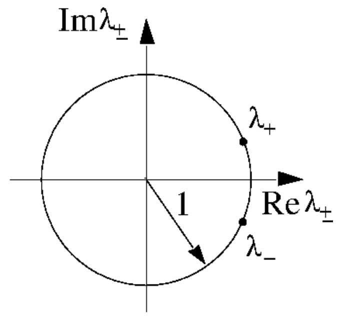
$$\sigma_0 = 180^\circ/\text{period}$$

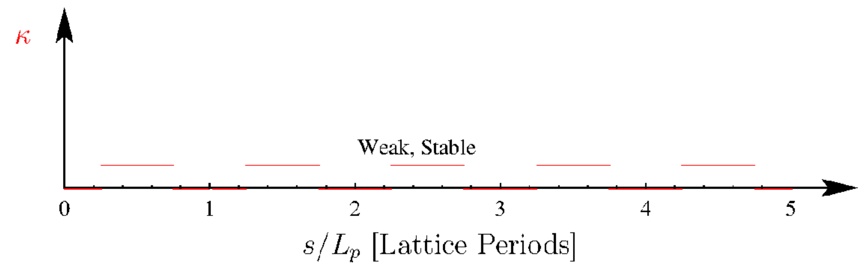
- Limited class of possibilities simplifies analysis of focusing lattices

Eigenvalue structure as focusing strength is increased

Weak Focusing:

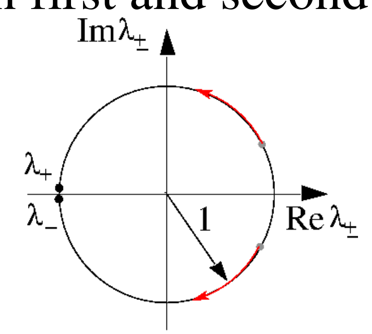
- Make κ as small as needed (low phase advance σ_0)
- Always first eigenvalue case: $|\lambda_\pm| = 1, \quad \lambda_+ = 1/\lambda_- = \lambda_-^*$

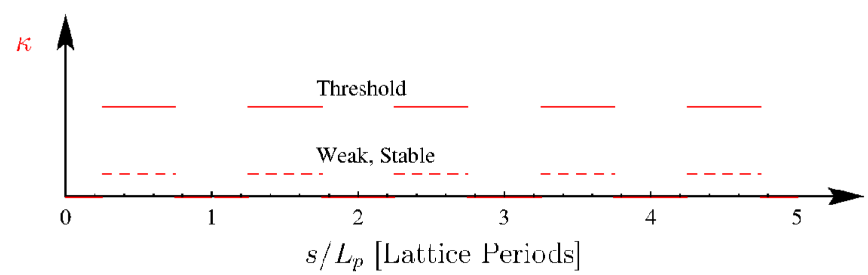




Stability Threshold:

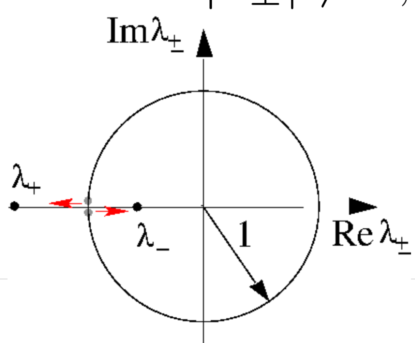
- Increase κ o stability limit (phase advance $\sigma_0 = 180^\circ/\mathrm{Period}$)
- Transition between first and second eigenvalue case: $\lambda_\pm = -1$

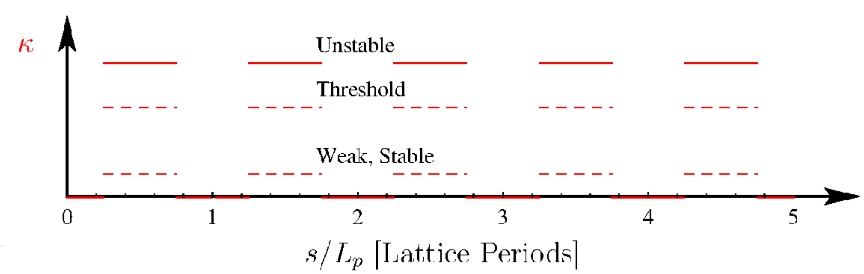




Instability:

- Increase κ beyond threshold (phase advance $\sigma_0 = 180^\circ/\mathrm{Period}$)
- Second eigenvalue case: $|\lambda_\pm| \neq 1, \quad \lambda_+ = 1/\lambda_-$ $\lambda_\pm$ both real and negative





Comments:

- As κ becomes stronger and stronger it is not necessarily the case that instability persists. There can be (typically) narrow ranges of stability within a mostly unstable range of parameters.
  - Example: Stability/instability bands of the Matheiu equation commonly studied in mathematical physics which is a special case of Hills' equation.
- Higher order regions of stability past the first instability band likely make little sense to exploit because they require higher field strength (to generate larger κ) and generally lead to larger particle oscillations than for weaker fields below the first stability threshold.

S6: Hill's Equation: Floquet's Theorem and the Phase-Amplitude Form of the Particle Orbit

S6A: Introduction

In this section we consider Hill's Equation:

$$\boxed{x''(s) + \kappa(s)x(s) = 0}$$

subject to a periodic applied focusing function

$$\boxed{\begin{aligned} &\kappa(s + L_p) = \kappa(s) \\ &L_p = \text{Lattice Period} \end{aligned}}$$

- Many results will also hold in more complicated form for a non-periodic $\kappa(s)$
  - Results less clean in this case
    (initial conditions not removable to same degree as periodic case)

S6B: Floquet's Theorem

Floquet's Theorem (proof: see standard Mathematics and Mathematical Physics Texts)

The solution to Hill's Equation $x(s)$ has two linearly independent solutions that can be expressed as:

$$x_1(s) = w(s)e^{i\mu s}$$

$$x_2(s) = w(s)e^{-i\mu s}$$

$$i = \sqrt{-1}$$

$$\mu = \frac{1}{2}\mathrm{Tr}\,\mathbf{M}(s_i + L_p|s_i) = \cos\sigma_0$$

$$= \mathrm{const} = \mathrm{Characteristic\ Exponent}$$

Where $w(s)$ is a periodic function:

$$w(s + L_p) = w(s)$$

- Theorem as written only applies for **M** with non-degenerate eigenvalues. But a similar theorem applies in the degenerate case.
- A similar theorem is also valid for non-periodic focusing functions
  - Expression not as simple but has analogous form

S6C: Phase-Amplitude Form of Particle Orbit

As a consequence of Floquet's Theorem, any (stable or unstable) nondegenerate solution to Hill's Equation can be expressed in phase-amplitude form as:

$$x(s) = A(s)\cos\psi(s) \qquad A(s) = \text{Real-Valued Amplitude Function}$$
$$A(s + L_p) = A(s) \qquad \psi(s) = \text{Real-Valued Phase Function}$$

Derive equations of motion for A, ψ by taking derivatives of the phase-amplitude form for $x(s)$:

$$x = A\cos\psi$$
$$x' = A'\cos\psi - A\psi'\sin\psi$$
$$x'' = A''\cos\psi - 2A'\psi'\sin\psi - A\psi''\sin\psi - A\psi'^2\cos\psi$$

then substitute in Hill's Equation:

$$x'' + \kappa x = \left[A'' + \kappa A - A\psi'^2\right]\cos\psi - \left[2A'\psi' + A\psi''\right]\sin\psi = 0$$

$$\boxed{x'' + \kappa x = \left[A'' + \kappa A - A\psi'^2\right]\cos\psi - \left[2A'\psi' + A\psi''\right]\sin\psi = 0}$$

We are free to introduce an additional constraint between *A* and ψ:

- Two functions A, ψ to represent one function *x* allows a constraint

Choose:

Eq. (1) $$\boxed{2A'\psi' + A\psi'' = 0} \qquad \Longrightarrow \qquad \text{Coefficient of } \sin\psi \text{ zero}$$

Then to satisfy Hill's Equation for all ψ, the coefficient of $\cos\psi$ must also vanish giving:

Eq. (2) $$\boxed{A'' + \kappa A - A\psi'^2 = 0} \qquad \Longrightarrow \qquad \text{Coefficient of } \cos\psi \text{ zero}$$

Eq. (1) Analysis (coefficient of $\sin\psi$): $2A'\psi' + A\psi'' = 0$

Simplify:

$$2A'\psi' + A\psi'' = \frac{(A^2\psi')'}{A} = 0$$

$$\Longrightarrow \quad (A^2\psi')' = 0$$

Assume for moment:

$A \neq 0$

Will show later that this assumption met for all *s*

Integrate once:

$$\boxed{A^2\psi' = \text{const}}$$

One commonly rescales the amplitude *A*(*s*) in terms of an auxiliary amplitude function *w*(*s*):

$$\boxed{A(s) = A_i w(s) \qquad A_i = \text{const} = \text{Initial Amplitude}}$$

such that

$$\boxed{w^2\psi' \equiv 1}$$

This equation can then be integrated to obtain the phase-function of the particle:

$$\boxed{\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})} \qquad \psi_i = \text{const} = \text{Initial Phase}}$$

Eq. (2) Analysis (coefficient of $\cos\psi$): $A'' + \kappa A - A\psi'^2 = 0$

With the choice of amplitude rescaling, $A = A_i w$ and $w^2\psi' = 1$, Eq. (2) becomes:

$$\boxed{w'' + \kappa w - \frac{1}{w^3} = 0}$$

Floquet's theorem tells us that we are free to restrict *w* to be a periodic solution:

$$\boxed{w(s + L_p) = w(s)}$$

Reduced Expressions for *x* and *x*':

Using $A = A_i w$ and $w^2\psi' = 1$:

$$x = A\cos\psi$$

$$x' = A'\cos\psi - A\psi'\sin\psi$$

$$\Longrightarrow \boxed{\begin{aligned} x &= A_i w\cos\psi \\ x' &= A_i w'\cos\psi - \frac{A_i}{w}\sin\psi \end{aligned}}$$

S6D: Summary: Phase-Amplitude Form of Solution to Hill's Eqn

$$x(s) = A_i w(s) \cos\psi(s)$$

$$x'(s) = A_i w'(s) \cos\psi(s) - \frac{A_i}{w(s)} \sin\psi(s)$$

$A_i = \text{const} = \text{Initial Amplitude}$

$\psi_i = \text{const} = \text{Initial Phase}$

where $w(s)$ and $\psi(s)$ are amplitude- and phase-functions satisfying:

Amplitude Equations

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s + L_p) = w(s)$$

$$w(s) > 0$$

Phase Equations

$$\psi'(s) = \frac{1}{w^2(s)}$$

$$\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

$$\psi(s) = \psi_i + \Delta\psi(s)$$

Initial ($s = s_i$) amplitudes are constrained by the particle initial conditions as:

$$x(s = s_i) = A_i w_i \cos\psi_i$$

$$x'(s = s_i) = A_i w_i' \cos\psi_i - \frac{A_i}{w_i} \sin\psi_i$$

or

$$A_i \cos\psi_i = x(s = s_i)/w_i$$

$$A_i \sin\psi_i = x(s = s_i)w_i' - x'(s = s_i)w_i$$

$$w_i \equiv w(s = s_i)$$

$$w_i' \equiv w'(s = s_i)$$

S6E: Points on the Phase-Amplitude Formulation

1) $w(s)$ can be taken as positive definite

$$w(s) > 0$$

/// Proof: Sign choices in w:

Let $w(s)$ be positive at some point. Then the equation:

$$w'' + \kappa w - \frac{1}{w^3} = 0$$

Insures that w can never vanish or change sign. This follows because whenever w becomes small, $w'' \simeq 1/w^3 \gg 0$ can become arbitrarily large to turn w before it reaches zero. Thus, to fix phases, we conveniently require that $w > 0$. ///

- Proof verifies assumption made in analysis that $A = A_i w \neq 0$
- Conversely, one could choose w negative and it would always remain negative for analogous reasons. This choice is *not* commonly made.
- Sign choice removes ambiguity in relating initial conditions $x(s_i), \; x'(s_i)$ to $A_i, \; \psi_i$

2) $w(s)$ is a unique periodic function

- Can be proved using a connection between w and the principal orbit functions C and S (see: Appendix A and S7)
- $w(s)$ can be regarded as a special, periodic function describing the lattice focusing function $\kappa(s)$

3) The amplitude parameters

$$w_i = w(s = s_i)$$
$$w_i' = w'(s_i)$$

depend *only* on the periodic lattice properties and are *independent* of the particle initial conditions $x(s_i),\ x'(s_i)$

4) The change in phase

$$\Delta\psi(s) = \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

depends on the choice of initial condition s_i. However, the phase-advance through one lattice period

$$\Delta\psi(s_i + L_p) = \int_{s_i}^{s_i+L_p} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

is independent of s_i since w is a periodic function with period L_p

- Will show that (see later in this section)

$$\Delta\psi(s_i + L_p) \equiv \sigma_0$$

is the undepressed phase advance of particle oscillations

5) $w(s)$ has dimensions $[[w]]$ = Sqrt[meters]

- Can prove inconvenient in applications and motivates the use of an alternative "betatron" function β

$$\beta(s) \equiv w^2(s)$$

with dimension $[[\beta]]$ = meters (see: S7 and S8)

6) On the surface, what we have done: Transform the linear Hill's Equation to a form where a solution to nonlinear axillary equations for w and ψ are needed via the phase-amplitude method seems insane ..... why do it?

