S1: Particle Equations of Motion

S1A: Introduction: The Lorentz Force Equation

The *Lorentz force equation* of a charged particle is given by (MKS Units):

$$\frac{d}{dt}\mathbf{p}_i(t) = q_i\left[\mathbf{E}(\mathbf{x}_i, t) + \mathbf{v}_i(t)\times\mathbf{B}(\mathbf{x}_i, t)\right]$$

$m_i, \quad q_i$ .... particle mass, charge

$\mathbf{x}_i(t)$ .... particle coordinate

$\mathbf{p}_i(t) = m_i\gamma_i(t)\mathbf{v}_i(t)$ .... particle momentum

$\mathbf{v}_i(t) = \frac{d}{dt}\mathbf{x}_i(t) = c\vec{\beta}_i(t)$ .... particle velocity

$\gamma_i(t) = \frac{1}{\sqrt{1-\beta_i^2(t)}}$ .... particle gamma factor

$i =$ particle index

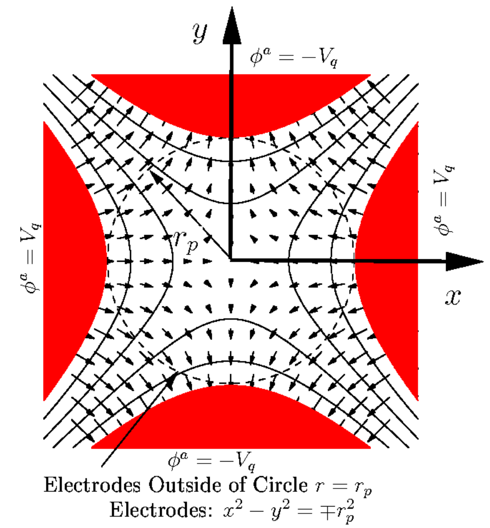
$t =$ time

	Total		Applied		Self
Electric Field:	$\mathbf{E}(\mathbf{x}, t)$	$=$	$\mathbf{E}^a(\mathbf{x}, t)$	$+$	$\mathbf{E}^s(\mathbf{x}, t)$
Magnetic Field:	$\mathbf{B}(\mathbf{x}, t)$	$=$	$\mathbf{B}^a(\mathbf{x}, t)$	$+$	$\mathbf{B}^s(\mathbf{x}, t)$

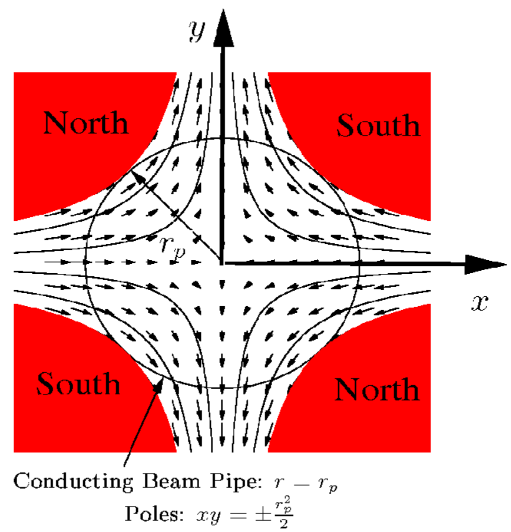
S1B: Applied Fields used to Focus, Bend, and Accelerate Beam

Transverse optics for focusing:

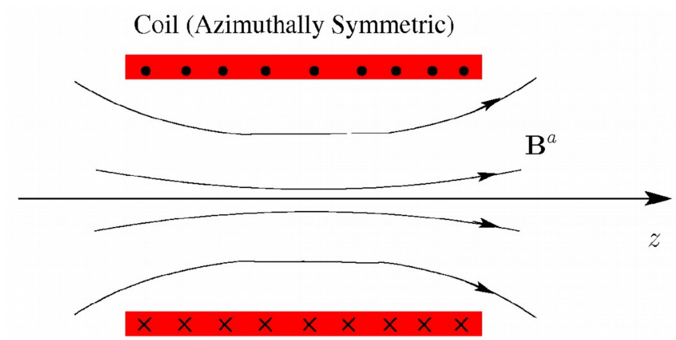
Electric Quadrupole



Magnetic Quadrupole

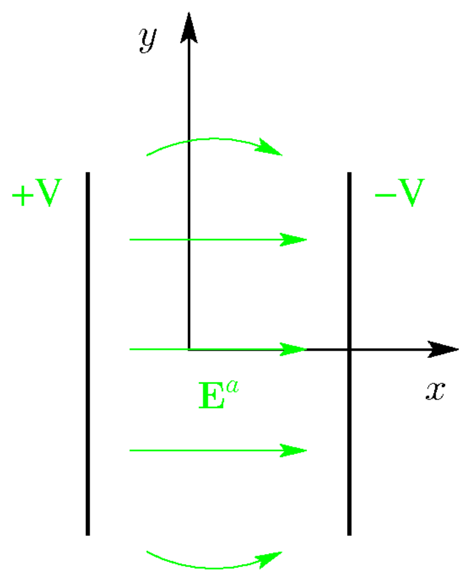


Solenoid

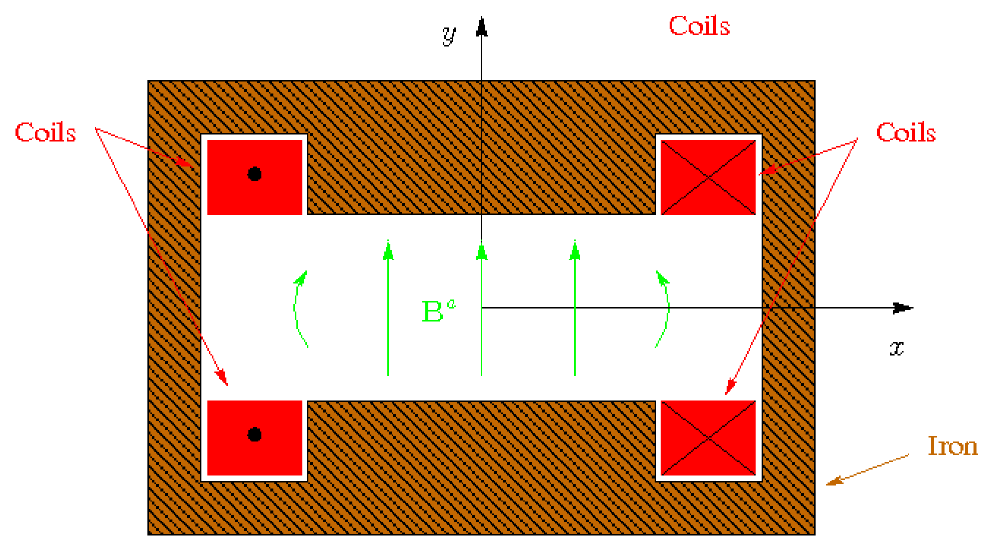


Dipole Bends:

