S9: Momentum Spread Effects and Bending
S9A: Formulation

Except for brief digressions in S1 and S4, we have concentrated on particle dynamics where all particles have the design longitudinal momentum at a value of s in the lattice:

$$p_s = m\gamma_b\beta_b c = \text{same for every particle}$$

Realistically, there will always be a finite spread of particle momentum within a beam slice, so we take:

$$p_s = p_0 + \delta p$$

$$p_0 \equiv m\gamma_b\beta_b c = \text{Design Momentum}$$

$$\delta p \equiv \text{Off Momentum}$$

Typical values of momentum spread in a beam with a single species of particles with conventional sources and accelerating structures:

$$\frac{|\delta p|}{p_0} \sim 10^{-2} \rightarrow 10^{-6}$$

The spread of particle momentum can modify particle orbits, particularly when dipole bends are present since the bend radius depends strongly on the particle momentum

To better understand this effect, we analyze the particle equations of motion with leading-order momentum spread (see: S1H) effects retained:

$$x''(s) + \left[\frac{1}{R^2(s)}\frac{1-\delta}{1+\delta} + \frac{\kappa_x(s)}{(1+\delta)^n}\right] x(s) = \frac{\delta}{1+\delta}\frac{1}{R(s)}$$

$$y''(s) + \frac{\kappa_y(s)}{(1+\delta)^n} y(s) = 0$$

Magnetic Dipole Bend

$$\frac{1}{R(s)} = \frac{B_y^a|_{\text{dipole}}}{[B\rho]}$$

$$[B\rho] = \frac{p_0}{q}$$

$R(s) =$ Local Bend Radius for design momentum p_0 ($R \to \infty$ in straight sections)

$$\delta \equiv \frac{\delta p}{p_0}$$

$\kappa_{x,y} =$ Focusing Functions (using design momentum)

$$n = \begin{cases} 1, & \text{Magnetic Quadrupoles} \\ 2, & \text{Solenoids, Electric Quadrupoles} \end{cases}$$

Neglects:

- Space-charge: $\phi \to 0$
- Nonlinear applied focusing: $\mathbf{E}^a$, $\mathbf{B}^a$ contain only linear focus terms
- Acceleration: $p_0 = mc\gamma_b\beta_b = \text{const}$

In the equations of motion, it is important to understand that B_y^a of the magnetic bends are set from the radius *R* required by the design particle orbit
(see: S1 for details)

- Equation relating *R* to fields must be modified for electric bends (see S1)
- *y*-plane bends also require modification of eqns (analogous to *x*-plane case)

The focusing strengths are defined with respect to the design momentum:

$$\kappa_x = \begin{cases} \dfrac{qG}{m\gamma_b\beta_b^2c^2}, & G = -\partial E_x^a/\partial x = \partial E_y^a/\partial y = \text{Electric Quad. Grad.} \\ \dfrac{qG}{m\gamma_b\beta_b c}, & G = \partial B_x^a/\partial y = \partial B_y^a/\partial x = \text{Magnetic Quad. Grad.} \\ \dfrac{qB_{z0}}{4m\gamma_b^2\beta_b^2c^2}, & B_{z0} = \text{Solenoidal Magnetic Field} \end{cases}$$

$$\gamma_b,\ \beta_b \text{ calculated from } p_0$$

Terms in the equations of motion associated with momentum spread (δ) can be lumped into two classes:

S.9B: Dispersive -- Associated with Dipole Bends
S.9C: Chromatic -- Associated with Focusing

S9B: Dispersive Effects

Present only in the *x*-equation of motion and result from bending. Neglecting chromatic terms:

$$x''(s) + \left[\frac{1}{R^2(s)}\frac{1-\delta}{1+\delta} + \kappa_x(s)\right] x(s) = \frac{\delta}{1+\delta}\frac{1}{R(s)}$$

Term 1 Term 2

Particles are bent at different radii when the momentum deviates from the design value ($\delta \neq 0$) leading to changes in the particle orbit