- Method will help identify the useful Courant-Snyder invariant which will aid interpretation of the dynamics (see: S7)
- Decoupling of initial conditions in the phase-amplitude method will help simplify understanding of bundles of particles in the distribution

S6F: Relation between Principal Orbit Functions and Phase-Amplitude Form Orbit Functions

The transfer matrix **M** of the particle orbit can be expressed in terms of the principal orbit functions C and S as (see: S4):

$$\boxed{\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \mathbf{M}(s|s_i)\cdot\begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix}\cdot\begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix}}$$

Use of the phase-amplitude forms and some algebra identifies (see problem sets):

$$C(s|s_i) = \frac{w(s)}{w_i}\cos\Delta\psi(s) - w_i' w(s)\sin\Delta\psi(s)$$

$$S(s|s_i) = w_i w(s)\sin\Delta\psi(s)$$

$$C'(s|s_i) = \left(\frac{w'(s)}{w_i} - \frac{w_i'}{w(s)}\right)\cos\Delta\psi(s) - \left(\frac{1}{w_i w(s)} + w_i' w'(s)\right)\sin\Delta\psi(s)$$

$$S'(s|s_i) = \frac{w_i}{w(s)}\cos\Delta\psi(s) + w_i w'(s)\sin\Delta\psi(s)$$

$$\Delta\psi(s) \equiv \int_{s_i}^{s}\frac{d\tilde{s}}{w^2(\tilde{s})} \qquad \begin{aligned} w_i &\equiv w(s=s_i) \\ w_i' &\equiv w'(s=s_i) \end{aligned}$$

/// Aside: Alternatively, it can be shown (see: Appendix A) that $w(s)$ can be related to the principal orbit functions calculated over one Lattice period by:

$$\boxed{\begin{aligned} w^2(s) = \beta(s) = \sin\sigma_0 \frac{S(s|s_i)}{S(s_i+L_p|s_i)} \\ + \frac{S(s_i+L_p|s_i)}{\sin\sigma_0}\left[C(s|s_i) + \frac{\cos\sigma_0 - C(s|s_i)}{S(s_i+L_p|s_i)}S(s|s_i)\right]^2 \\ \sigma_0 \equiv \int_{s_i}^{s_i+L_p} \frac{d\tilde{s}}{w^2(\tilde{s})} \end{aligned}}$$

The formula for σ_0 in terms of principal orbit functions is useful:

- σ_0 (phase advance, see: S6G) is often specified for the lattice and the focusing function $\kappa(s)$ is tuned to achieve the specified value
- Shows that $w(s)$ can be constructed from two principal orbit integrations over one lattice period
  - Integrations must generally be done numerically for C and S
  - No root finding required for initial conditions to construct periodic $w(s)$
  - s_i can be anywhere in the lattice period and $w(s)$ will be independent of the specific choice of s_i

- The form of $w^2(s)$ suggests an underlying Courant-Snyder Invariant (see: S7 and Appendix A)
- $w^2 = \beta$ can be applied to calculate max beam particle excursions in the absence of space-charge effects (see: S8)
  - Useful in machine design
  - Exploits Courant-Snyder Invariant

///

S6G: Undepressed Particle Phase Advance

We can now concretely connect σ_0 for a stable orbit to the change in particle oscillation phase $\Delta\psi$ through one lattice period:

From S5D:

$$\boxed{\cos\sigma_0 \equiv \frac{1}{2}\mathrm{Tr}\,\mathbf{M}(s_i + L_p|s_i)}$$

Apply the principal orbit representation of **M**

$$\mathrm{Tr}\,\mathbf{M}(s_i + L_p|s_i) = C(s_i + L_p|s_i) + S'(s_i + L_p|s_i)$$

and use the phase-amplitude identifications of C and S' calculated in S6F:

$$\frac{1}{2}\mathrm{Tr}\,\mathbf{M}(s_i + L_p|s_i) = \frac{1}{2}\left(\frac{w(s_i + L_p)}{w_i} + \frac{w_i}{w(s_i + L_p)}\right)\cos\Delta\psi(s_i + L_p)$$
$$+ \frac{1}{2}\left(w_i w'(s_i + L_p) - w_i' w(s_i + L_p)\right)\sin\Delta\psi(s_i + L_p)$$

By periodicity:

$$\begin{aligned} w(s_i + L_p) &= w(s_i) = w_i \\ w'(s_i + L_p) &= w'(s_i) = w_i' \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &\text{coefficient of } \cos\Delta\psi = 1 \\ &\text{coefficient of } \sin\Delta\psi = 0 \end{aligned}$$

Applying these results gives:

$$\boxed{\cos\sigma_0 = \cos\Delta\psi(s_i + L_p) = \frac{1}{2}\text{Tr}\,\mathbf{M}(s_i + L_p|s_i)}$$

Thus, σ_0 is identified as the phase advance of a stable particle orbit through one lattice period:

$$\boxed{\sigma_0 = \Delta\psi(s_i + L_p) = \int_{s_i}^{s_i+L_p} \frac{ds}{w^2(s)}}$$

- Again verifies that σ_0 is independent of s_i since w(s) is periodic with period L_p
- The stability criterion (see: S5)

$$\frac{1}{2}|\text{Tr}\,\mathbf{M}(s_i + L_p|s_i)| = |\cos\sigma_0| \leq 1$$

is concretely connected to the particle phase advance through one lattice period providing a useful physical interpretation

Consequence:

Any periodic lattice with undepressed phase advance satisfying

$$\sigma_0 < \pi/\text{period} = 180^\circ/\text{period}$$

will have stable single particle orbits.

Discussion:

The phase advance σ_0 is an extremely useful dimensionless measure to characterize the focusing strength of a periodic lattice. Much of conventional accelerator physics centers on focusing strength and the suppression of resonance effects. The phase advance is a natural parameter to employ in many situations to allow ready interpretation of results in a generalizable manner.

We present phase advance formulas for several simple classes of lattices to help build intuition on focusing strength:

1) Continuous Focusing
2) Periodic Solenoidal Focusing
3) Periodic Quadrupole Doublet Focusing
   - FODO Quadrupole Limit

Several of these will be derived in the problem sets

- Lattices analyzed as "hard-edge" with piecewise-constant $\kappa(s)$ and lattice period L_p
- Results are summarized only with derivations guided in the problem sets.

4) Thin Lens Limits
   - Useful for analysis of scaling properties

1) Continuous Focusing

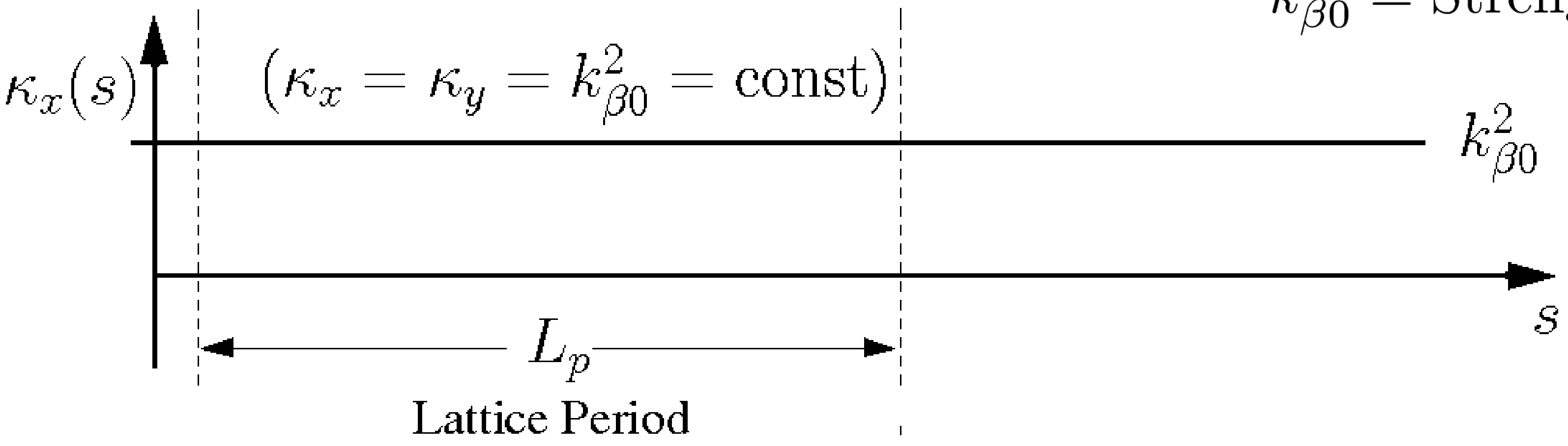
"Lattice period" L_p is an arbitrary length for phase accumulation

$$\kappa(s) = k_{\beta 0}^2 = \text{const} > 0$$

Parameters:

$L_p = \text{Lattice Period}$

$k_{\beta 0}^2 = \text{Strength}$



Apply phase advance formulas:

$$w'' + k_{\beta 0}^2 w - \frac{1}{w^3} = 0 \quad \Longrightarrow \quad w = \frac{1}{\sqrt{k_{\beta 0}}}$$

$$\sigma_0 = \int_{s_i}^{s_i+L_p} \frac{ds}{w^2} = k_{\beta 0} L_p$$

$$\boxed{\sigma_0 = k_{\beta 0} L_p}$$

- Always stable
  - Energy cannot pump into or out of particle orbit

Rescaled Principal Orbit Evolution:

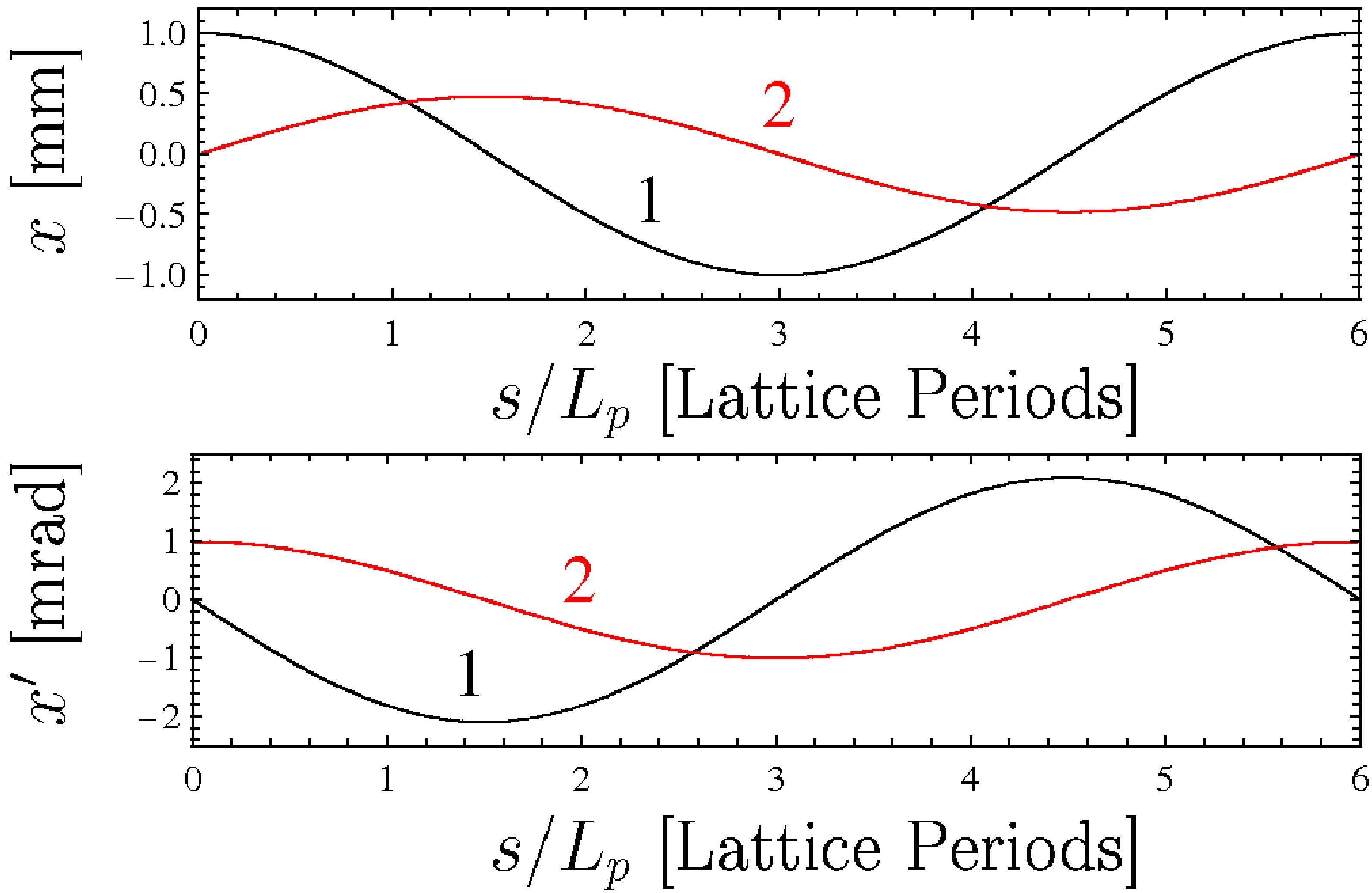
$$L_p = 0.5 \text{ m}$$
$$\sigma_0 = \pi/3 = 60^\circ$$
$$k_{\beta 0} = (\pi/6) \text{ rad/m}$$