Electric *x*-direction bend

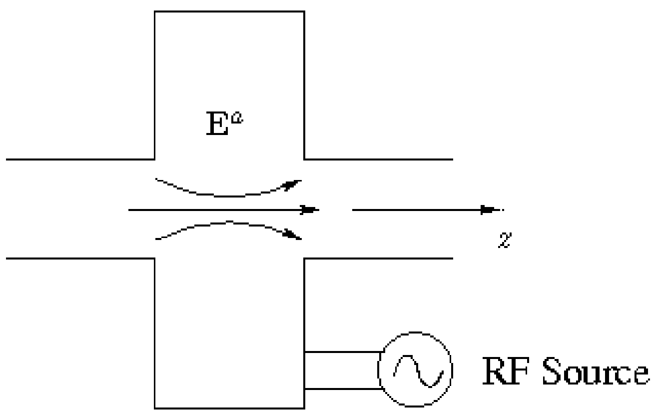


Magnetic *x*-direction bend

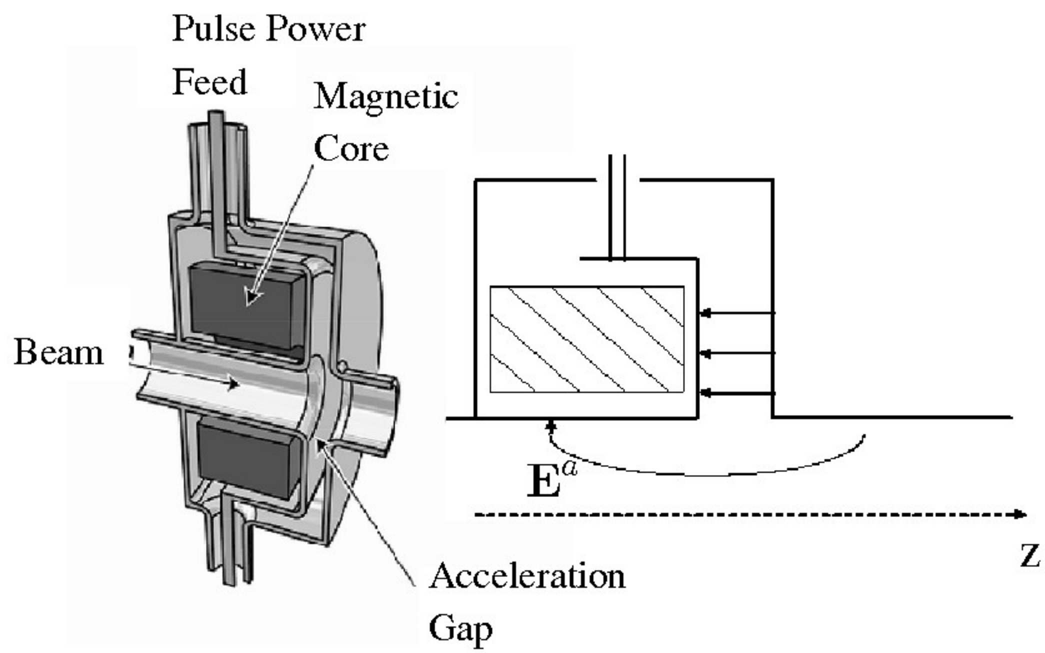


Longitudinal Acceleration:

RF Cavity



Induction Cell

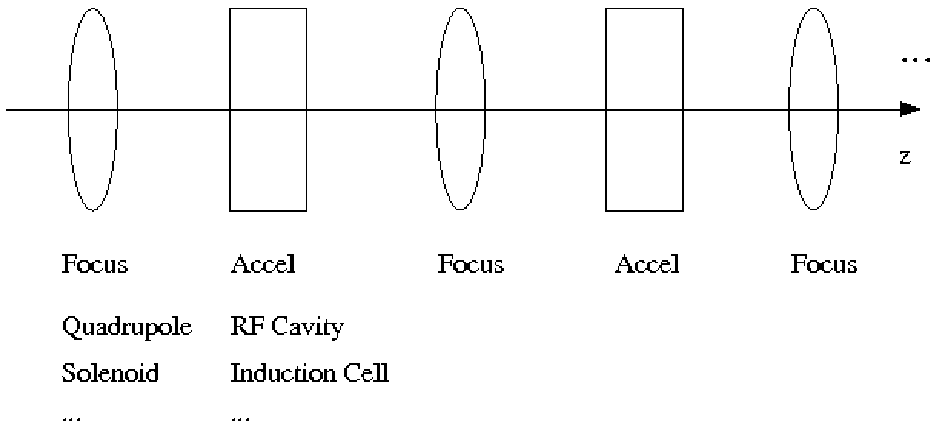


We will cover primarily transverse dynamics. Lectures by J.J. Barnard will cover acceleration and longitudinal physics:

- Acceleration influences transverse dynamics – not possible to fully decouple

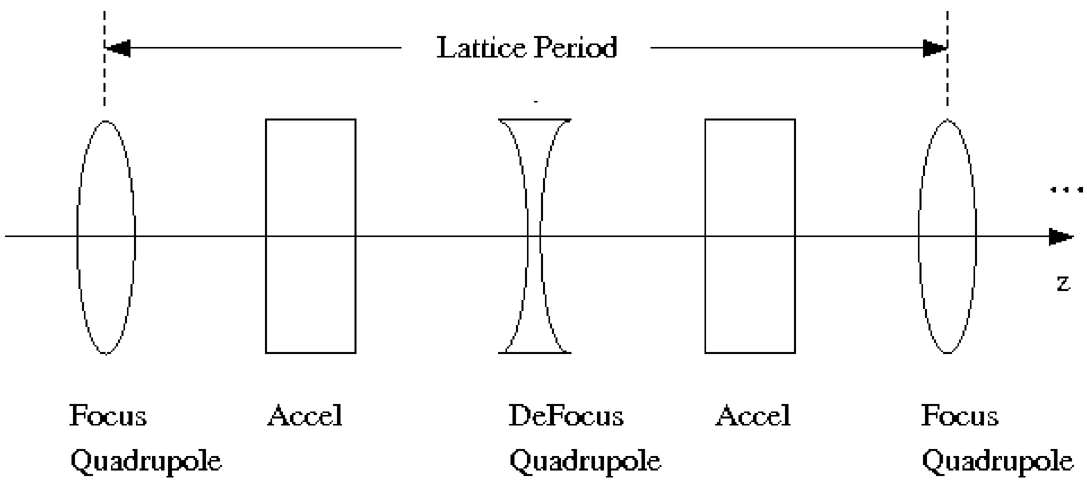
S1C: Machine Lattice

Applied field structures are often arraigned in a regular (periodic) lattice for beam transport/acceleration:

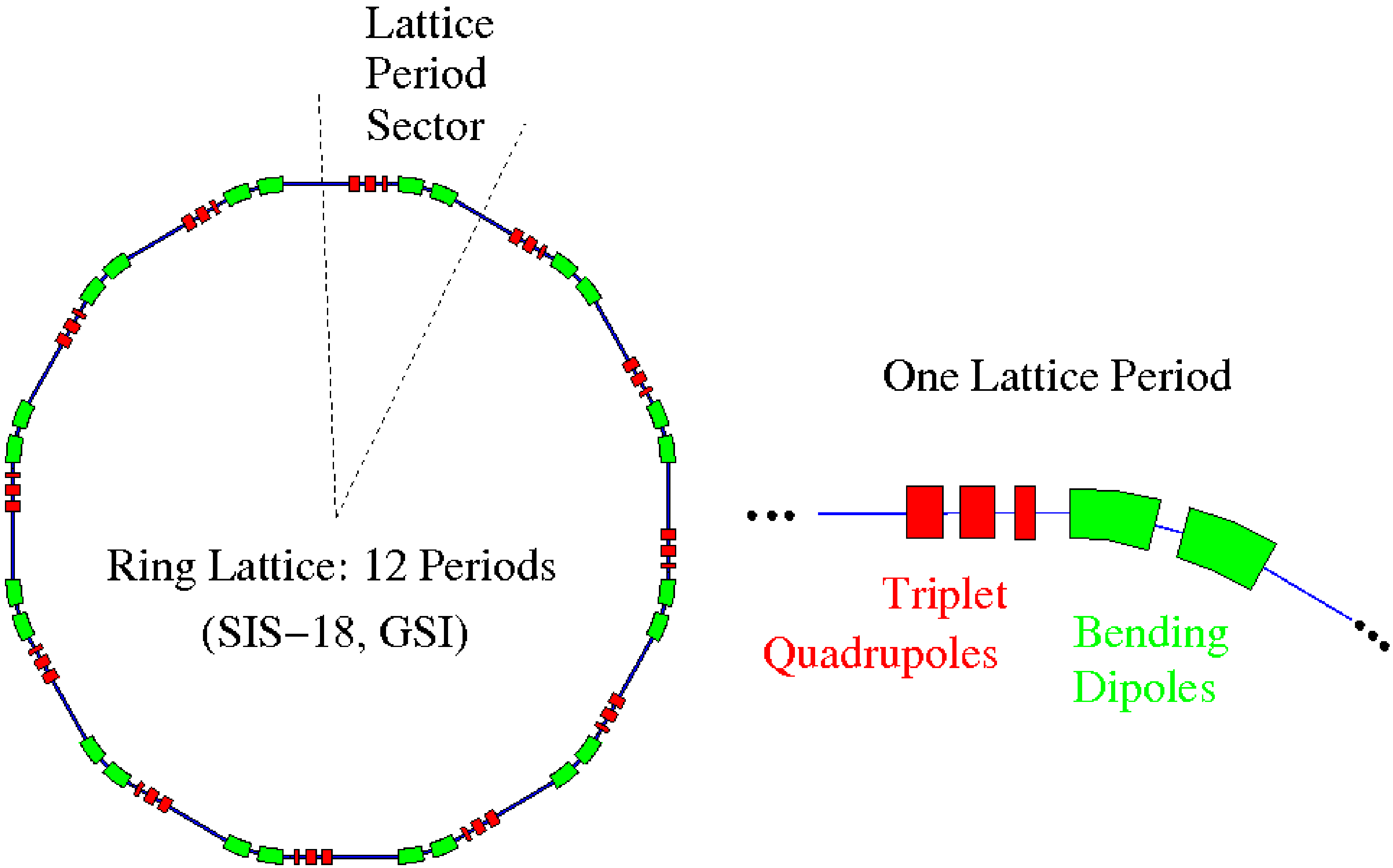


- Sometimes functions like bending/focusing are combined into a single element

Example – Linear FODO lattice (symmetric quadrupole doublet)



Lattices for rings and some beam insertion/extraction sections also incorporate bends and more complicated periodic structures:



- Elements to insert beam into and out of ring further complicate lattice
- Acceleration cells also present
  (typically several RF cavities at one or more location)

S1D: Self fields

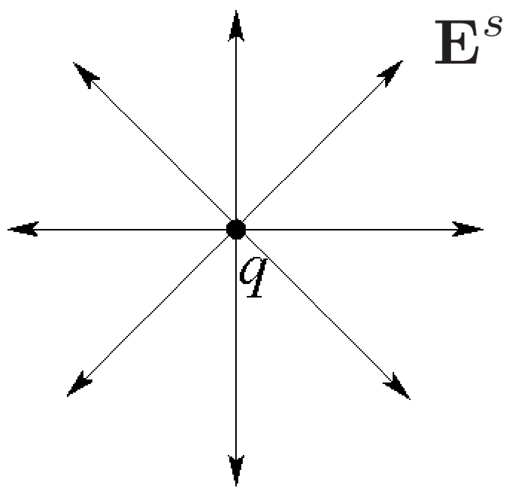
Self-fields are generated by the distribution of beam particles:

Charges

Currents

Particle at Rest

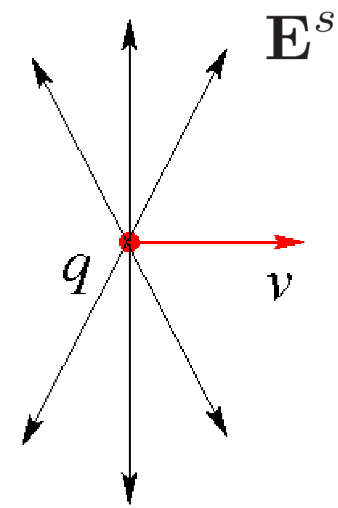
(pure electrostatic)

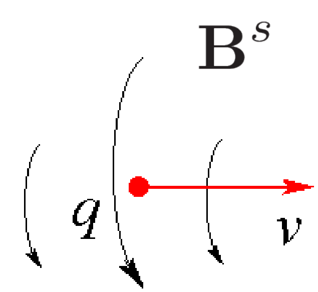


$\mathbf{B}^s = 0$

Particle in Motion

Obtain from Lorentz boost of rest-frame field: see Jackson, *Classical Electrodynamics*





- Superimpose for all particles in the beam distribution
- Accelerating particles also radiate
  - We neglect electromagnetic radiation in this class (see: J.J. Barnard, Intro Lectures)

