Dispersive and Chromatic Effects: Particle Equations of Motion

Particles do not necessarily have the design momentum p

$$\boxed{p = p_0 + \delta p}$$

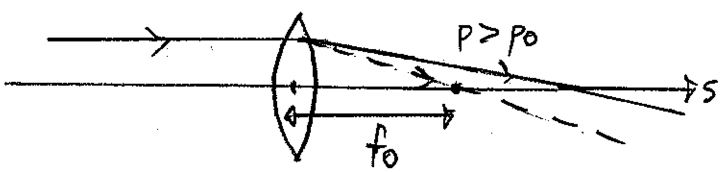
p_0 = Design momentum
δp = "Off Momentum" (momentum deviation)

This results in different focal lengths and bend radii in optical elements since the particle rigidity $(B\rho)$ changes:

$$\boxed{\begin{aligned}(B\rho) &= \frac{p}{q} = \left(\frac{p_0}{q}\right)\left(\frac{p}{p_0}\right) = (B\rho)_0\left(\frac{p}{p_0}\right) \\ (B\rho)_0 &= \frac{q}{p_0} = \text{Design Rigidity}\end{aligned}}$$

Focusing
(Thin Lens)

$$\frac{1}{f} \simeq K\ell = \frac{B'\ell}{(B\rho)}$$

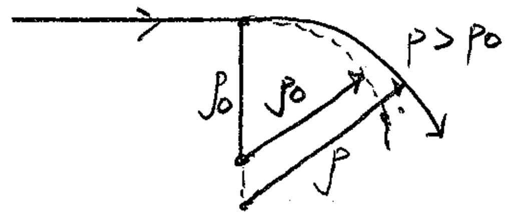


design $\frac{1}{f_0} = \frac{B_y'\ell}{(B\rho)_0}$

$f > f_0 \quad p > p_0$
$f < f_0 \quad p < p_0$

Bending

$$\frac{1}{\rho} = \frac{B_y(0)}{(B\rho)}$$



$\frac{1}{\rho_0} = \frac{B_y(0)}{(B\rho)_0}$

$\rho_0 < \rho \quad p > p_0$
$\rho_0 > \rho \quad p < p_0$

Need to account for this effect in the particle equations of motion. This is relatively straightforward to do: Simply retrace steps in derivation of design momentum equations presented in lecture note set 04.lecture.pdf and allow momentum deviations while measuring s along the reference (design) trajectory.

We previously obtained

$$\boxed{\begin{aligned} \hat{x}: \quad & x'' - \frac{(\rho_0 + x)}{\rho_0^2} = -\frac{B_y}{(B\rho)}\left(1 + \frac{x}{\rho_0}\right)^2 \\ \hat{y}: \quad & y'' = \frac{B_x}{(B\rho)}\left(1 + \frac{x}{\rho_0}\right)^2 \end{aligned}}$$

Here we take

Fields:

$$\boxed{\begin{aligned} B_x &= B' y \\ B_y &= \underbrace{B_y(0)}_{\text{Dipole Bend}} + \underbrace{B' x}_{\text{Quadropole}} \end{aligned}}$$

Design Bend:

$$\boxed{\frac{1}{\rho_0} = \frac{B_y(0)}{(B\rho)_0}}$$

★ Comment: Denoting design bend with subscript "0" to avoid confusion.

Rigidity:

$$\boxed{(B\rho) = \underbrace{(B\rho)_0}_{\text{Design}} \frac{p}{p_0}}$$

Insert these expressions

$$\boxed{\begin{aligned} x'' - \frac{(\rho_0 + x)}{\rho_0^2} &= -\frac{(B_y(0) + B'x)}{(B\rho)_0}\left(\frac{p_0}{p}\right)\left(1 + \frac{x}{\rho_0}\right)^2 \\ y'' &= \frac{B'y}{(B\rho)_0}\left(\frac{p_0}{p}\right)\left(1 + \frac{x}{\rho_0}\right)^2 \end{aligned}}$$

Expand keeping only linear order terms:

$$\boxed{\begin{aligned} x'' + \left[-\frac{1}{\rho_0^2} + \frac{2B_y(0)}{\rho_0 (B\rho)_0}\left(\frac{p_0}{p}\right) + \frac{B'}{(B\rho)_0}\left(\frac{p_0}{p}\right)\right] x &= \frac{1}{\rho_0} - \frac{B_y(0)}{(B\rho)_0}\left(\frac{p_0}{p}\right) \\ y'' - \frac{B'}{(B\rho)_0}\left(\frac{p_0}{p}\right) y &= 0 \end{aligned}}$$

Denote:

$$K \equiv \frac{B'}{(B\rho)_0} = \text{Quadrupole coupling defined wrt design momentum.}$$

and apply

$$\frac{1}{\rho_0} \equiv \frac{B_y(0)}{(B\rho)_0}$$

to obtain:

$$\boxed{\begin{aligned} x'' + \left[\frac{1}{\rho_0^2}\left(-1 + 2\left(\frac{p_0}{p}\right)\right) + \frac{K}{(p/p_0)}\right] x &= \frac{1}{\rho_0}\left[1 - \frac{p_0}{p}\right] \\ y'' - \frac{K}{(p/p_0)} y &= 0 \end{aligned}}$$

Use:

$$\left(\frac{p}{p_0}\right) = \frac{p_0+\delta p}{p_0} = 1 + \frac{\delta p}{p_0} \equiv 1+\delta \qquad \boxed{\delta \equiv \frac{\delta p}{p_0} = \text{Fractional Momentum Error}}$$

Then

$$-1 + 2\left(\frac{p_0}{p}\right) = \frac{-p + 2p_0}{p} = \frac{-(p_0+\delta p)+2p_0}{p_0+\delta p} = \frac{p_0-\delta p}{p_0+\delta p} = \frac{1-\delta}{1+\delta}$$

$$1 - \frac{p_0}{p} = \frac{p-p_0}{p} = \frac{p_0+\delta p-p_0}{p_0+\delta p} = \frac{\delta p}{p_0+\delta p} = \frac{\delta}{1+\delta}$$

and the equations of motion including off design momentum become:

$$\boxed{\begin{aligned} x'' + \left[\frac{1}{\rho_0^2}\frac{1-\delta}{1+\delta} + \frac{K}{1+\delta}\right] x &= \frac{\delta}{1+\delta}\frac{1}{\rho_0} \\ y'' - \frac{K}{1+\delta}\, y &= 0 \end{aligned}}$$

$$K \equiv \frac{B'}{(B\rho)_0} \qquad \frac{1}{\rho_0} = \frac{B_y(0)}{(B\rho)_0} \qquad (B\rho)_0 = \frac{p_0}{q}$$

☆ Often the "0" subscript is dropped on p_0 and $(B\rho)_0$ and the quantities are taken implicitly to be defined wrt the design momentum.