Cosine-Like

1: $x(0) = 1$ mm
$x'(0) = 0$ mrad

Sine-Like

2: $x(0) = 0$ mm
$x'(0) = 1$ mrad



Phase-Space Evolution (see also S7):

- Phase-space ellipse stationary and aligned along *x*, *x*' axes for continuous focusing

$$w = \sqrt{1/k_{\beta 0}} = \text{const}$$

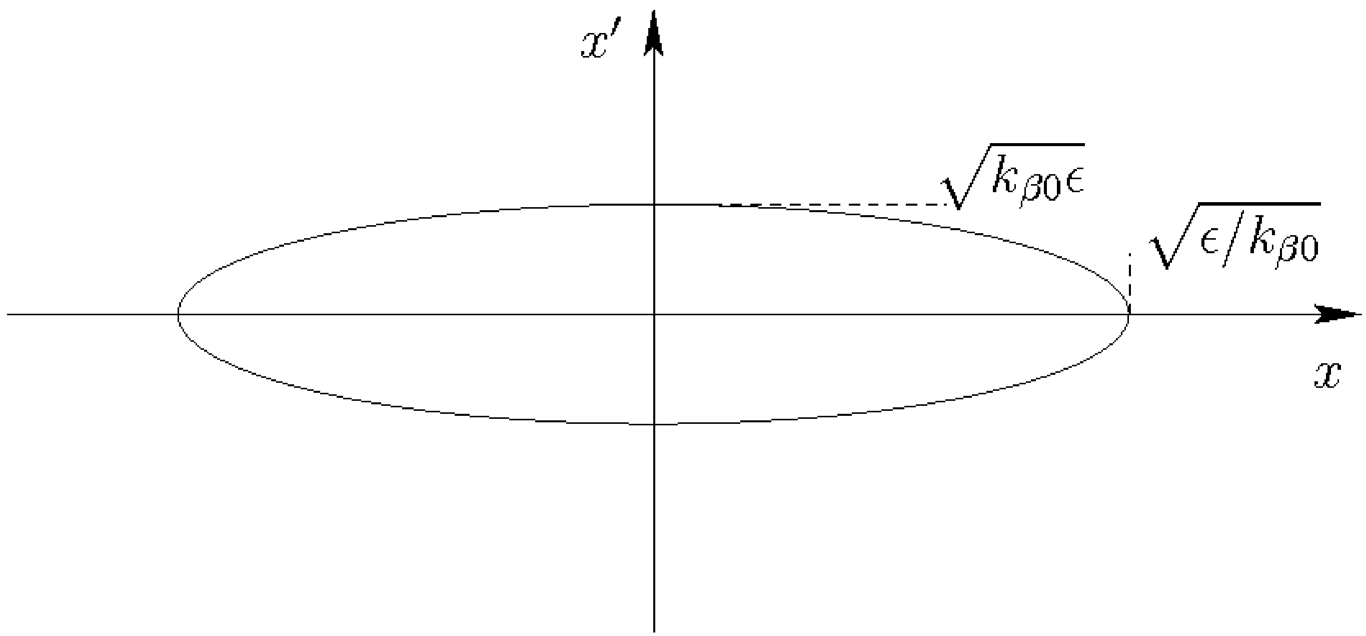
$$w' = 0$$

$$\gamma = \frac{1}{w^2} = k_{\beta 0} = \text{const}$$

$$\alpha = -ww' = 0$$

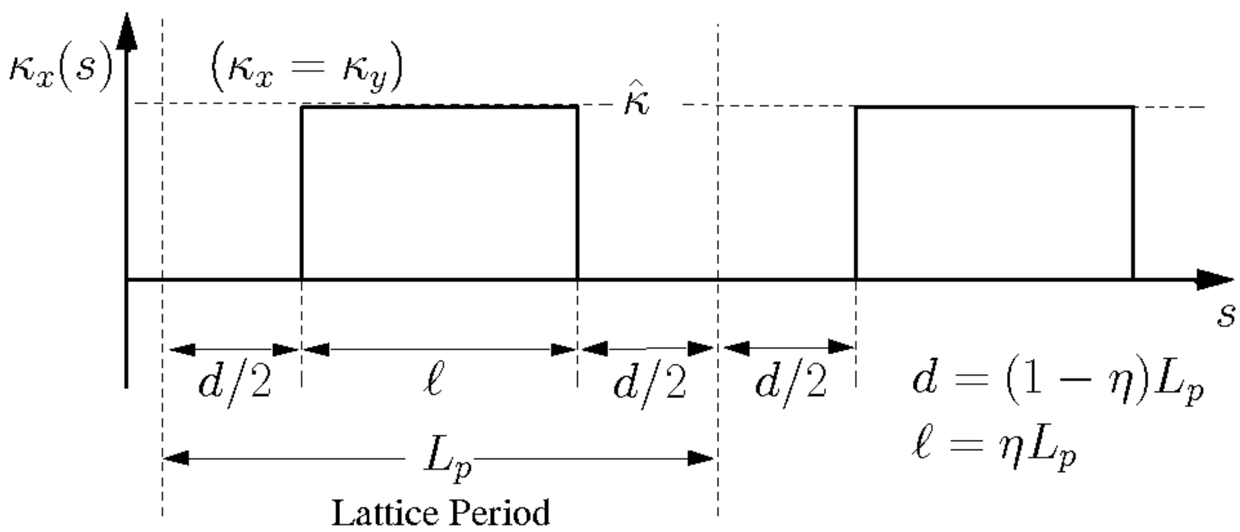
$$\beta = w^2 = 1/k_{\beta 0} = \text{const}$$

$$k_{\beta 0} x^2 + x'^2/k_{\beta 0} = \epsilon = \text{const}$$



2) Periodic Solenoidal Focusing

Results are interpreted in the rotating Larmor frame (see S2 and Appendix A)



Parameters:

L_p = Lattice Period

$\eta \in (0, 1]$ = Occupancy

$\hat{\kappa}$ = Strength

Characteristics:

ηL_p = Optic Length

$(1-\eta)L_p$ = Drift Length

Calculation gives:

$$\boxed{\cos\sigma_0 = \cos(2\Theta) - \frac{1-\eta}{\eta}\Theta\sin(2\Theta) \qquad \Theta \equiv \frac{\eta}{2}\sqrt{\hat{\kappa}}L_p}$$

- Can be unstable when $\hat{\kappa}$ becomes large
  - Energy can pump into or out of particle orbit

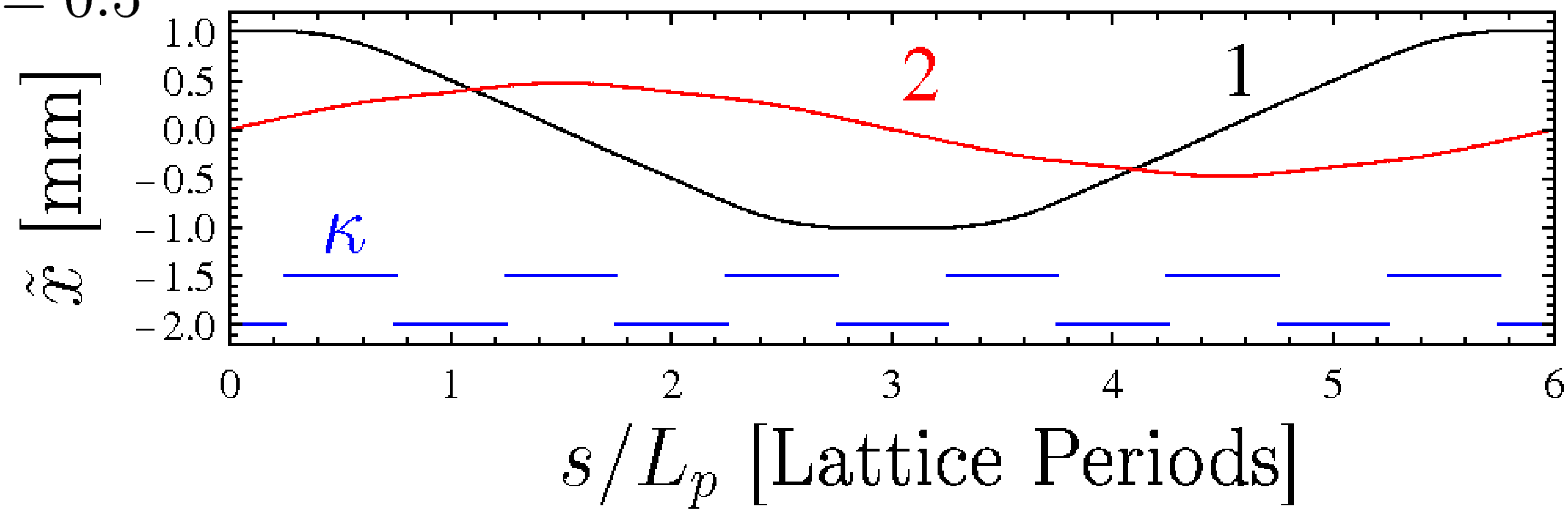
Rescaled Larmor-Frame Principal Orbit Evolution Solenoid Focusing:

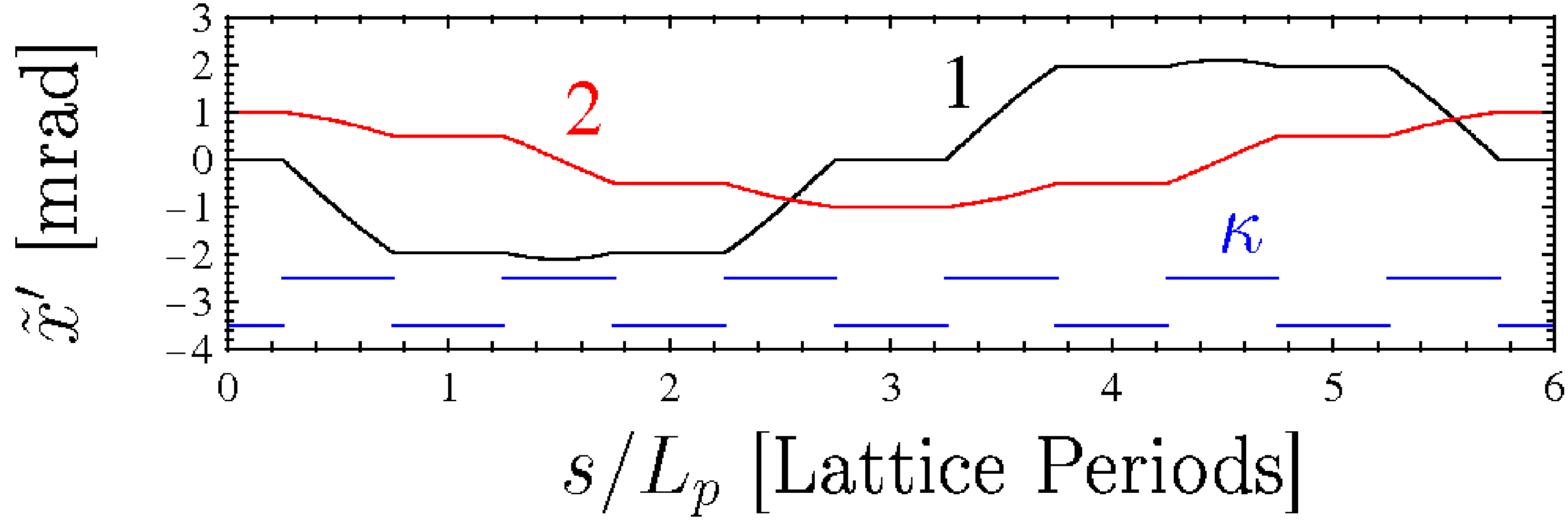
$L_p = 0.5$ m

$\sigma_0 = \pi/3 = 60^\circ$ ($\kappa = 8.558$ m^{-2})

$\eta = 0.5$

Cosine-Like	Sine-Like
1: $\tilde{x}(0) = 1$ mm	2: $\tilde{x}(0) = 0$ mm
$\tilde{x}'(0) = 0$ mrad	$\tilde{x}'(0) = 1$ mrad