The electric ($\mathbf{E}^a$) and magnetic ($\mathbf{B}^a$) fields satisfy the Maxwell Equations. The linear structure of the Maxwell equations can be exploited to resolve the field into Applied and Self-Field components:

$$\mathbf{E} = \mathbf{E}^a + \mathbf{E}^s$$

$$\mathbf{B} = \mathbf{B}^a + \mathbf{B}^s$$

Applied Fields (often quasi-static $\partial/\partial t \simeq 0$) $\mathbf{E}^a, \ \mathbf{B}^a$

Generated by elements in lattice

$$\nabla \cdot \mathbf{E}^a = \frac{\rho^a}{\epsilon_0} \qquad \nabla \times \mathbf{B}^a = \mu_0 \mathbf{J}^a + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^a$$

$$\nabla \times \mathbf{E}^a = -\frac{\partial}{\partial t}\mathbf{B}^a \qquad \nabla \cdot \mathbf{B}^a = 0$$

ρ^a = applied charge density

$\mathbf{J}^a$ = applied current density

$$\frac{1}{\mu_0 \epsilon_0} = c^2$$

+ Boundary Conditions on $\mathbf{E}^a$ and $\mathbf{B}^a$

- Boundary conditions depend on the total fields $\mathbf{E}, \ \mathbf{B}$ and if separated into Applied and Self-Field components, care can be required
- System often solved as static boundary value problem and source free in the vacuum transport region of the beam

/// Aside: Notation:

$$\nabla \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

- Cartesian Representation

$$= \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

- Cylindrical Representation

$$x = r\cos\theta \qquad \hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta$$

$$y = r\sin\theta \qquad \hat{\theta} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta$$

$$= \frac{\partial}{\partial \mathbf{x}}$$

- Abbreviated Representation

$$= \frac{\partial}{\partial \mathbf{x}_\perp} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

- Resolved Abbreviated Representation

Resolved into Perpendicular $(\perp)$ and Parallel (z) components

$$\mathbf{x} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$$
$$= \mathbf{x}_\perp + \hat{\mathbf{z}}z$$

$$\mathbf{x}_\perp \equiv \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$$

In integrals, we denote:

$$\int d^3x \; \cdots = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; \cdots = \int d^2x_\perp \int_{-\infty}^{\infty} dz \; \cdots$$

$$\int d^2x_\perp \; \cdots = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \cdots = \int_0^{\infty} dr\, r \int_{-\pi}^{\pi} d\theta \; \cdots$$

///

Self-Fields (dynamic, evolve with beam)

Generated by particle of the beam rather than (applied) sources outside beam

$$\nabla \cdot \mathbf{E}^s = \frac{\rho^s}{\epsilon_0} \qquad \nabla \times \mathbf{B}^s = \mu_0 \mathbf{J}^s + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^s$$

$$\nabla \times \mathbf{E}^s = -\frac{\partial}{\partial t}\mathbf{B}^s \qquad \nabla \cdot \mathbf{B}^s = 0$$

$$\rho^s = \text{beam charge density}$$
$$= \sum_{i=1}^{N} q_i \delta[\mathbf{x} - \mathbf{x}_i(t)]$$

$$\mathbf{J}^s = \text{beam current density}$$
$$= \sum_{i=1}^{N} q_i \mathbf{v}_i(t) \delta[\mathbf{x} - \mathbf{x}_i(t)]$$

i = particle index (N particles)

q_i = particle charge

$\mathbf{x}_i$ = particle coordinate

$\mathbf{v}_i$ = particle velocity

$$\delta(\mathbf{x}) \equiv \delta(x)\delta(y)\delta(z)$$

$\delta(x) \equiv$ Dirac-delta function

$\sum_{i=1}^{N} \cdots$ = sum over beam particles

+ Boundary Conditions on $\mathbf{E}^s$ and $\mathbf{B}^s$ from material structures, radiation conditions, etc.

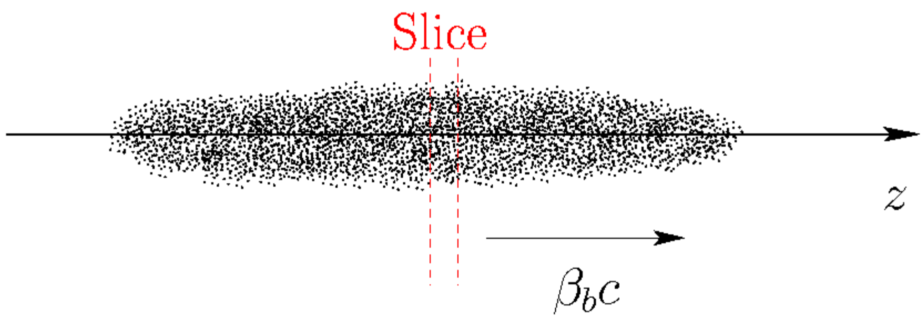
In accelerators, typically there is ideally a single species of particle:

$$q_i \rightarrow q$$
$$m_i \rightarrow m$$

Large Simplification!

Multi-species results in more complex collective effects

Motion of particles within axial slices of the "bunch" are highly directed:



$$\beta_b(z)c \equiv \frac{1}{N'} \sum_{i=1}^{N'} \mathbf{v}_i \cdot \hat{\mathbf{z}}$$

$=$ Mean axial velocity of N' particles in beam slice

$$\frac{d}{dt}\mathbf{x}_i(t) = \mathbf{v}_i(t) = \hat{\mathbf{z}}\beta_b(z)c + \delta\mathbf{v}_i$$

$$|\delta\mathbf{v}_i| \ll |\beta_b|c$$

Paraxial Approximation

There are typically many particles: (see S13, Vlasov Models for more details)

$$\rho^s = \sum_{i=1}^{N} q_i \delta[\mathbf{x} - \mathbf{x}_i(t)]$$

$\simeq \rho(\mathbf{x}, t)$ continuous charge-density

$$\mathbf{J}^s = \sum_{i=1}^{N} q_i \mathbf{v}_i(t) \delta[\mathbf{x} - \mathbf{x}_i(t)]$$

$\simeq \beta_b c \rho(\mathbf{x}, t)\hat{\mathbf{z}}$ continuous axial current-density

The beam evolution is typically sufficiently slow (for heavy ions) where we can neglect radiation and approximate the self-field Maxwell Equations as:

See: J. J. Barnard, Intro. Lectures: Electrostatic Approximation

$$\mathbf{E}^s = -\nabla\phi$$

$$\mathbf{B}^s = \nabla \times \mathbf{A} \qquad \mathbf{A} = \hat{\mathbf{z}}\frac{\beta_b}{c}\phi$$

$$\nabla^2\phi = \frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{x}}\phi = -\frac{\rho^s}{\epsilon_0}$$

$$+ \text{ Boundary Conditions on } \phi$$

Vast Reduction of self-field model:

Approximation equiv to electrostatic interactions in frame moving with beam

But still complicated!