- Dispersive terms contain the bend radius *R*

Generally, the bend radii *R* are large and δ is small, and we can take to leading order:

Term 1: $\left[\frac{1}{R^2}\frac{1-\delta}{1+\delta} + \kappa_x\right] x \simeq \kappa_x x$

Need to be careful here if *R* not large

Term 2: $\frac{\delta}{1+\delta}\frac{1}{R} \simeq \frac{\delta}{R} + \Theta(\delta^2)$

The equations of motion then become:

$$x''(s) + \kappa_x(s)x(s) = \frac{\delta}{R(s)}$$

$$y''(s) + \kappa_y(s)y(s) = 0$$

- The y-equation is not changed from the usual Hill's Equation

The *x*-equation is typically solved for *periodic* ring lattices by exploiting the linear structure of the equation and linearly resolving:

$$x(s) = x_h(s) + x_p(s)$$

$$x_h \equiv \text{Homogeneous Solution}$$

$$x_p \equiv \text{Particular Solution}$$

where x_h is the *general* solution to the Hill's Equation:

$$x_h''(s) + \kappa_x(s)x_h(s) = 0$$

and x_p is the *periodic* solution to:

$$x_p = \delta \cdot D \qquad D'' + \kappa_x(s)D(s) = \frac{1}{R(s)}$$

$$D \equiv \text{Dispersion Function} \qquad D(s + L_p) = D(s)$$

This convenient resolution of the orbit *x*(*s*) can *always* be made because the homogeneous solution will be adjusted to match any initial condition

Note that x_p provides a measure of the offset of the particle orbit relative to the design orbit resulting from a small deviation of momentum (δ)

x(*s*) = 0 defines the design orbit

$$[[D]] = \text{meters}$$

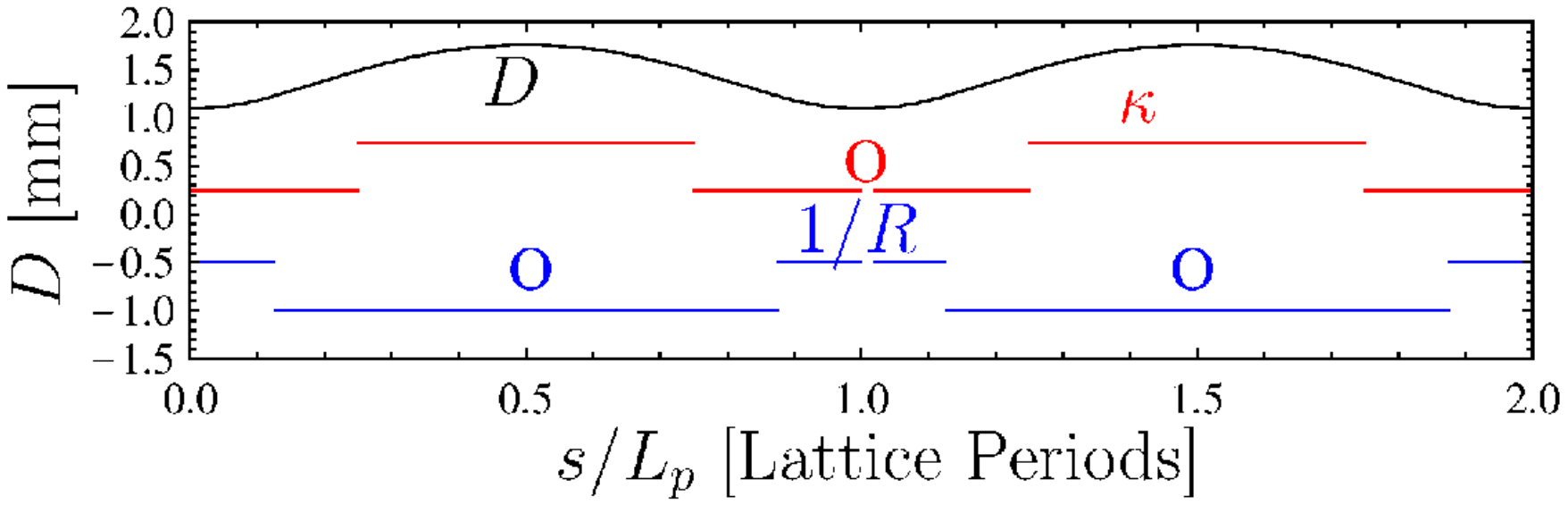
$$\delta \cdot D = \text{Dispersion induced orbit offset in meters}$$