- Principal orbits in $\tilde{y} - \tilde{y}'$ phase-space are identical

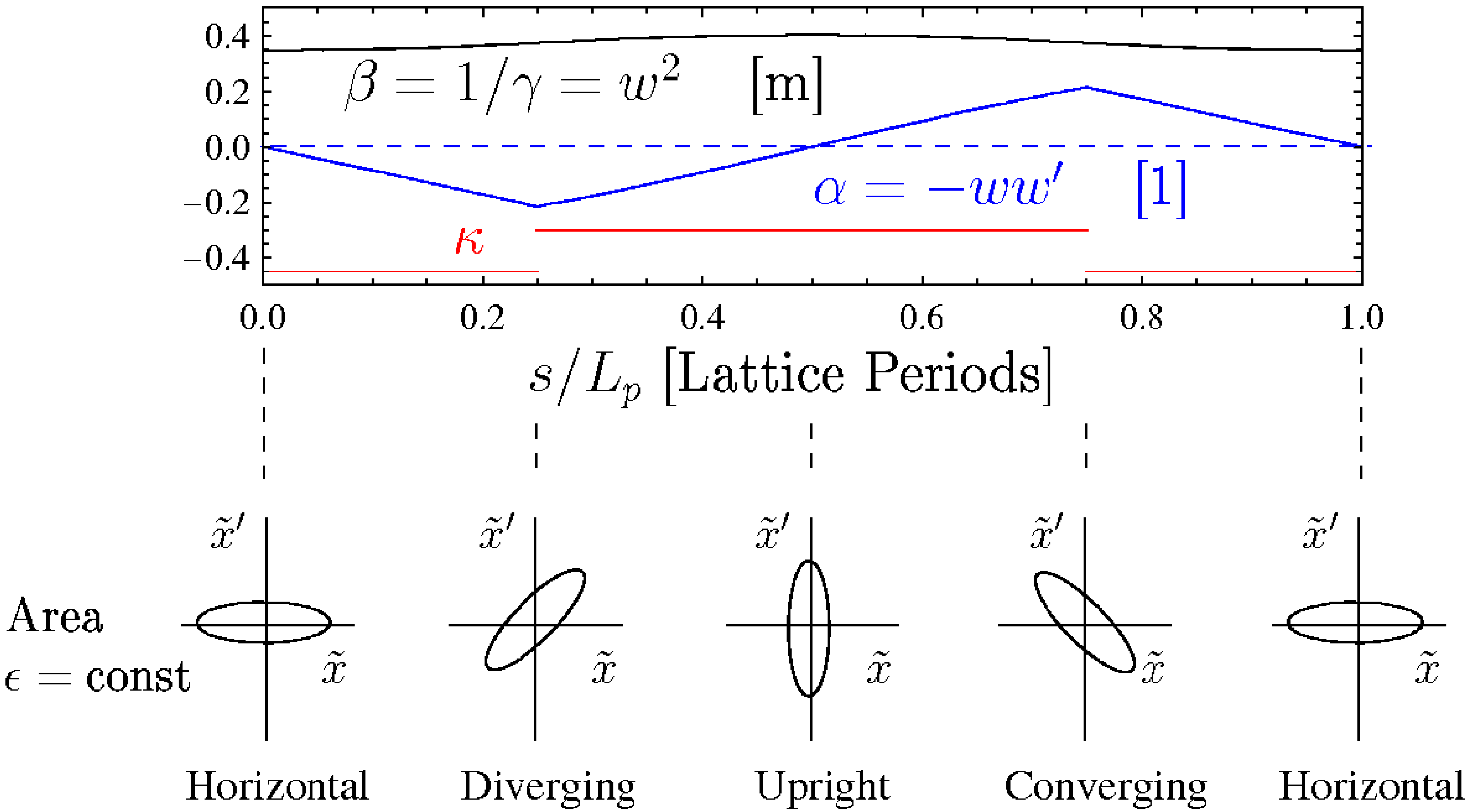
Phase-Space Evolution in the Larmor frame (see also: S7):

- Phase-Space ellipse rotates and evolves in periodic lattice

  $\tilde{y} - \tilde{y}'$ phase-space properties same as in $\tilde{x} - \tilde{x}'$

  - Phase-space structure in *x*-*x*', *y*-*y*' phase space is complicated

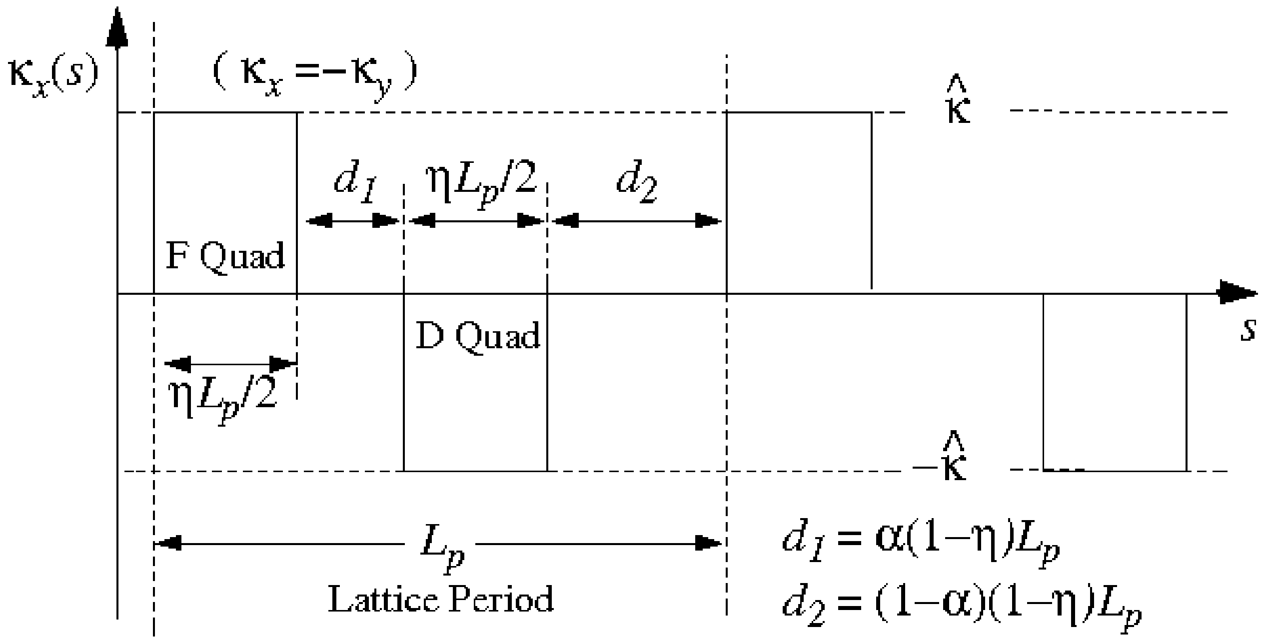
$$\gamma \tilde{x}^2 - 2\alpha \tilde{x}\tilde{x}' + \beta \tilde{x}'^2 = \epsilon = \text{const}$$



Comments on periodic solenoid results:

- Larmor frame analysis greatly simplifies results
  - 4D coupled orbit in x-x', y-y' phase-space will be much more intricate in structure
- Phase-Space ellipse rotates and evolves in periodic lattice
- Periodic structure of lattice changes orbits from simple harmonic

3) Periodic Quadrupole Doublet Focusing



Parameters:

$$L_p = \text{Lattice Period}$$
$$\eta \in (0, 1] = \text{Occupancy}$$
$$\alpha \in [0, 1] = \text{Syncopation}$$
$$\hat{\kappa} = \text{Strength}$$

Characteristics:

$$\eta L_p/2 = \text{F/D Len}$$
$$\alpha(1-\eta)L_p = \text{Drift Len } d_1$$
$$(1-\alpha)(1-\eta)L_p = \text{Drift Len } d_2$$

Calculation gives:

$$\begin{aligned}\cos\sigma_0 = \cos\Theta\cosh\Theta &+ \frac{1-\eta}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta) \\ &- 2\alpha(1-\alpha)\frac{(1-\eta)^2}{\eta^2}\Theta^2\sin\Theta\sinh\Theta\end{aligned} \qquad \Theta \equiv \frac{\eta}{2}\sqrt{|\hat{\kappa}|}L_p$$

- Can be unstable when $\hat{\kappa}$ becomes large
  - Energy can pump into or out of particle orbit

Comments on Parameters:

- The "syncopation" parameter α measures how close the Focusing (F) and DeFocusing (D) quadrupoles are to each other in the lattice

$$\alpha \in [0,1] \qquad \begin{array}{llll} \alpha = 0 & \Longrightarrow & d_1 = 0 & d_2 = (1-\eta)L_p \\ \alpha = 1 & \Longrightarrow & d_1 = (1-\eta)L_p & d_2 = 0 \end{array}$$

The range $\alpha \in [1/2, 1]$ can be mapped to $\alpha \in [0, 1/2]$ by simply relabeling quantities. Therefore, we can take:

$$\alpha \in [0, 1/2]$$

- The special case of a doublet lattice with $\alpha = 1/2$ corresponds to equal drift lengths between the F and D quadrupoles and is called a FODO lattice

$$\alpha = 1/2 \qquad \Longrightarrow \qquad d_1 = d_2 \equiv d = (1-\eta)L_p/2$$

Phase advance constraint will be derived for FODO case in problems (algebra much simpler than doublet case)

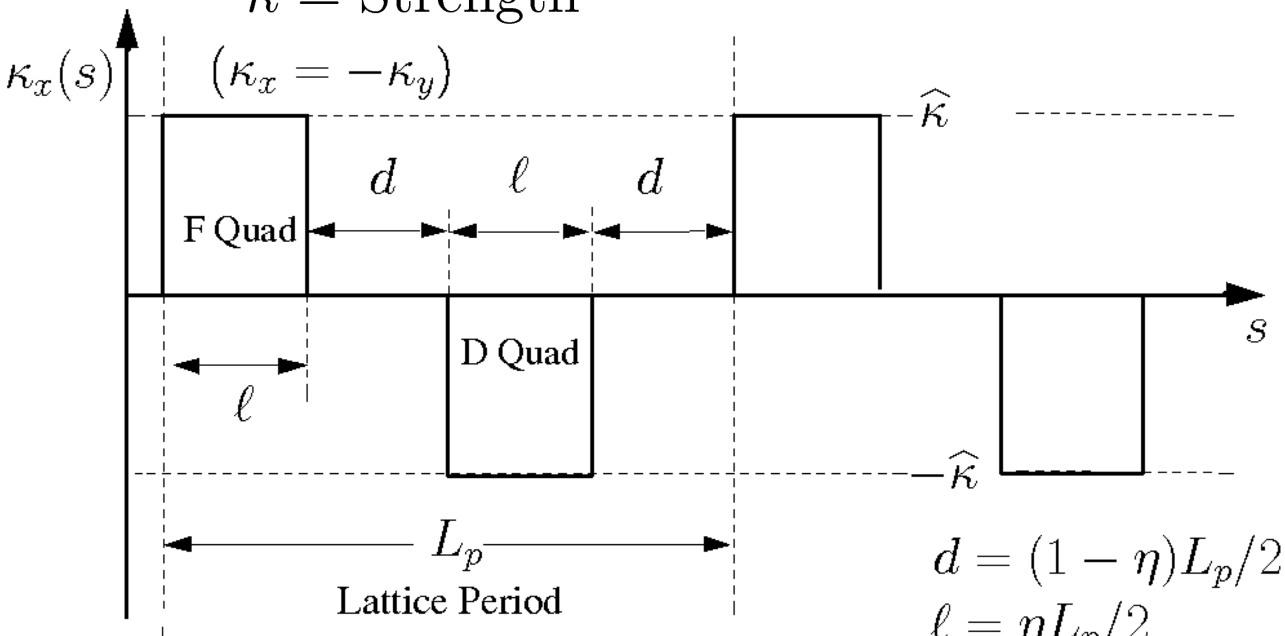
Special Case Doublet Focusing: Periodic Quadrupole FODO Lattice

Parameters:

$$L_p = \text{Lattice Period}$$
$$\eta \in (0,1] = \text{Occupancy}$$
$$\hat{\kappa} = \text{Strength}$$

Characteristics:

$$\eta L_p/2 = \ell = \text{F/D Len}$$
$$(1-\eta)L_p/2 = d = \text{Drift Len}$$



Phase advance formula reduces to:

$$\cos\sigma_0 = \cos\Theta\cosh\Theta + \frac{1-\eta}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta) - \frac{(1-\eta)^2}{2\eta^2}\Theta^2\sin\Theta\sinh\Theta$$

$$\Theta \equiv \frac{\eta}{2}\sqrt{|\hat{\kappa}|}L_p$$

- Analysis shows FODO provides stronger focus for same integrated field gradients than doublet due to symmetry

Rescaled Principal Orbit Evolution FODO Quadrupole:

$L_p = 0.5$ m

$\sigma_0 = \pi/3 = 60^\circ$ ($\kappa = 39.24\ \mathrm{m}^{-2}$)