Resolve the Lorentz force acting on beam particles into Applied and Self-Field terms:

$$\mathbf{F}_i(\mathbf{x}_i, t) = q\mathbf{E}(\mathbf{x}_i, t) + q\mathbf{v}_i(t) \times \mathbf{B}(\mathbf{x}_i, t)$$

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{F}_i^s$$

$$\mathbf{E} = \mathbf{E}^a + \mathbf{E}^s$$

$$\mathbf{B} = \mathbf{B}^a + \mathbf{B}^s$$

Applied:

$$\mathbf{F}_i^a = q\mathbf{E}_i^a + q\mathbf{v}_i \times \mathbf{B}_i^a$$

$$\mathbf{E}^a(\mathbf{x}_i, t) \equiv \mathbf{E}_i^a \quad \text{etc.}$$

Self-Field:

$$\mathbf{F}_i^s = q\mathbf{E}_i^s + q\mathbf{v}_i \times \mathbf{B}_i^s$$

The self-field force can be simplified:

- See also: J.J. Barnard, Intro. Lectures

Plug in self-field forms:

$$\mathbf{F}_i^s = q\mathbf{E}_i^s + q\mathbf{v}_i \times \mathbf{B}_i^s$$

$$\simeq q\left[-\left.\frac{\partial \phi}{\partial \mathbf{x}}\right|_i + (\beta_b c\hat{\mathbf{z}} + \delta\mathbf{v}_i) \times \left.\left(\frac{\partial}{\partial \mathbf{x}} \times \hat{\mathbf{z}}\frac{\beta_b}{c}\phi\right)\right|_i \right]$$

~0 Neglect: Paraxial (for $\delta\mathbf{v}_i$)

$$\cdots\Big|_i \equiv \cdots\Big|_{\mathbf{x}=\mathbf{x}_i}$$

Resolve into transverse (x and y) and longitudinal (z) components and simplify:

$$\beta_b c\hat{\mathbf{z}} \times \left.\left(\frac{\partial}{\partial \mathbf{x}} \times \hat{\mathbf{z}}\frac{\beta_b}{c}\phi\right)\right|_i = \beta_b^2 \hat{\mathbf{z}} \times \left.\left(\frac{\partial}{\partial \mathbf{x}_\perp} \times \hat{\mathbf{z}}\phi\right)\right|_i$$

$$= \beta_b^2 \hat{\mathbf{z}} \times \left.\left(\frac{\partial \phi}{\partial y}\hat{\mathbf{x}} - \frac{\partial \phi}{\partial x}\hat{\mathbf{y}}\right)\right|_i$$

$$= \beta_b^2 \left.\left(\frac{\partial \phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{y}}\right)\right|_i$$

$$= \beta_b^2 \left.\frac{\partial \phi}{\partial \mathbf{x}_\perp}\right|_i$$

also

$$-\left.\frac{\partial \phi}{\partial \mathbf{x}}\right|_i = -\left.\frac{\partial \phi}{\partial \mathbf{x}_\perp}\right|_i - \left.\frac{\partial \phi}{\partial \mathbf{z}}\right|_i \hat{\mathbf{z}}$$

Together, these results give:

$$\mathbf{F}_i^s = -\frac{q}{\gamma_b^2}\left.\frac{\partial \phi}{\partial \mathbf{x}_\perp}\right|_i - \hat{\mathbf{z}} q \left.\frac{\partial \phi}{\partial z}\right|_i$$

Transverse Longitudinal

$$\gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}}$$

Axial relativistic gamma of beam

- Transverse and longitudinal forces have different axial gamma factors
- $1/\gamma_b^2$ factor in transverse force shows the space-charge forces become weaker as axial beam kinetic energy increases
  - Most important in low energy (nonrelativistic) beam transport
  - Strong in/near injectors before much acceleration

/// Aside: Singular Self Fields

In *free space*, the beam potential generated from the singular charge density:

$$\rho^s = \sum_{i=1}^{N} q_i \delta[\mathbf{x} - \mathbf{x}_i(t)]$$

is

$$\phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{1}{|\mathbf{x} - \mathbf{x}_i|}$$

Thus, the force of a particle at $\mathbf{x} = \mathbf{x}_i$ is:

$$\mathbf{F}_i = -q \left. \frac{\partial \phi}{\partial \mathbf{x}} \right|_i = \frac{q^2}{4\pi\epsilon_0} \sum_{j=1}^{N} \frac{(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^{3/2}}$$

Which diverges due to the $i = j$ term. This divergence is essentially "erased" when the continuous charge density is applied:

$$\rho^s = \sum_{i=1}^{N} q_i \delta[\mathbf{x} - \mathbf{x}_i(t)] \quad \longrightarrow \quad \rho(\mathbf{x}, t)$$

- Effectively removes effect of collisions
  See: J.J. Barnard, Intro Lectures for more details
  - Find collisionless Vlasov model of evolution is often adequate ///

The particle equations of motion in $\mathbf{x}_i - \mathbf{v}_i$ phase-space variables become:

- Separate parts of $q\mathbf{E}_i^a + q\mathbf{v}_i \times \mathbf{B}_i^a$ into transverse and longitudinal comp

Transverse

$$\frac{d}{dt}\mathbf{x}_{\perp i} = \mathbf{v}_{\perp i}$$

$$\frac{d}{dt}(m\gamma_i \mathbf{v}_{\perp i}) \simeq \underbrace{q\mathbf{E}_{\perp i}^a + q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}_{\perp i}^a + qB_{zi}^a \mathbf{v}_{\perp i} \times \hat{\mathbf{z}}}_{\text{Applied}} \underbrace{- q\frac{1}{\gamma_b^2}\frac{\partial \phi}{\partial \mathbf{x}_\perp}\bigg|_i}_{\text{Self}}$$

Longitudinal

$$\frac{d}{dt}z_i = v_{zi}$$

$$\frac{d}{dt}(m\gamma_i v_{zi}) \simeq \underbrace{qE_{zi}^a - q(v_{xi}B_{yi}^a - v_{yi}B_{xi}^a)}_{\text{Applied}} \underbrace{- q\frac{\partial \phi}{\partial z}\bigg|_i}_{\text{Self}}$$

In the remainder of this (and most other) lectures, we analyze Transverse Dynamics. Longitudinal Dynamics will be covered in J.J. Barnard lectures