/// Example:

For purposes of a simple illustration we here use an imaginary FO (Focus-Drift) piecewise-constant lattice where the *x*-plane focusing is like the focus-plane of a quadrupole with one thick lens focus optic per lattice period and a single drift with the bend in the middle of the drift

- Focus element implemented by $\kappa > 0$ *x*-plane quadrupole transfer matrix in S5B.

$$L_p = 0.5 \text{ m} \quad \kappa = 20/\text{m}^2 \text{ in Focusing}$$
$$\eta = 0.5 \quad R = 15 \text{ m, in bend, } 25\% \text{ Occupancy}$$



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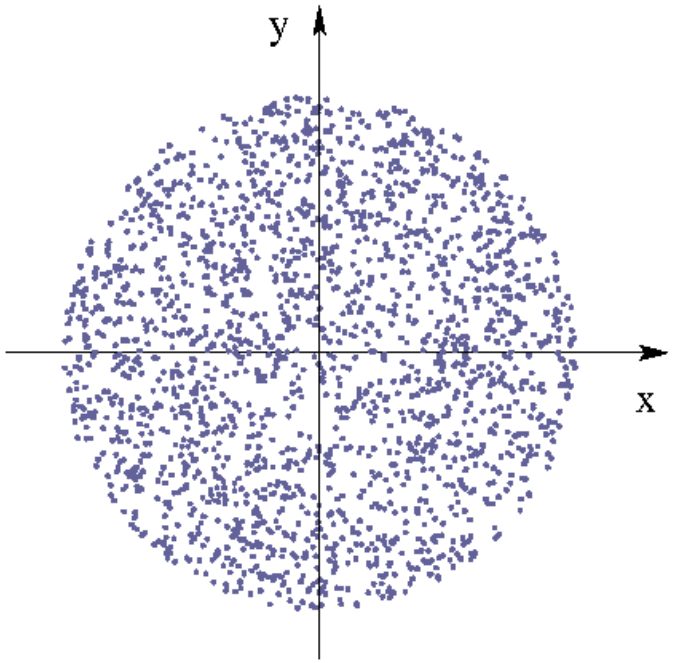
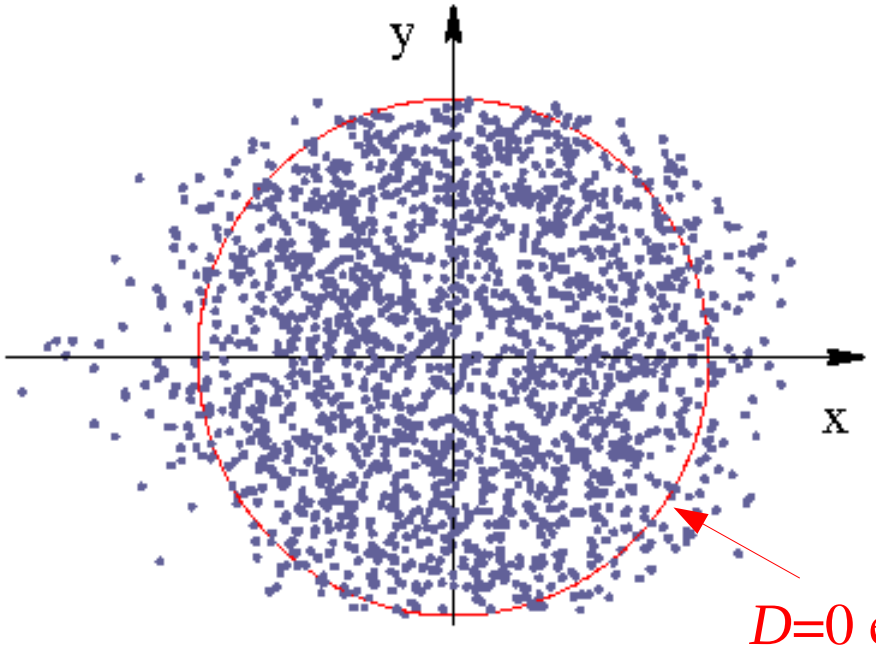
/// Example: Dispersion broadens the *x*-distribution