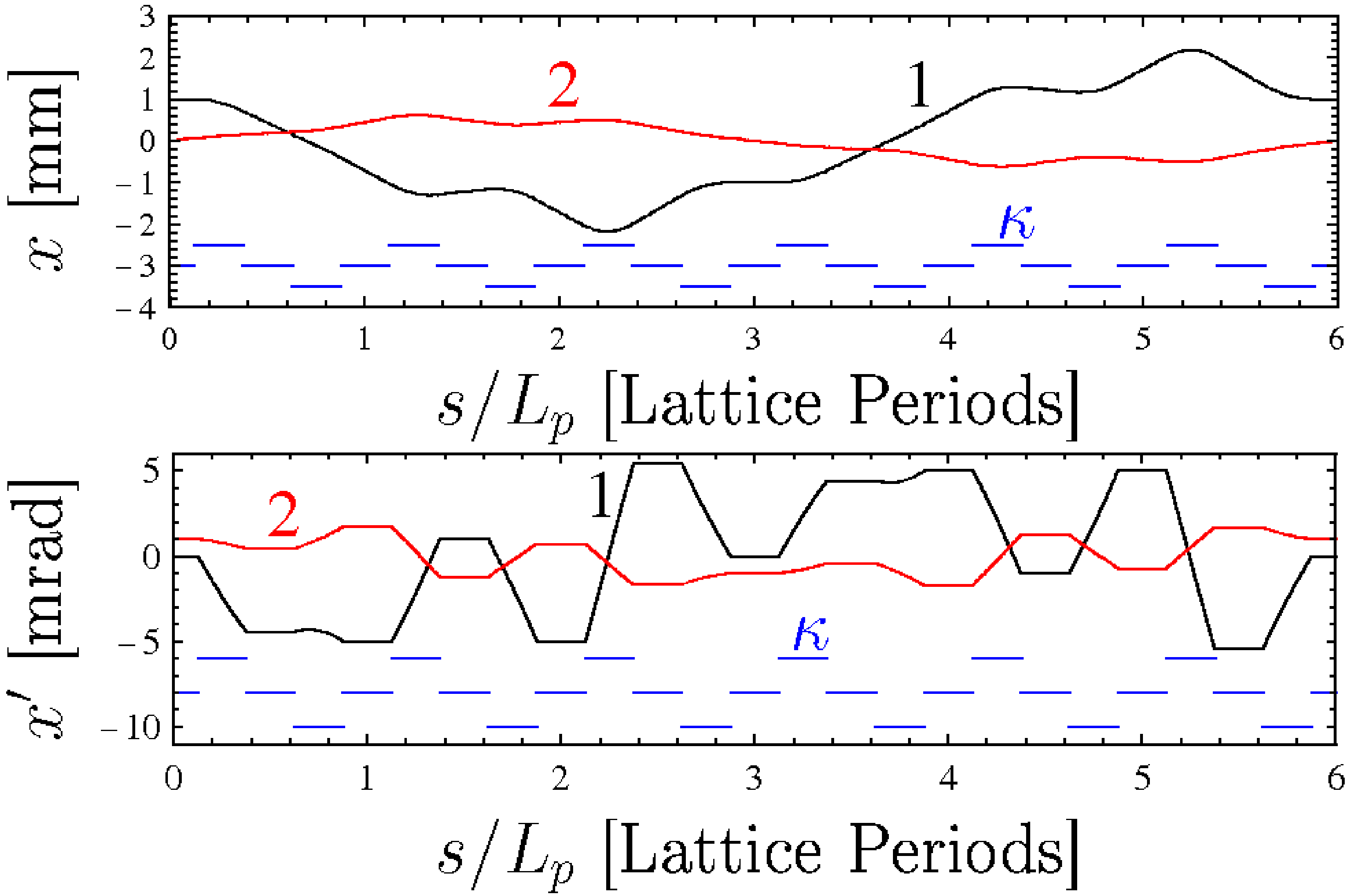
$\eta = 0.5$

Cosine-Like

1: $x(0) = 1$ mm, $x'(0) = 0$ mrad

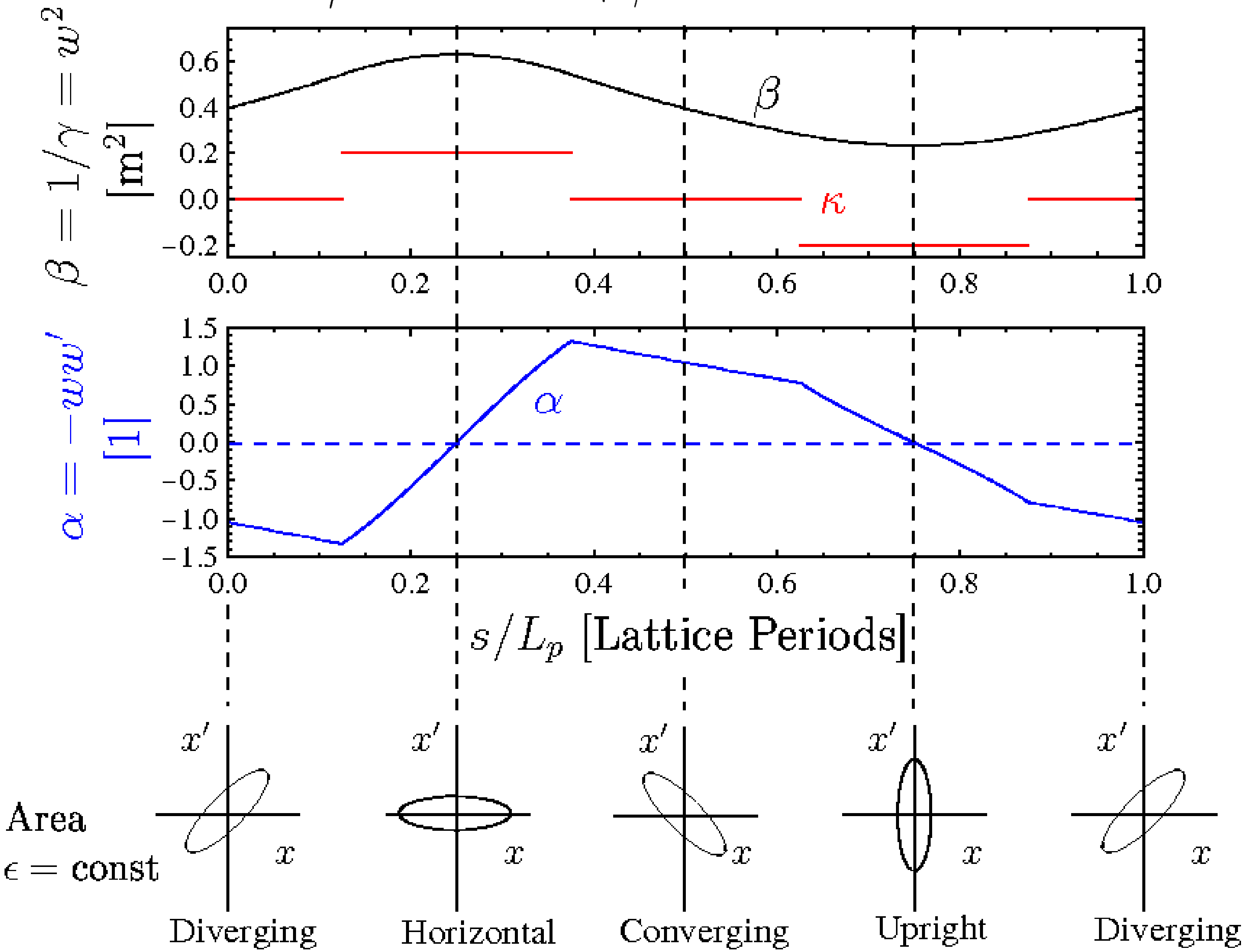
Sine-Like

2: $x(0) = 0$ mm, $x'(0) = 1$ mrad



Phase-Space Evolution (see also: S7):

$$\gamma x^2 - 2\alpha x x' + \beta x'^2 = \epsilon = \text{const}$$



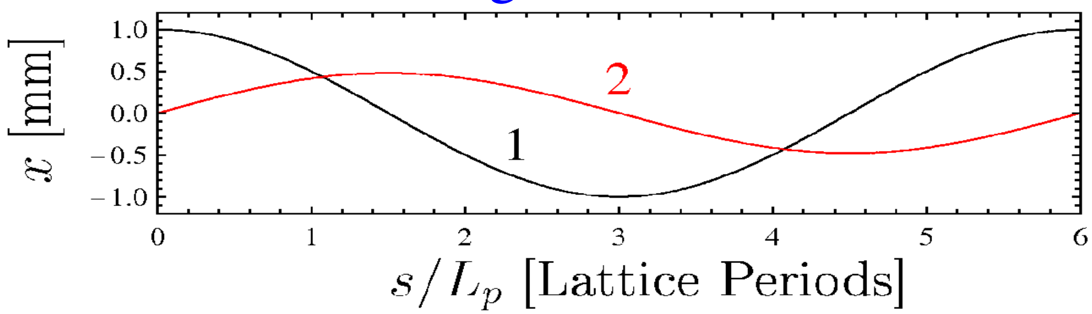
Comments on periodic FODO quadrupole results:

- Phase-Space ellipse rotates and evolves in periodic lattice
  - Evolution more intricate for Alternating Gradient (AG) focusing than for solenoidal focusing in the Larmor frame
- Harmonic content of orbits larger for AG focusing than solenodial focusing
- Orbit and phase space evolution analogous in y-y' plane
  - Simply related by an shift in s of the lattice

Contrast of Principal Orbits for different focusing:

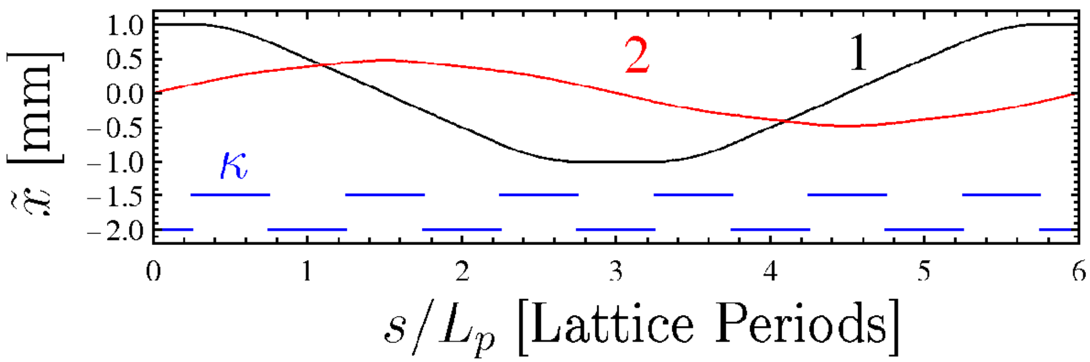
- Use previous examples with "equivalent" focusing strength $\sigma_0 = 60^\circ$
- Note that periodic focusing adds harmonic structure

1) Continuous Focusing



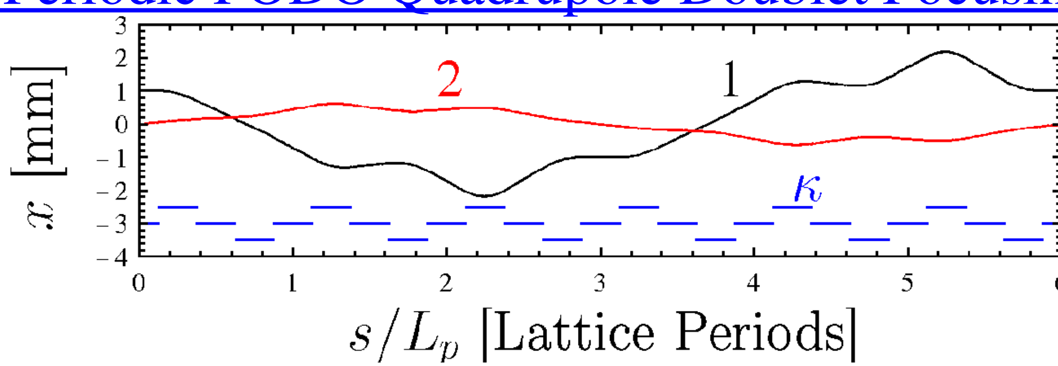
Simple Harmonic Oscillator

2) Periodic Solenoidal Focusing (Larmor Frame)



Simple harmonic oscillations modified with additional harmonics due to periodic focus