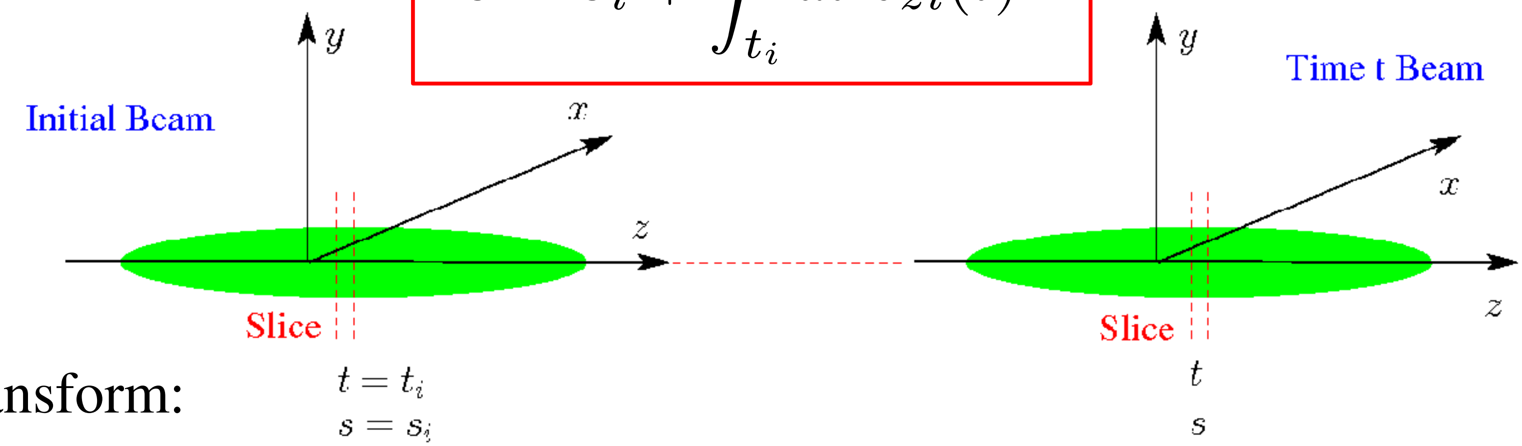
- Except near injector, acceleration is typically slow
  - Fractional change in γ_b, β_b small over characteristic transverse dynamical scales such as lattice period and betatron oscillation periods
- Regard γ_b, β_b as specified functions given by the "acceleration schedule"

S1E: Equations of Motion in *s* and the Paraxial Approximation

In transverse accelerator dynamics, it is convenient to employ the axial coordinate (s) of a particle in the accelerator as the independent variable:

- Need fields at lattice location of particle to integrate equations for particle trajectories

$$s \equiv s_i + \int_{t_i}^{t} d\tilde{t}\; v_{zi}(\tilde{t})$$



Transform:

$$v_{zi} = \frac{ds}{dt} \implies v_{xi} = \frac{dx_i}{dt} = \frac{ds}{dt}\frac{dx_i}{ds} = v_{zi}\frac{dx_i}{ds} = (\beta_b c + \delta v_{zi})\frac{dx_i}{ds}$$

Neglect

$$\simeq \beta_b c \frac{dx_i}{ds}$$

Denote:

$$' \equiv \frac{d}{ds} \qquad v_{xi} = \frac{dx_i}{dt} \simeq \beta_b c x_i' \qquad v_{yi} = \frac{dy_i}{dt} \simeq \beta_b c y_i'$$

Neglecting term consistent with assumption of small longitudinal momentum spread (paraxial approximation)

- Procedure becomes more complicated when bends present: see S1H

In the paraxial approximation, x' and y' can be interpreted as the (small magnitude) angles that the particles make with the longitudinal-axis:

$$x - \text{angle} = \frac{v_{xi}}{v_{zi}} \simeq \frac{v_{xi}}{\beta_b c} = x'_i$$

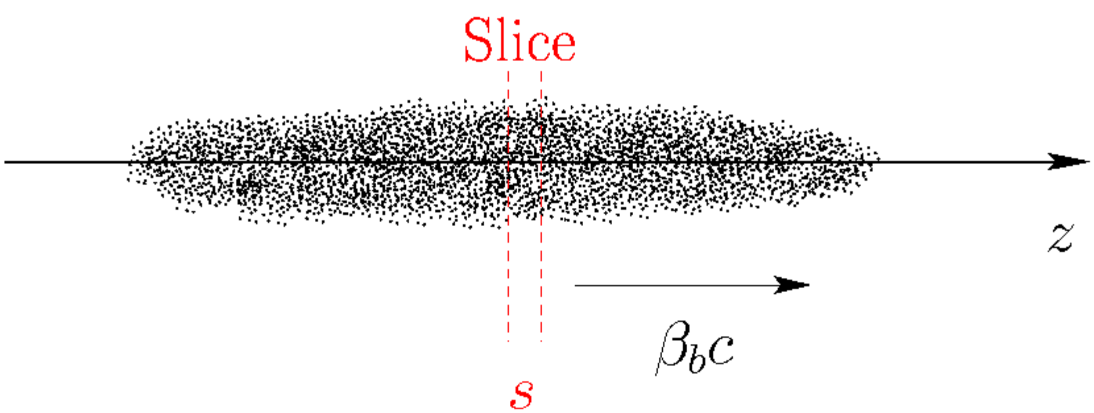
$$y - \text{angle} = \frac{v_{yi}}{v_{zi}} \simeq \frac{v_{yi}}{\beta_b c} = y'_i$$

Typical accel lattice values:
$|x'| < 50$ mrad

The angles will be *small* in the paraxial approximation:

$$v_{xi}^2,\ v_{yi}^2 \ll \beta_b^2 c^2 \quad \Longrightarrow \quad x_i'^2,\ y_i'^2 \ll 1$$

Since the spread of axial momentum/velocities is small in the paraxial approximation, a thin axial slice of the beam maps to a thin axial slice and s can also be thought of as the axial coordinate of the slice in the accelerator lattice



$$\beta_b \equiv \sum_{i=1}^{N'} \frac{v_{zi}}{c}$$

slice

$$s \simeq s_i + \int_{t_i}^{t} d\tilde{t}\ \beta_b(\tilde{t})$$

$$s \simeq s_i + \int_{t_i}^{t} d\tilde{t}\ \beta_b(\tilde{t})$$

The coordinate *s* can alternatively be interpreted as the axial coordinate of a reference (design) particle moving in the lattice

- Design particle has no momentum spread

It is often desirable to express the particle equations of motion in terms of *s* rather than the time *t*

- Makes it clear where you are in the lattice of the machine
- Sometimes easier to use t in codes when including many effects to high order

Transform transverse particle equations of motion to s rather than t derivatives

$$\frac{d}{dt}(m\gamma_i \mathbf{v}_{\perp i}) \simeq q\mathbf{E}^a_{\perp i} + q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}^a_{\perp i} + qB^a_{zi}\mathbf{v}_{\perp i} \times \hat{\mathbf{z}} - q\frac{1}{\gamma_b^2}\frac{\partial \phi}{\mathbf{x}_\perp}\bigg|_i$$