Uniform Bundle of particles D = 0



Same Bundle of particles D nonzero

- Gaussian distribution of momentum spread distorts the *x*-*y* distribution extents in *x* but not in *y*



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/// Example: Continuous Focusing in a Continuous Bend

$$\kappa_x(s) = k_{\beta 0}^2 = \text{const}$$
$$R(s) = R = \text{const}$$

Dispersion equation becomes:

$$D'' + k_{\beta 0}^2 D = \frac{1}{R}$$

With constant solution:

$$D = \frac{1}{k_{\beta 0}^2 R} = \text{const}$$

From this result we can crudely estimate the average value of the dispersion function in a ring with periodic focusing by taking:

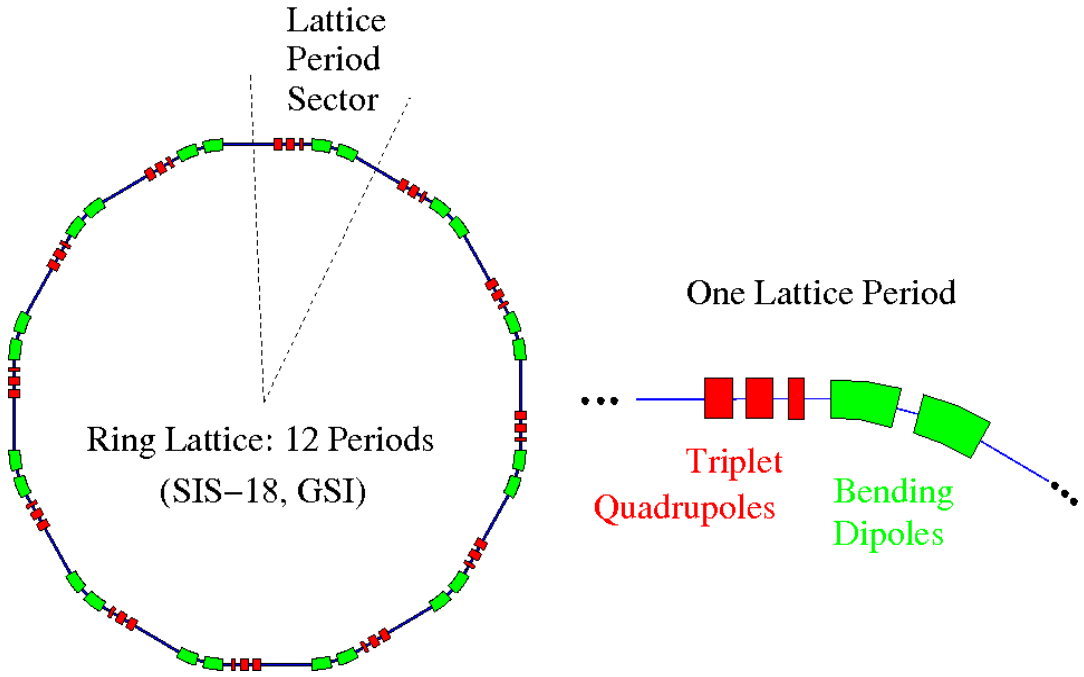
$$R = \text{Avg Radius Ring}$$
$$L_p = \text{Lattice Period (Focusing)}$$
$$\sigma_{0x} = x\text{-Plane Phase Advance}$$

$$\Longrightarrow \quad k_{\beta 0} \sim \frac{\sigma_0}{L_p} \qquad \Longrightarrow \quad D \sim \frac{L_p^2}{\sigma_0^2 R}$$