3) Periodic FODO Quadrupole Doublet Focusing



Simple harmonic oscillations more strongly modified due to periodic AG focus

4) Thin Lens Limits

Convenient to simply understand analytic scaling

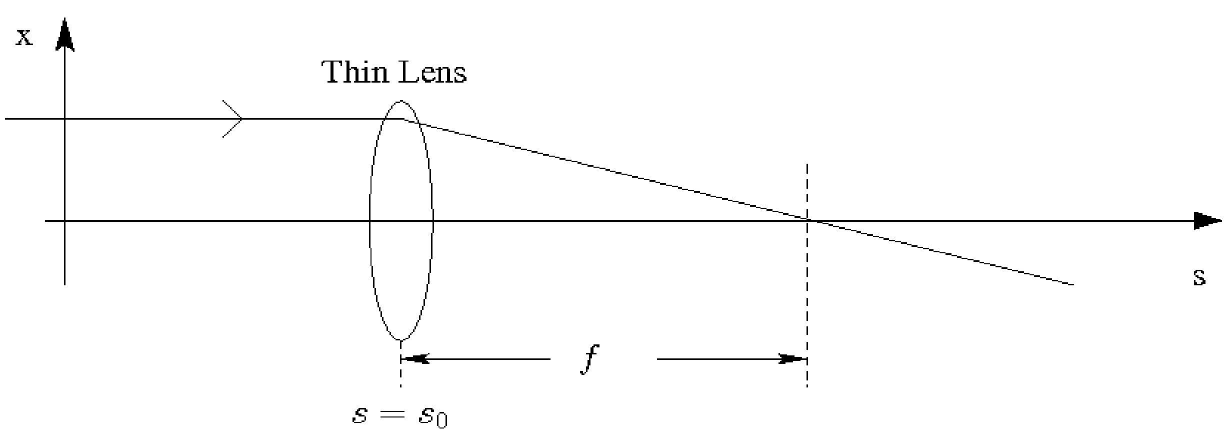
$$\kappa_x(s) = \frac{1}{f}\delta(s - s_0)$$

$$s_0 = \text{Optic Location} = \text{const}$$
$$f = \text{focal length} = \text{const}$$

Transfer Matrix:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s=s_0^+} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s=s_0^-}$$

Graphical Interpretation:



The thin lens limit of "thick" hard-edge solenoid and quadrupole focusing lattices presented can be obtained by taking:

Solenoids: $\hat{\kappa} \equiv \dfrac{1}{\eta f L_p}$ then take $\lim_{\eta \to 0}$

Quadrupoles: $\hat{\kappa} \equiv \dfrac{2}{\eta f L_p}$ then take $\lim_{\eta \to 0}$

This obtains when applied in the previous formulas:

$$\boxed{\cos\sigma_0 = \begin{cases} 1 - \frac{1}{2}\frac{L_p}{f}, & \text{thin-lens periodic solenoid} \\ 1 - \frac{\alpha}{2}(1-\alpha)\left(\frac{L_p}{f}\right)^2, & \text{thin-lens quadrupole doublet} \\ & \alpha = \dfrac{1}{2} \Longrightarrow \text{FODO} \end{cases}}$$

These formulas can also be derived directly from the drift and thin lens transfer matrices as

Periodic Solenoid

$$\cos\sigma_0 = \frac{1}{2}\mathrm{Tr}\begin{bmatrix} 1 & L_p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = 1 - \frac{1}{2}\frac{L_p}{f}$$

Periodic Quadrupole Doublet

$$\cos\sigma_0 = \frac{1}{2}\mathrm{Tr}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & \alpha L_p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & (1-\alpha)L_p \\ 0 & 1 \end{bmatrix} = 1 - \frac{\alpha}{2}(1-\alpha)\left(\frac{L_p}{f}\right)^2$$

Expanded phase advance formulas (thin lens type limit and similar) can be useful in system design studies

- Desirable to derive simple formulas relating magnet parameters to σ_0
  - Clear analytic scaling trends clarify design trade-offs
- For hard edge periodic lattices, expand formula for $\cos\sigma_0$ to leading order in $\Theta = \sqrt{|\hat{\kappa}|\eta L_p/2}$

/// Example: Periodic Quadrupole Doublet Focusing:

Expand previous phase advance formula for syncopated quadrupole doublet to obtain:

$$\cos\sigma_0 = 1 - \frac{(\eta\hat{\kappa}L_p^2)^2}{32}\left[\left(1-\frac{2}{3}\eta\right) - 4\left(\alpha - \frac{1}{2}\right)^2(1-\eta)^2\right]$$

where:

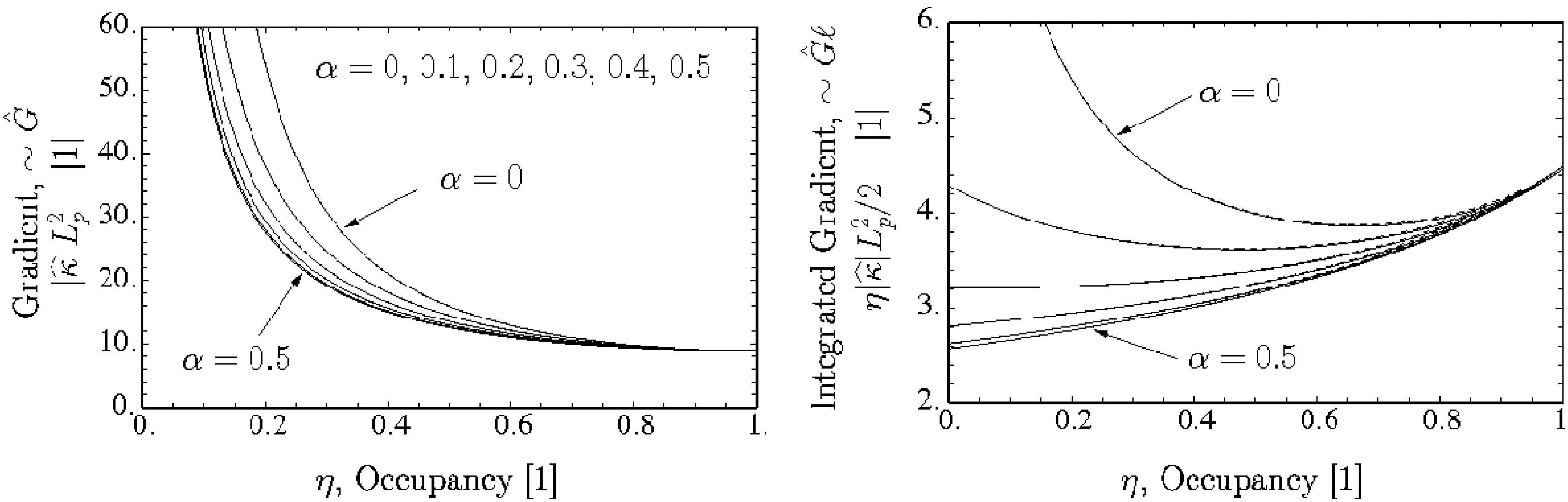
$$\hat{\kappa} = \begin{cases} \frac{\hat{G}}{[B\rho]}, & \text{Magnetic Quadrupoles} \\ \frac{\hat{G}}{\beta_b c[B\rho]}, & \text{Electric Quadrupoles} \end{cases} \qquad \hat{G} = \begin{array}{l}\text{Hard-Edge} \\ \text{Field Gradient}\end{array}$$

Using these results, plot the Field Gradient and Integrated Gradient for quadrupole doublet focusing needed for $\sigma_0 = 80^\circ$ per lattice period

Gradient ~ $|\hat{\kappa}|L_p^2 \sim \hat{G}$

Integrated Gradient ~ $\eta|\hat{\kappa}|L_p^2/2 \sim \hat{G}\ell$

$\sigma_0 = 80^\circ$ /(Lattice Period) Quadrupole Doublet



- Exact (non-expanded) solutions plotted dashed (almost overlay)
- Gradient and integrated gradient required depend only weakly on syncopation factor α when α is near or larger than ½
- Stronger gradient required for low occupancy η but integrated gradient varies comparatively less with η except for small α

///

Appendix A: Calculation of $w(s)$ from Principal Orbit Functions

Evaluate principal orbit expressions of the transfer matrix through one lattice period using

$$w(s_i + L_p) = w_i$$
$$w'(s_i + L_p) = w'_i$$

and

$$\Delta\psi(s_i + L_p) = \int_{s_i}^{s_i+L_p} \frac{ds}{w^2(s)} = \sigma_0$$

to obtain (see principal orbit formulas expressed in phase-amplitude form):

$$C(s_i + L_p|s_i) = \cos\sigma_0 - w_i w'_i \sin\sigma_0$$
$$S(s_i + L_p|s_i) = w_i^2 \sin\sigma_0$$
$$C'(s_i + L_p|s_i) = -\left(\frac{1}{w_i^2} + w_i w'_i\right)\sin\sigma_0$$
$$S'(s_i + L_p|s_i) = \cos\sigma_0 + w_i w'_i \sin\sigma_0$$

A1

Giving:

$$\boxed{\begin{aligned} w_i &= \sqrt{\frac{S(s_i + L_p|s_i)}{\sin\sigma_0}} \\ w_i' &= \frac{\cos\sigma_0 - C(s_i + L_p|s_i)}{\sqrt{S(s_i + L_p|s_i)\sin\sigma_0}} \end{aligned}}$$

Apply $C(s|s_i)$ Eqn.

Apply $S(s|s_i)$ Eqn.
$+ \; w_i$ Result Above

Or in terms of the betatron formulation (see: S7 and S8) with

$$\beta = w^2, \;\; \beta' = 2ww'$$

$$\boxed{\begin{aligned} \beta_i &= w_i^2 = \frac{S(s_i + L_p|s_i)}{\sin\sigma_0} \\ \beta_i' &= 2w_i w_i' = \frac{2[\cos\sigma_0 - C(s_i + L_p|s_i)]}{\sin\sigma_0} \end{aligned}}$$

Next, calculate w from the principal orbit expression in phase-amplitude form:

$$\frac{S}{w_i w} = \sin\Delta\psi$$

$$S \equiv S(s|s_i) \;\; \text{etc.}$$

$$\frac{w_i}{w}C + \frac{w_i'}{w}S = \cos\Delta\psi$$

A2

Square and add equations:

$$\left(\frac{S}{w_i w}\right)^2 + \left(\frac{w_i C}{w} + \frac{w_i' S}{w}\right)^2 = 1$$

- This result reflects the structure of the underlying Courant-Snyder invariant (see: S7)

Gives:

$$w^2 = \left(\frac{S}{w_i}\right)^2 + (w_i C + w_i' S)^2$$

Use $w_i, \; w_i'$ previously identified and write out result:

$$\boxed{\begin{aligned} w^2(s) = \beta(s) = \sin\sigma_0 \frac{S^2(s|s_i)}{S(s_i + L_p|s_i)} \\ + \frac{S(s_i + L_p|s_i)}{\sin\sigma_0}\left[C(s|s_i) + \frac{\cos\sigma_0 - C(s_i + L_p|s_i)}{S(s_i + L_p|s_i)} S(s|s_i)\right]^2 \end{aligned}}$$

- Formula shows that for a given σ_0 (used to specify lattice focusing strength), $w(s)$ is given by two linear principal orbits calculated over one lattice period
  - Easy to apply numerically

A3

An alternative way to calculate $w(s)$ is as follows. 1st apply the phase-amplitude formulas for the principal orbit functions with:

$$s_i \longrightarrow s$$
$$s \longrightarrow s + L_p$$

$$\Longrightarrow \quad \begin{aligned} C(s+L_p|s) &= \cos\sigma_0 - w(s)w'(s)\sin\sigma_0 \\ S(s+L_p|s) &= w^2(s)\sin\sigma_0 \end{aligned}$$

$$\boxed{w^2(s) = \beta(s) = \frac{S(s+L_p|s)}{\sin\sigma_0} = \frac{\mathbf{M}_{12}(s+L_p|s)}{\sin\sigma_0}}$$

- Formula requires calculation of $S(s+L_p|s)$ at every value of s within lattice period
- Previous formula requires one calculation of $C(s|s_i),\ S(s|s_i)$ for $s_i \leq s \leq s_i + L_p$ and any value of s_i

A4

Matrix algebra can be applied to simplify this result:

$$
\begin{aligned}
\mathbf{M}(s+L_p|s) &= \mathbf{M}(s+L_p|s_i+L_p)\cdot\mathbf{M}(s_i+L_p|s) \\
&= \mathbf{M}(s|s_i)\cdot\mathbf{M}(s_i+L_p|s)\cdot[\mathbf{M}(s|s_i)\cdot\mathbf{M}^{-1}(s|s_i)] \\
&= \mathbf{M}(s|s_i)\cdot\mathbf{M}(s_i+L_p|s_i)\cdot\mathbf{M}^{-1}(s|s_i)
\end{aligned}
$$

$$\boxed{\mathbf{M}(s+L_p|s) = \mathbf{M}(s|s_i)\cdot\mathbf{M}(s_i+L_p|s_i)\cdot\mathbf{M}^{-1}(s|s_i)}$$

- Using this result with the previous formula allows the transfer matrix to be calculated only once per period from any initial condition
- Using:

$$\mathbf{M} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \qquad \mathbf{M}^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

Apply Wronskian condition:

$$\det \mathbf{M} = 1$$

The matrix formula can be shown to the equivalent to the previous one

- Methodology applied in: Lund, Chilton, and Lee, PRSTAB **9** 064201 (2006) to construct a fail-safe iterative matched envelope including space-charge A5

S7: Hill's Equation: The Courant-Snyder Invariant and Single Particle Emittance

S7A: Introduction

Constants of the motion can simplify the interpretation of dynamics in physics

- Desirable to identify constants of motion for Hill's equation for improved understanding of focusing in accelerators
- Constants of the motion are not immediately obvious for Hill's Equation due to s-varying focusing forces related to $\kappa(s)$ can add and remove energy from the particle
    - Wronskian symmetry is one useful symmetry
    - Are there other symmetries?