Term 1 Term 2

Transform Terms 1 and 2 in the particle equation of motion:

$$\frac{d}{dt} = v_{zi}\frac{d}{ds}$$

Term 1:
$$\frac{d}{dt}\left(m\gamma_i \frac{d\mathbf{x}_{\perp i}}{dt}\right) = mv_{zi}\frac{d}{ds}\left(\gamma_i v_{zi}\frac{d}{ds}\mathbf{x}_{\perp i}\right)$$
$$= m\gamma_i v_{zi}^2 \frac{d^2}{ds^2}\mathbf{x}_{\perp i} + mv_{zi}\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_i v_{zi})$$

Term 1A Term 1B

Approximate:

Term 1A:
$$m\gamma_i v_{zi}^2 \frac{d^2}{ds^2}\mathbf{x}_{\perp i} \simeq m\gamma_b\beta_b^2 c^2 \frac{d^2}{ds^2}\mathbf{x}_{\perp i} = m\gamma_b\beta_b^2 c^2 \mathbf{x}''_{\perp i}$$

Term 1B:
$$mv_{zi}\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_i v_{zi}) \simeq m\beta_b c\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_b\beta_b c)$$
$$\simeq m\beta_b c^2(\gamma_b\beta_b)'\mathbf{x}'_{\perp i}$$

Using the approximations 1A and 1B gives for Term 1:

$$m\frac{d}{dt}\left(\gamma_i \frac{d\mathbf{x}_{\perp i}}{dt}\right) \simeq m\gamma_b\beta_b^2c^2\left[\mathbf{x}''_{\perp i} + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_{\perp i}\right]$$

Similarly we approximate in Term 2:

$$qB^a_{zi}\mathbf{v}_{\perp i} \times \hat{\mathbf{z}} \simeq qB^a_{zi}\beta_b c\mathbf{x}'_{\perp i} \times \hat{\mathbf{z}}$$

Using the simplified expressions for Terms 1 and 2 obtain the reduced transverse equation of motion:

$$\mathbf{x}''_{\perp i} + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_{\perp i} = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}^a_{\perp i} + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}} \times \mathbf{B}^a_{\perp i}$$
$$+ \frac{qB^a_{zi}}{m\gamma_b\beta_b c}\mathbf{x}'_{\perp i} \times \hat{\mathbf{z}} - \frac{q}{m\gamma_b^3\beta_b^2c^2}\left.\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right|_i$$

- Will be analyzed extensively in lectures that follow in various limits to better understand solution properties

S1F: Axial Particle Kinetic Energy

Relativistic particle kinetic energy is:

$$\gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}$$

$$\mathcal{E} = (\gamma - 1)mc^2$$

$$\mathbf{v} = (\beta_b + \delta\beta_z)c\hat{\mathbf{z}} + \beta_\perp c\hat{\mathbf{x}}_\perp$$
$$= \text{Particle Velocity (3D)}$$

For a directed paraxial beam with motion primarily along the machine axis the kinetic energy is essentially the axial kinetic energy $\mathcal{E}_b$:

$$\mathcal{E} = (\gamma_b - 1)mc^2 \; + \; \Theta\left(\frac{|\delta\beta_z|}{\beta_b}, \frac{\beta_\perp^2}{\beta_b^2}\right)$$

$$\mathcal{E} \simeq \mathcal{E}_b \equiv (\gamma_b - 1)mc^2$$

In nonrelativistic limit: $\beta_b^2 \ll 1$

$$\mathcal{E}_b \equiv (\gamma_b - 1)mc^2 = \frac{1}{2}m\beta_b^2c^2 \; + \; \frac{3}{8}m\beta_b^4c^2 \; + \; \cdots$$
$$\simeq \frac{1}{2}m\beta_b^2c^2 \; + \; \Theta(\beta_b^4)$$

Convenient units:

Electrons:

$$m = m_e = 511 \; \frac{\text{keV}}{c^2}$$

Electrons rapidly relativistic due to relatively low mass

Ions/Protons:

$$m = (\text{atomic mass}) \cdot m_u$$

$$m_u \equiv \text{Atomic Mass Unit} = 931.49 \; \frac{\text{MeV}}{c^2}$$

Note:

$$m_p = \text{Proton Mass} = 938.27 \; \frac{\text{MeV}}{c^2}$$

$$m_n = \text{Neutron Mass} = 939.57 \; \frac{\text{MeV}}{c^2}$$

$$m_p \simeq m_n \simeq 940 \frac{\text{MeV}}{c^2}$$

Approximate roughly for ions:

$$m \simeq A m_u \qquad A = \text{Mass Number (Number of Nucleons)}$$

$$m_u \gg m_e$$

Protons/ions take much longer to become relativistic than electrons

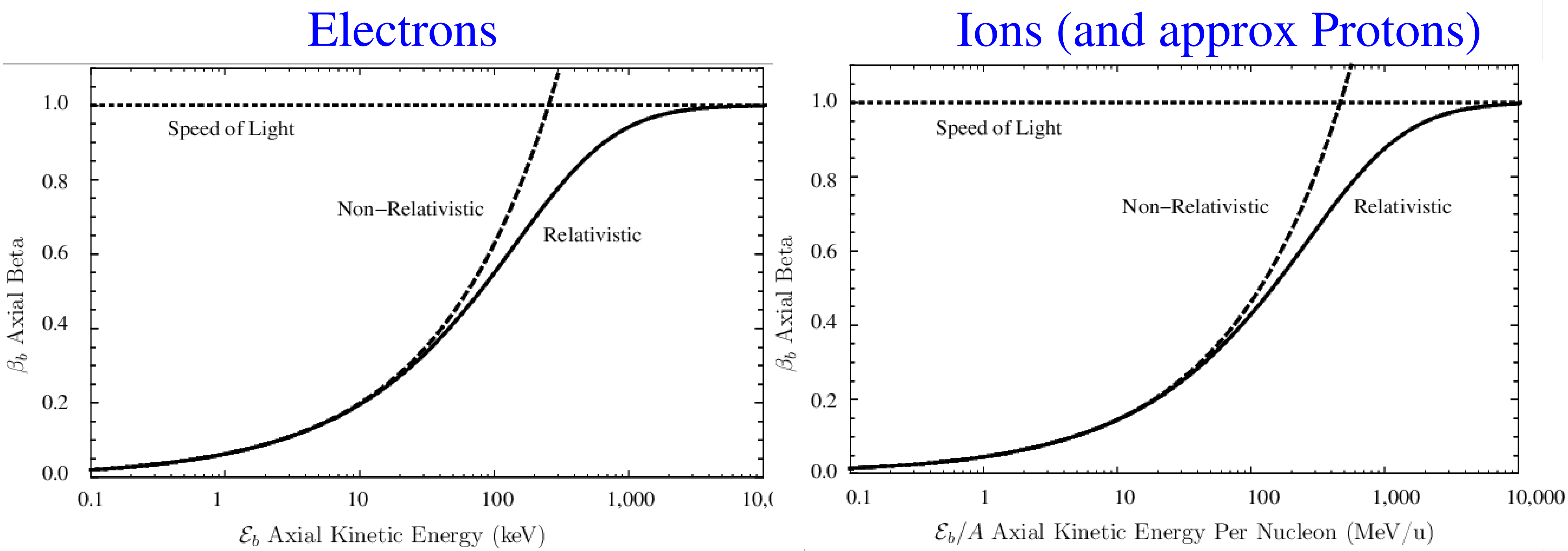
$m_p, m_n > m_u$ due to nuclear binding energy