///

Many rings are designed to focus the dispersion function $D(s)$ to small values in straight sections even though the lattice has strong bends

- Desirable since it allows smaller beam sizes at locations near where $D = 0$ and these locations can be used to insert and extract (kick) the beam into and out of the ring with minimal losses
  - Since average value of D is dictated by ring size and focusing strength (see example next page) this variation in values can lead to D being larger in other parts of the ring
- Quadrupole triplet focusing lattices are often employed in rings since the optics allows sufficient flexibility to tune D while simultaneously allowing particle phase advances to also be adjusted



Dispersive Effects in Bends

It is common that a beam is transported through a single or series of bends in applications rather than a periodic ring lattice. In such situations, dispersive corrections to the particle orbit are analyzed differently. In this case, the same particular + homogeneous solution decomposition is used as in the ring case with the Dispersion function satisfying:

$$D''(s) + \kappa_x(s) D(s) = \frac{1}{R(s)}$$

However, in this case D is solved from an initial condition. Usually from a dispersion-free initial condition $s = s_i$ upstream of the bending section with:

$$D(s_i) = 0 = D'(s_i)$$

If the bend can be configured such that on transport through the bend $(s = s_d)$ that

$$D(s_d) = 0 = D'(s_d)$$

Then the bend system is first order achromatic meaning there will be no orbit deviation to 1st order in δ

This equation has the form of a *Driven Hill's Equation*:

$$x'' + \kappa(s)x = p(s)$$

$$x \to D$$

$$p \to 1/R$$

The general solution to this equation can be solved analytically using a Green function method (see Appendix A) as:

$$x(s) = x(s_i)\mathcal{C}(s|s_i) \; + \; x'(s_i)\mathcal{S}(s|s_i) \; + \; \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})p(\tilde{s})$$

$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

Cosine-Like Solution

$$\mathcal{C}''(s|s_i) + \kappa(s)\mathcal{C}(s|s_i) = 0$$

$$\mathcal{C}(s_i|s_i) = 1$$

$$\mathcal{C}'(s_i|s_i) = 0$$

Sine-Like Solution

$$\mathcal{S}''(s|s_i) + \kappa(s)\mathcal{S}(s|s_i) = 0$$

$$\mathcal{S}(s_i|s_i) = 0$$

$$\mathcal{S}'(s_i|s_i) = 1$$

$$x(s_i) = \text{Initial value } x$$

$$x'(s_i) = \text{Initial value } x'$$

Green function effectively casts driven equation in terms of homogeneous solution projections.

Using this Green function solution from the dispersion-free initial condition gives

$$D(s) = \mathcal{S}(s|s_i) \int_{s_i}^{s} d\tilde{s}\ \frac{1}{R(\tilde{s})} \mathcal{C}(\tilde{s}|s_i) \ - \ \mathcal{C}(s|s_i) \int_{s_i}^{s} d\tilde{s}\ \frac{1}{R(\tilde{s})} \mathcal{S}(\tilde{s}|s_i)$$

$$\mathcal{C}(s|s_i) = \text{Cosine-like Principal Trajectory}$$
$$\mathcal{S}(s|s_i) = \text{Sine-like Principal Trajectory}$$

So for a 1st order achromatic system we requite for no leading-order dispersive corrections that

$$0 = \int_{s_i}^{s_d} d\tilde{s}\ \frac{1}{R(\tilde{s})} \mathcal{C}(\tilde{s}|s_i)$$
$$0 = \int_{s_i}^{s_d} d\tilde{s}\ \frac{1}{R(\tilde{s})} \mathcal{S}(\tilde{s}|s_i)$$

Various lattices consisting of regular combinations of bends and focusing optics can be made achromatic to 1st order by meeting these criteria.