/// Illustrative Example: Continuous Focusing/Simple Harmonic Oscillator

Equation of motion:

$$x'' + k_{\beta 0}^2 x = 0 \qquad\qquad k_{\beta 0}^2 = \text{const} > 0$$

Constant of motion is the well-know Hamiltonian/Energy:

$$H = \frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2 \ = \text{const}$$

which shows that the particle moves on an ellipse in *x*-*x*' phase-space with:

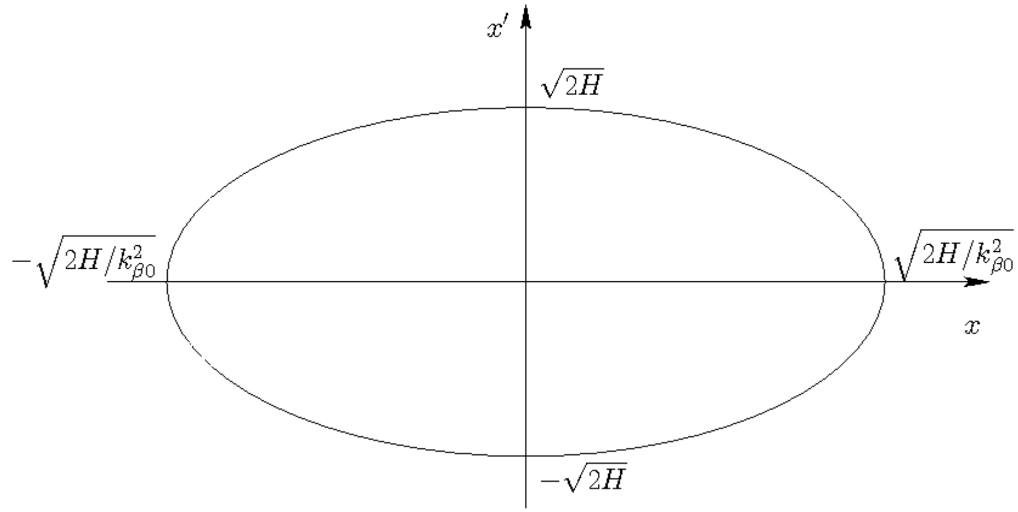
- Location of particle on ellipse set by initial conditions
- All initial conditions with same energy/H give same ellipse

$$\text{Max/Min}[x] \Leftrightarrow x' = 0$$

$$\text{Max/Min}[x] = \pm\sqrt{2H/k_{\beta 0}^2}$$

$$\text{Max/Min}[x'] \Leftrightarrow x = 0$$

$$\text{Max/Min}[x'] = \pm\sqrt{2H}$$



///

Question:

For Hill's equation:

$$x'' + \kappa(s)x = 0$$

does a quadratic invariant exist that can aid interpretation of the dynamics?

Answer we will find:

Yes, the Courant-Snyder invariant

Comments:

- Very important in accelerator physics
  - Helps interpretation of linear dynamics
- Named in honor of Courant and Snyder who popularized it's use in Accelerator physics while co-discovering alternating gradient (AG) focusing in a single seminal (and very elegant) paper:

  Courant and Snyder, *Theory of the Alternating Gradient Synchrotron*, Annals of Physics **3**, 1 (1958).
  - Christofolos also understood AG focusing in the same period using a more heuristic analysis
- Easily derived using phase-amplitude form of orbit solution
  - Can be much harder using other methods

S7B: Derivation of Courant-Snyder Invariant

The phase amplitude method described in S6 makes identification of the invariant elementary. Use the phase amplitude form of the orbit:

$$\boxed{\begin{aligned} x(s) &= A_i w(s) \cos\psi(s) \\ x'(s) &= A_i w'(s) \cos\psi(s) - \frac{A_i}{w(s)} \sin\psi(s) \end{aligned}}$$

$A_i, \ \psi_i = \psi(s_i)$
set by initial
at $s = s_i$

where

$$w'' + \kappa(s) w - \frac{1}{w^3} = 0$$

Re-arrange the phase-amplitude trajectory equations:

$$\frac{x}{w} = A_i \cos\psi$$

$$wx' - w'x = A_i \sin\psi$$

square and add the equations to obtain the Courant-Snyder invariant:

$$\boxed{\begin{aligned} \left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 &= A_i^2(\cos^2\psi + \sin^2\psi) \\ &= A_i^2 = \text{const} \end{aligned}}$$

Comments on the Courant-Snyder Invariant:

- Simplifies interpretation of dynamics (will show how shortly)
- Extensively used in accelerator physics
- Quadratic structure in *x*-*x*' defines a rotated ellipse in *x*-*x*' phase space.
- Because
$$w^2 \left(\frac{x}{w}\right)' = wx' - w'x$$
  the Courant-Snyder invariant can be alternatively expressed as:
$$\left(\frac{x}{w}\right)^2 + \left[w^2 \left(\frac{x}{w}\right)'\right]^2 = \text{const}$$
- *Cannot* be interpreted as a conserved energy!

The point that the Courant-Snyder invariant is *not* a conserved energy should be elaborated on. The equation of motion:

$$x'' + \kappa(s)x = 0$$

Is derivable from the Hamiltonian

$$H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 \quad \Longrightarrow \quad \begin{aligned} \frac{d}{ds}x &= \frac{\partial H}{\partial x'} = x' \\ \frac{d}{ds}x' &= -\frac{\partial H}{\partial x} = -\kappa x \end{aligned} \quad \Longrightarrow \quad x'' + \kappa x = 0$$

H is the energy:

$$H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 = T + V$$

$$T = \frac{1}{2}x'^2 \quad = \text{Kinetic "Energy"}$$

$$V = \frac{1}{2}\kappa x^2 \quad = \text{Potential "Energy"}$$

Apply the chain-Rule with $H = H(x,x';s)$:

$$\frac{dH}{ds} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial x}\frac{dx}{ds} + \frac{\partial H}{\partial x'}\frac{dx'}{ds}$$

Apply the equation of motion in Hamiltonian form:

$$\frac{d}{ds}x = \frac{\partial H}{\partial x'} \qquad \frac{d}{ds}x' = -\frac{\partial H}{\partial x}$$

$$\frac{dH}{ds} = \frac{\partial H}{\partial s} \overbrace{- \frac{dx'}{ds}\frac{dx}{ds} + \frac{dx}{ds}\frac{dx'}{ds}}^{0} = \frac{\partial H}{\partial s} = \frac{1}{2}\kappa' x^2 \neq 0$$

$$\Longrightarrow \quad \boxed{H \neq \text{const}}$$

- Energy of a "kicked" oscillator with $\kappa(s) \neq \text{const}$ is not conserved
- Energy should not be confused with the Courant-Snyder invariant

/// Aside: Only for the special case of continuous focusing (i.e., a simple Harmonic oscillator) are the Courant-Snyder invariant and energy simply related:

Continuous Focusing: $\kappa(s) = k_{\beta 0}^2 = \text{const}$

$$\Longrightarrow \quad H = \frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2 = \text{const}$$

w equation: $w'' + k_{\beta 0}^2 w - \frac{1}{w^3} = 0$

$$\Longrightarrow \quad w = \sqrt{\frac{1}{k_{\beta 0}}} = \text{const}$$

Courant-Snyder Invariant: $\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = \text{const}$

$$\begin{aligned}
\Longrightarrow \quad \left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 &= k_{\beta 0}x^2 + \frac{x'^2}{k_{\beta 0}} \\
&= \frac{2}{k_{\beta 0}}\left(\frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2\right) \\
&= \frac{2H}{k_{\beta 0}} = \text{const}
\end{aligned}$$

///

Interpret the Courant-Snyder invariant:

$$\boxed{\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2 = \text{const}}$$

by expanding and isolating terms quadratic terms in *x*-*x*' phase-space variables:

$$\left[\frac{1}{w^2} + w'^2\right] x^2 + 2[-ww']xx' + [w^2]x'^2 = A_i^2 = \text{const}$$

The three coefficients in [...] are functions of *w* and *w*' only and therefore are *functions of the lattice only* (not particle initial conditions). They are commonly called "Twiss Parameters" and are expressed denoted as:

$$\boxed{\begin{aligned} &\gamma x^2 + 2\alpha x x' + \beta x'^2 = A_i^2 = \text{const} \\ &\gamma(s) \equiv \frac{1}{w^2(s)} + [w'(s)]^2 = \frac{1+\alpha^2(s)}{\beta(s)} \\ &\beta(s) \equiv w^2(s) \\ &\alpha(s) \equiv -w(s)w'(s) \end{aligned}}$$

$$\gamma\beta = 1 + \alpha^2$$

- All Twiss "parameters" are specified by $w(s)$
- Given w and w' at a point (s) any 2 Twiss parameters give the 3rd

The area of the invariant ellipse is:

- Analytic geometry formulas: $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \pi A_i^2 \rightarrow Area = A_i^2/\sqrt{\gamma\beta - \alpha^2}$
- For Courant-Snyder ellipse: $\gamma\beta = 1 + \alpha^2$

$$\boxed{\text{Phase-Space Area} = \int_{\text{ellipse}} dx dx' \quad = \frac{\pi A_i^2}{\sqrt{\gamma\beta - \alpha^2}} = \pi A_i^2 \equiv \pi\epsilon}$$

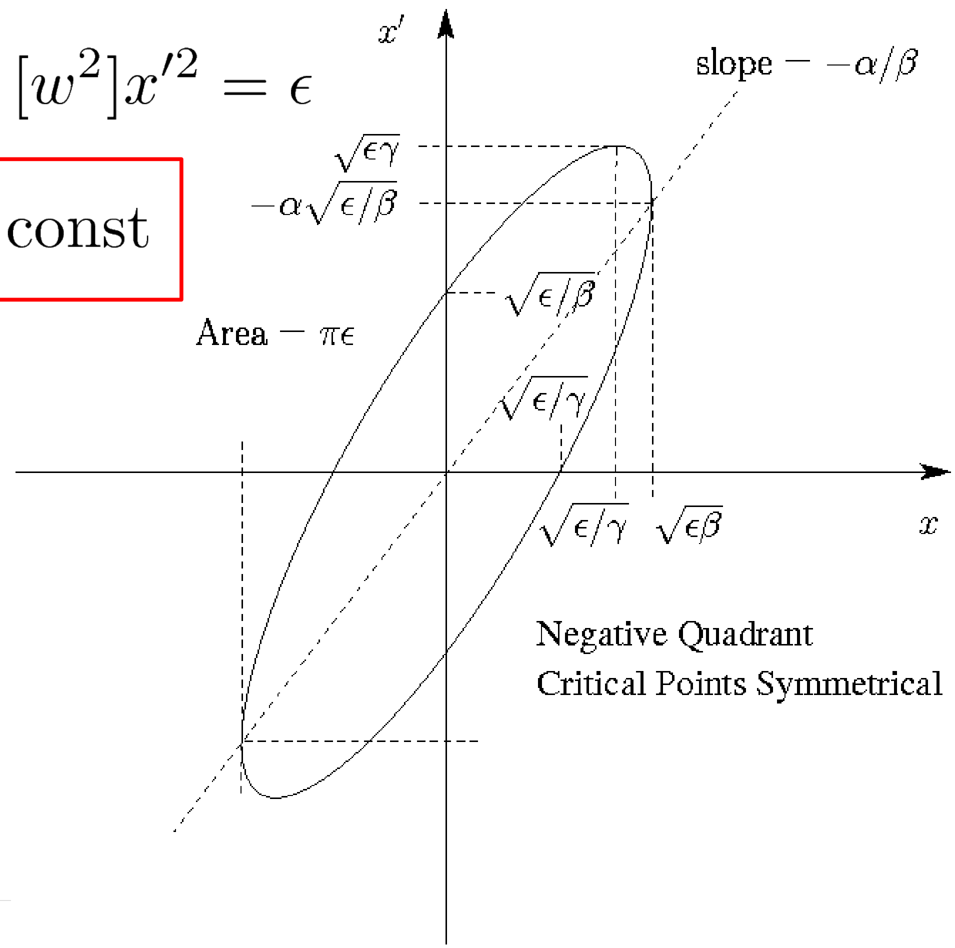
Where ϵ is the single-particle emittance:

- Emittance is the area of the orbit in *x*-*x*' phase-space divided by π

$$[1/w^2 + w'^2]x^2 + 2[-ww']xx' + [w^2]x'^2 = \epsilon$$

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon = \text{const}}$$

See problem sets
for critical point
calculation



/// Aside on Notation: Twiss Parameters and Emittance Units:

Twiss Parameters:

Use of $\alpha, \ \beta, \ \gamma$ should not create confusion with kinematic relativistic factors

- $\beta_b, \ \gamma_b$ are absorbed in the focusing function
- Contextual use of notation unfortunate reality .... not enough symbols!
- Notation originally due to Courant and Snyder, not Twiss, and might be more appropriately called "Courant-Snyder functions" or "lattice functions."

Emittance Units:

x has dimensions of length and *x*' is a dimensionless angle. So *x*-*x*' phase-space area has dimensions [[ϵ]] = length. A common choice of units is millimeters (mm) and milliradians (mrad), e.g.,

$$\epsilon = 10 \text{ mm-mrad}$$

The definition of the emittance employed is not unique and different workers use a wide variety of symbols. Some common notational choices:

$$\pi\epsilon \longrightarrow \epsilon \qquad \epsilon \longrightarrow \varepsilon \qquad \epsilon \longrightarrow E$$

Write the emittance values in units with a π, e.g.,

$$\epsilon = 10.5\,\pi - \text{mm-mrad}$$ (seems falling out of favor but still common)

Use caution! Understand conventions being used before applying results! ///

Properties of Courant-Snyder Invariant:

- The ellipse will rotate and change shape as the particle advances through the focusing lattice, but the instantaneous area of the ellipse ($\pi\epsilon = \mathrm{const}$) remains constant.
- The location of the particle on the ellipse and the size (area) of the ellipse depends on the initial conditions of the particle.
- The orientation of the ellipse is independent of the particle initial conditions. All particles move on nested ellipses.
- Quadratic in the *x*-*x*' phase-space coordinates, but is *not* the transverse particle energy (which is not conserved).

S7C: Lattice Maps

The Courant-Snyder invariant helps us understand the phase-space evolution of the particles. Knowing how the ellipse transforms (twists and rotates without changing area) is equivalent to knowing the dynamics of a *bundle* of particles. To see this:

General s:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

Initial $s = s_i$

$$\gamma_i x_i^2 + 2\alpha_i x_i x_i' + \beta_i x_i'^2 = \epsilon$$

$$\beta_i \equiv \beta(s = s_i) \qquad x_i \equiv x(s = s_i)$$
$$\alpha_i \equiv \alpha(s = s_i) \qquad x_i' \equiv x'(s = s_i)$$
$$\gamma_i \equiv \gamma(s = s_i)$$

Apply the components of the transport matrix:

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \mathbf{M}(s|s_i) \cdot \begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \cdot \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$

Invert 2x2 matrix and apply det **M** = 1 (Wronskian):

$$\Longrightarrow \qquad \begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \begin{bmatrix} S' & -S \\ -C' & C \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix} \qquad C \equiv C(s|s_i), \text{ etc.}$$

Insert expansion for $x_i, \ x_i'$ in the initial ellipse expression, collect factors of *x*^2, *xx*', and *x*'^2, and equate to general *s* ellipse expression:

$$\begin{aligned}[\gamma_i S'^2 - 2\alpha_i S'C' + \beta_i C'^2]x^2 \\ +2[-\gamma_i SS' + \alpha_i(CS' + SC') - \beta_i CC']xx' \\ +[\gamma_i S^2 - 2\alpha_i SC + \beta_i C^2]x'^2 \\ = \gamma x^2 + 2\alpha x x' + \beta x'^2\end{aligned}$$

Collect coefficients of x^2, xx', and x'^2 and summarize in matrix form:

$$\boxed{\begin{bmatrix} \gamma \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} S'^2 & -2C'S' & C'^2 \\ -SS' & CS' + SC' & -CC' \\ S^2 & -2CS & C^2 \end{bmatrix} \cdot \begin{bmatrix} \gamma_i \\ \beta_i \\ \alpha_i \end{bmatrix}}$$

This result can be applied to illustrate how a bundle of particles will evolve from an initial location in the lattice subject to the linear focusing optics in the machine using only principal orbits *C*, *S*, *C*', and *S*'

- Principal orbits will generally need to be calculated numerically
  - Intuition can be built up using simple analytical results (hard edge etc)

/// Example: Ellipse Evolution in a simple kicked focusing lattice

Drift:

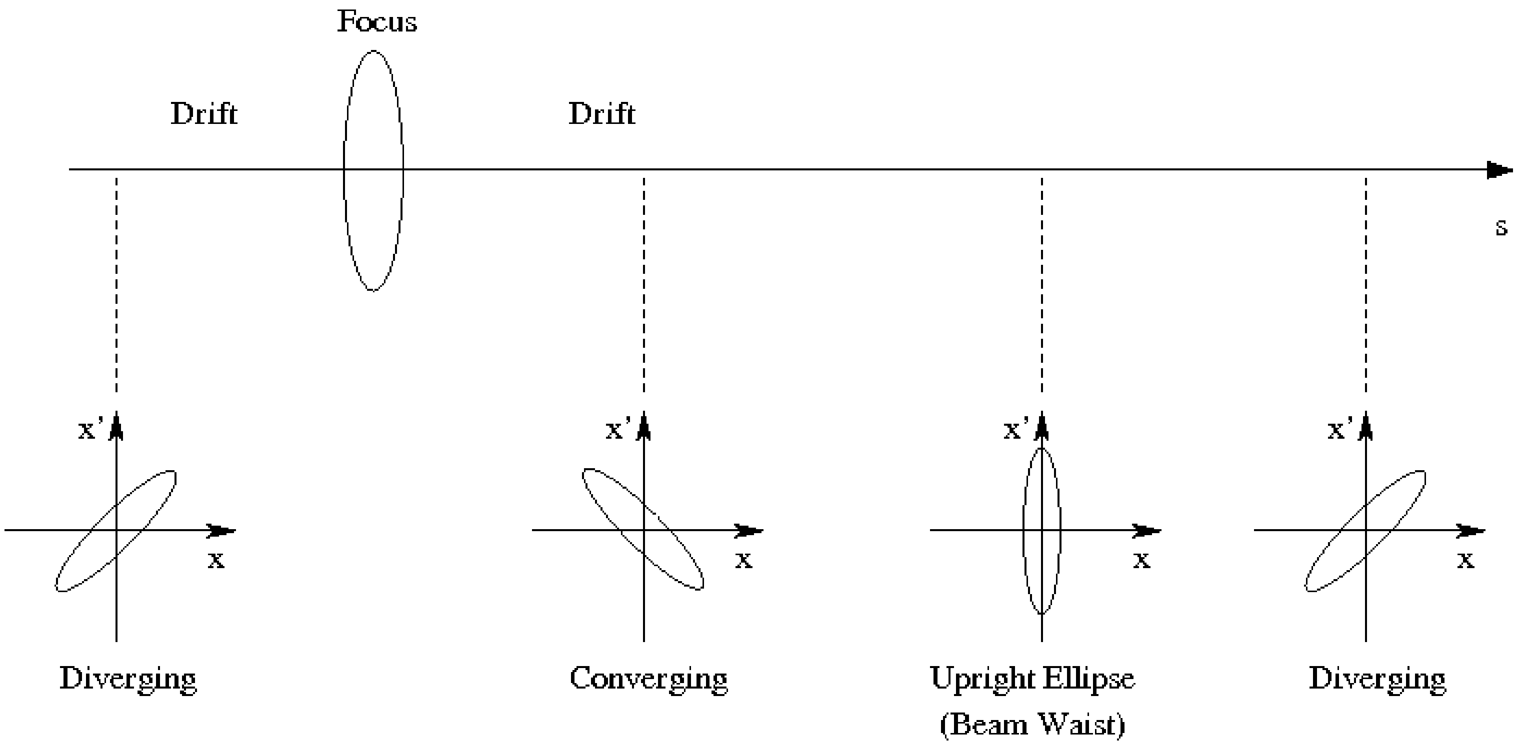
$$\begin{bmatrix} C & S \\ C' & S' \end{bmatrix} = \begin{bmatrix} 1 & s - s_i \\ 0 & 1 \end{bmatrix}$$

$$\gamma = \gamma_i$$
$$\alpha = -\gamma_i(s - s_i) + \alpha_i$$
$$\beta = \gamma_i(s - s_i)^2 - 2\alpha_i(s - s_i) + \beta_i$$

Thin Lens:
focal length f

$$\begin{bmatrix} C & S \\ C' & S' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

$$\gamma = \gamma_i + 2\alpha_i/f + \beta_i/f^2$$
$$\alpha = -\beta_i/f + \alpha_i$$
$$\beta = \beta_i$$



For further examples of phase-space ellipse evolutions in standard lattices, see previous examples given in: S6G ///

S8: Hill's Equation: The Betatron Formulation of the Particle Orbit and Maximum Orbit Excursions S8A: Formulation

The phase-amplitude form of the particle orbit analyzed in S6 of

$$\boxed{x(s) = A_i w(s) \cos\psi(s) = \sqrt{\epsilon} w(s) \cos\psi(s)} \qquad [[w]] = (\text{meters})^{1/2}$$

is not a unique choice. Here, *w* has dimensions sqrt(meters), which can render it inconvenient in applications. Due to this and the utility of the Twiss parameters used in describing orientation of the phase-space ellipse associated with the Courant-Snyder invariant (see: S7) on which the particle moves, it is convenient to define an alternative, Betatron representation of the orbit with:

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)} \cos\psi(s)$$

Betatron function: $\beta(s) \equiv w^2(s)$

Single-Particle Emittance: $\epsilon \equiv A_i^2 = \text{const}$

Phase: $\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{\beta(\tilde{s})} = \psi_i + \Delta\psi(s)$

- The betatron function is a Twiss "parameter" with dimension [[β]] = meters

Comments:

- Use of the symbol β for the betatron function does not result in confusion with relativistic factors such as β_b since the context of use will make clear
  - Relativistic factors often absorbed in lattice focusing function and do not directly appear in the dynamical descriptions
- The change in phase $\Delta\psi$ is the same for both formulations:

$$\Delta\psi(s) = \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})} = \int_{s_i}^{s} \frac{d\tilde{s}}{\beta(\tilde{s})}$$

From the equation for *w*:

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s + L_p) = w(s) \qquad\qquad w(s) > 0$$

the betatron function is described by:

$$\boxed{\begin{aligned} &\frac{1}{2}\beta(s)\beta''(s) - \frac{1}{4}\beta'^2(s) + \kappa(s)\beta^2(s) = 1 \\ &\beta(s + L_p) = \beta(s) \qquad\qquad \beta(s) > 0 \end{aligned}}$$

- The betatron function represents, analogously to the *w*-function, a special function defined by the periodic lattice. Similar to *w*(*s*) it is a unique function of the lattice.
- The equation is still nonlinear but we can apply our previous analysis of *w*(*s*) (see S6 Appendix A) to solve analytically in terms of the principle orbits

S8B: Maximum Orbit Excursions

From the orbit equation

$$x = \sqrt{\epsilon\beta}\cos\psi$$

the maximum and minimum possible particle excursions occur where:

$$\cos\psi = +1 \quad \longrightarrow \quad \mathrm{Max}[x] = \sqrt{\epsilon\beta(s)} = \sqrt{\epsilon}w(s)$$

$$\cos\psi = -1 \quad \longrightarrow \quad \mathrm{Min}[x] = -\sqrt{\epsilon\beta(s)} = -\sqrt{\epsilon}w(s)$$

Thus, the max radial extent of *all* particle oscillations $\mathrm{Max}[x] \equiv x_m$ in the beam distribution occurs for the particle with the max single particle emittance since the particles move on nested ellipses:

$$\boxed{\begin{aligned} &\mathrm{Max}[\epsilon] \equiv \epsilon_m \\ &x_m(s) = \sqrt{\epsilon_m\beta(s)} = \sqrt{\epsilon_m}w(s) \end{aligned}}$$

In terms of Twiss parameters:

$$\boxed{\begin{aligned} x_m &= \sqrt{\epsilon_m}w = \sqrt{\epsilon_m\beta} \\ x'_m &= \sqrt{\epsilon_m}w' = -\sqrt{\frac{\epsilon_m}{\beta}}\alpha \end{aligned}}$$

- Assumes sufficient numbers of particles to populate all possible phases
- x_m corresponds to the min possible machine aperture to prevent particle losses

  \- Practical aperture choice influenced by: resonance effects due to nonlinear applied fields, space-charge, scattering, finite particle lifetime, ...

From:

$$w''(s) + \kappa(s) w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s + L_p) = w(s) \qquad\qquad w(s) > 0$$

We immediately obtain an equation for the maximum locus (envelope) of radial particle excursions $x_m = \sqrt{\epsilon_m} w$ as:

$$\boxed{\begin{aligned} & x_m''(s) + \kappa(s) x_m(s) - \frac{\epsilon_m^2}{x_m^3(s)} = 0 \\ & x_m(s + L_p) = x_m(s) \qquad\qquad x_m(s) > 0 \end{aligned}}$$

Comments:

- Equation is analogous to the statistical envelope equation derived by J.J. Barnard in the Intro Lectures when a space-charge term is added and the max single particle emittance is interpreted as a statistical emittance
  - correspondence will become more concrete in later lectures
- This correspondence will be developed more extensively in later lectures on Transverse Centroid and Envelope Descriptions of Beam Evolution and Transverse Equilibrium Distributions