- Higher-order achromats also possible under more detailed analysis

This topic is beyond the scope of these lectures but detailed examples can be found in the literature

- See, for examples: Rusthoi and Wadlinger, 1991 PAC, 607

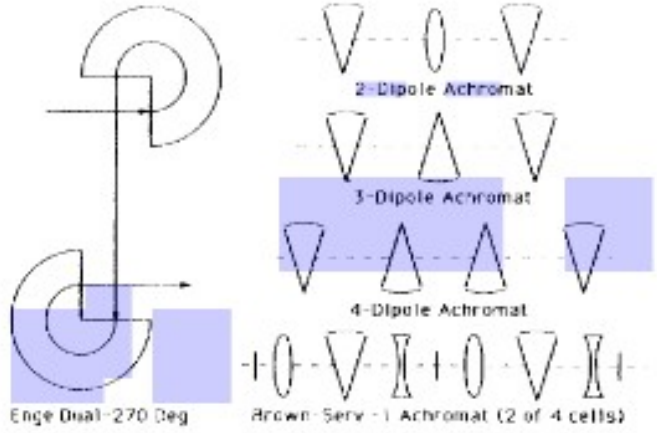


Figure 1. First-order achromats.

S9C: Chromatic Effects

Present in both *x*- and *y*-equations of motion and result from applied focusing strength changing with deviations in momentum:

$$x''(s) + \frac{\kappa_x(s)}{(1+\delta)^n} x(s) = 0$$

$$y''(s) + \frac{\kappa_y(s)}{(1+\delta)^n} y(s) = 0$$

$R \to \infty$

to neglect bending terms

$\kappa_{x,y}$ = Focusing Functions with γ_b, β_b calculated from p_0

- Generally of lesser importance (smaller corrections) relative to dispersive terms (S9C) in linacs *except* where the beam is focused onto a target (small spot) or when momentum spreads are large
- Can be important in rings where precise control of tunes (betratron oscillations per ring lap) are needed to avoid resonances: see Transverse Particle Resonances
- J.J. Barnard in Application Lectures: Heavy Ion Fusion and Final Focusing will overview consequences of chromatic effects on the achievable beam spot in his analysis on final focus optics

Can analyze by redefining kappa function to incorporate off-momentum:

$$\frac{\kappa_x(s)}{(1+\delta)^n} \rightarrow \kappa_{x,\mathrm{new}}(s)$$

However, this would require calculating new amplitude/betatron functions for each particle off-momentum value δ in the distribution to describe the evolution of the orbits. That would not be efficient.

Rather, need a perturbative formula to calculate the small amplitude correction to the nominal particle orbit with design momentum due to the small amplitude correction due to the off-momentum δ .

Either the *x*- and *y*-equations of motion can be put in the form:

$$x''(s) + \frac{\kappa(s)}{(1+\delta)^n} x(s) = 0$$

Expand to leading order in δ :

$$x''(s) + \kappa(s)(1 - n\delta)x(s) = 0$$

Set:

$$x(s) = x_0(s) + \eta(s)$$

$$x_0(s) = \text{Orbit Solution for } \delta = 0$$
$$\eta(s) = \text{Orbit Correction to } x_0 \text{ for } \delta \neq 0$$

Giving:

$$x_0'' + \kappa x_0 = 0$$

$$(x_0 + \eta)'' + \kappa(1 - n\delta)(x_0 + \eta) = 0$$

Subtract equations and neglect 2nd order term in $\delta \cdot \eta$ to obtain a linear equation for η :

$$\eta'' + \kappa\eta = n\delta\kappa x_0$$

This equation has the form of a *Driven Hill's Equation*:

$$x'' + \kappa(s)x = p(s)$$

$$x \to \eta$$

$$p \to n\delta\kappa x_0$$

The general solution to this equation can be solved analytically using a Green function method (see Appendix A) as:

$$x(s) = x(s_i)\mathcal{C}(s|s_i) \; + \; x'(s_i)\mathcal{S}(s|s_i) \; + \; \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})p(\tilde{s})$$

$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

Cosine-Like Solution

$$\mathcal{C}''(s|s_i) + \kappa(s)\mathcal{C}(s|s_i) = 0$$

$$\mathcal{C}(s_i|s_i) = 1$$

$$\mathcal{C}'(s_i|s_i) = 0$$

Sine-Like Solution

$$\mathcal{S}''(s|s_i) + \kappa(s)\mathcal{S}(s|s_i) = 0$$

$$\mathcal{S}(s_i|s_i) = 0$$

$$\mathcal{S}'(s_i|s_i) = 1$$

$$x(s_i) = \text{Initial value } x$$

$$x'(s_i) = \text{Initial value } x'$$

Using this result, the general solution for the chromatic correction to the particle orbit can be expressed as:

$$\eta(s) = \eta(s_i)\mathcal{C}(s|s_i) + \eta'(s_i)\mathcal{S}(s|s_i) + n\delta \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})\kappa(\tilde{s})x_0(\tilde{s})$$

$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

$$\eta(s_i) = \text{Initial value } \eta$$

$$\eta'(s_i) = \text{Initial value } \eta'$$

Chromatic orbit perturbations are typically measured from a point in the lattice where they are initially zero like a drift where the orbit was correct before focusing quadrupoles. In this context, can take:

$$\eta(s_i) = 0 = \eta'(s_i)$$

$$\eta(s) = n\delta \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})\kappa(\tilde{s})x_0(\tilde{s})$$

The Green function can be simplified using results from S6F:

$$C(s|s_i) = \frac{w(s)}{w_i}\cos\Delta\psi(s) - w_i' w(s)\sin\Delta\psi(s)$$

$$S(s|s_i) = w_i w(s)\sin\Delta\psi(s)$$

$$\Delta\psi(s) \equiv \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

$$w_i \equiv w(s = s_i)$$

$$w_i' \equiv w'(s = s_i)$$

Giving after some algebra:

$$\begin{aligned} G(s,\tilde{s}) &= \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i) \\ &= w(s)w(\tilde{s})[\sin\Delta\psi(s)\cos\Delta\psi(\tilde{s}) - \cos\Delta\psi(s)\sin\Delta\psi(\tilde{s})] \\ &= w(s)w(\tilde{s})\sin[\Delta\psi(s) - \Delta\psi(\tilde{s})] \end{aligned}$$

Using this and the phase amplitude form of the orbit:

$$\begin{aligned} x_0(s) &= A_i w(s) \cos[\psi(s)] \\ &= \sqrt{\epsilon} w(s) \cos[\Delta\psi(s) + \psi_i] \end{aligned}$$

- Initial phase ψ_i implicitly chosen (can always do) for initial amplitude $A_i \geq 0$

the orbit deviation from chromatic effects can be calculated as:

$$\begin{aligned} \eta(s) &= n\delta \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s}) \kappa(\tilde{s}) x_0(\tilde{s}) \\ &= n\delta\sqrt{\epsilon} w(s) \int_{s_i}^{s} d\tilde{s}\; w^2(\tilde{s}) \sin[\Delta\psi(s) - \Delta\psi(\tilde{s})] \cos[\Delta\psi(\tilde{s}) + \psi_i] \end{aligned}$$

Formula applicable to all types of focusing lattices:

- Quadrupole: electric and magnetic
- Solenoid (Larmor frame)
- Linac and rings

Add examples in future editions of notes …

Comments:

- Perturbative formulas can be derived to calculate the effect on betatron tunes (particle oscillations per lap) in a ring based on integrals of the unpreturbed betatron function: see Wiedemann, *Particle Accelerator Physics*
- For magnetic quadrupole lattices further detailed analysis (see Steffen, *High Energy Beam Optics*) it can be shown that:
  - Impossible to make an achromatic focus in any quadrupole system.
    Here achromatic means if

$$\eta(s_i) = 0 = \eta'(s_i)$$

    there is some achromatic point $s = s_f$ post optics with

$$\eta(s_f) = 0 = \eta'(s_f)$$

- More detailed analysis of the chromatic correction to particle orbits in rings show that a properly oriented nonlinear sextupole inserted into the periodic ring lattice with correct azimuthal orientation at a large dispersion points can to leading order compensate for chromatic corrections. See Wille, *The Physics of Particle Accelerators* for details.
  - Correction introduces nonlinear terms for large amplitude
  - Correction often distributed around ring for practical reasons

Appendix A: Green Function for Driven Hill's Equation

Following Wiedemann (Particle Accelerator Physics, 1993, pp 106) first, consider more general *Driven Hill's Equation*

$$x'' + \kappa(s)x = p(s)$$

The corresponding homogeneous equation:

$$x'' + \kappa(s)x = 0$$

has principal solutions

$$x(s) = C_1\mathcal{C}(s|s_i) + C_2\mathcal{S}(s|s_i) \qquad C_1, C_2 = \text{constants}$$

where

Cosine-Like Solution

$$\mathcal{C}'' + \kappa(s)\mathcal{C} = 0$$

$$\mathcal{C}(s = s_i) = 1$$

$$\mathcal{C}'(s = s_i) = 0$$

Sine-Like Solution

$$\mathcal{S}'' + \kappa(s)\mathcal{S} = 0$$

$$\mathcal{S}(s = s_i) = 0$$

$$\mathcal{S}'(s = s_i) = 1$$

Recall that the homogeneous solutions have the Wronskian symmetry:

- See S5C

$$W(s) = \mathcal{C}(s)\mathcal{S}'(s) - \mathcal{C}'(s)\mathcal{S}(s) = 1 \qquad \mathcal{C}(s) \equiv \mathcal{C}(s|s_i) \quad \text{etc.}$$

A particular solution to the *Driven Hill's Equation* can be constructed using a Greens' function method:

$$x(s) = \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s}) p(\tilde{s})$$

$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

Demonstrate this works by first taking derivatives:

$$x = \mathcal{S}(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s})$$

$$x' = \mathcal{S}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s})$$

$$+ p(s)\left[\mathcal{S}(s)\mathcal{C}(s) - \mathcal{S}(s)\mathcal{C}(s)\right] \quad [\ldots] = 0$$

$$= \mathcal{S}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s})$$

$$x'' = \mathcal{S}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s})$$

$$+ p(s)\left[\mathcal{S}'(s)\mathcal{C}(s) - \mathcal{C}'(s)\mathcal{S}(s)\right] \quad [..] = 1$$

Wronskian Symmetry

$$= p(s) \;+\; \mathcal{S}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s})$$

Apply these results in the *Driven Hill's Equation*:

Definition of Principal Orbit Functions

$$x'' + \kappa(s)x = p(s) + \cancelto{0}{[\mathcal{S}'' + \kappa\mathcal{S}]} \int_{s_i}^{s} d\tilde{s}\, \mathcal{C}(\tilde{s})p(\tilde{s}) - \cancelto{0}{[\mathcal{C}'' + \kappa\mathcal{C}]} \int_{s_i}^{s} d\tilde{s}\, \mathcal{S}(\tilde{s})p(\tilde{s})$$

$$= p(s)$$

Thereby proving we have a valid particular solution. The general solution to the *Driven Hill's Equation* is then:

- Choose constants $C_1,\ C_2$ consistent with particle initial conditions at $s = s_i$

$$x(s) = x(s_i)\mathcal{C}(s|s_i) + x'(s_i)\mathcal{S}(s|s_i) + \int_{s_i}^{s} d\tilde{s}\, G(s,\tilde{s})p(\tilde{s})$$

$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$