$$\frac{\mathcal{E}_b/A}{m_u c^2} \simeq \gamma_b - 1 \qquad \Longrightarrow \qquad \gamma_b = 1 + \frac{\mathcal{E}_b/A}{m_u c^2}$$

$$\beta_b = \sqrt{1 - 1/\gamma_b^2}$$

Energy/Nucleon $\mathcal{E}_b/A$ fixes β_b to set phase needs of RF cavities

Contrast beam relativistic β_b for electrons and protons/ions:



Notes: 1) plots do not overlay, scale changed

2) Ion plot slightly off for protons since $m_u \neq m_p$

- Electrons become relativistic easier relative to protons/ions due to light mass
- Space-charge more important for ions than electrons (see Sec. S1D)
  - Low energy ions near injector expected to have strongest space-charge

S1G: Summary: Transverse Particle Equations of Motion

$$\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}^a_\perp + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}}\times\mathbf{B}^a_\perp + \frac{qB^a_z}{m\gamma_b\beta_b c}\mathbf{x}'_\perp \times \hat{\mathbf{z}}$$

$$- \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial \mathbf{x}_\perp}\phi$$

$$\mathbf{E}^a = \text{Applied Electric Field}$$

$$\mathbf{B}^a = \text{Applied Magnetic Field}$$

$$' \equiv \frac{d}{ds} \qquad \gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}}$$

$$\nabla^2\phi = \frac{\partial}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{x}}\phi = -\frac{\rho}{\epsilon_0}$$

$$+ \text{ Boundary Conditions on } \phi$$

Drop particle *i* subscripts (in most cases) henceforth to simplify notation

Neglects axial energy spread, bending, and electromagnetic radiation

$\gamma-$factors different in applied and self-field terms:

In-q$\frac{}{m\gamma_b^3\beta_b^2c^2\frac{\partial}{\partial \mathbf{x}}\phi}$, contributions to γ_b^3:

$\gamma_b \Longrightarrow$ Kinematics

$\gamma_b^2 \Longrightarrow$ Self-Magnetic Field Corrections (leading order)

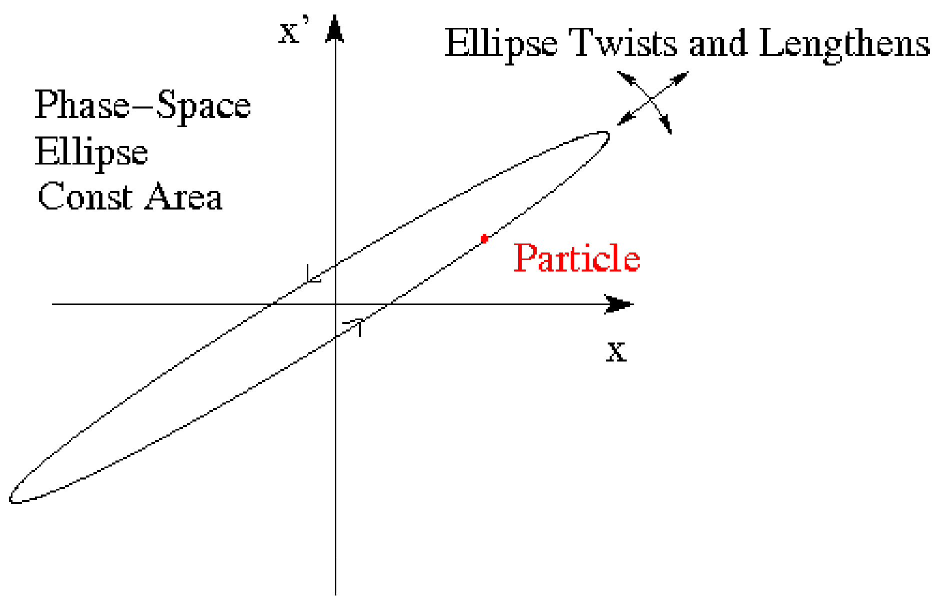
S1H: Preview: Analysis to Come

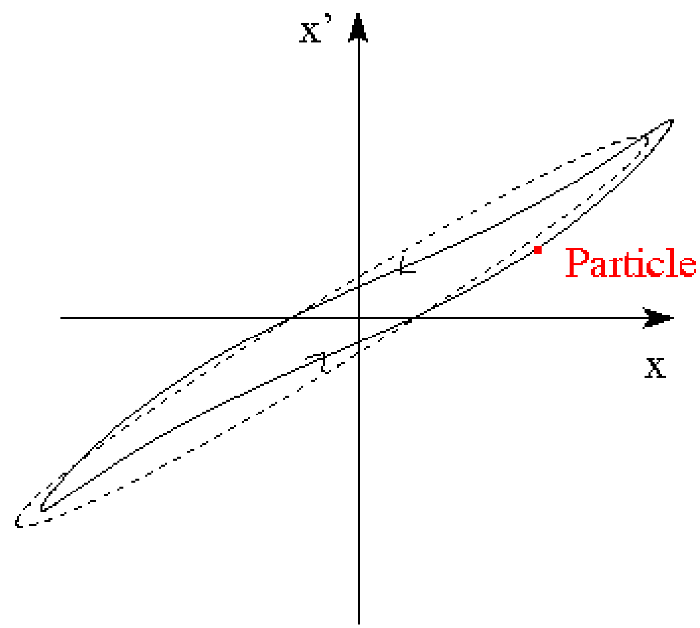
Much of transverse accelerator physics centers on understanding the evolution of beam particles in 4-dimensional *x*-*x*' and *y*-*y*' phase space.

Typically, restricted 2-dimensional phase-space projections in *x*-*x*' and/or *y*-*y*' are analyzed to simplify interpretations:

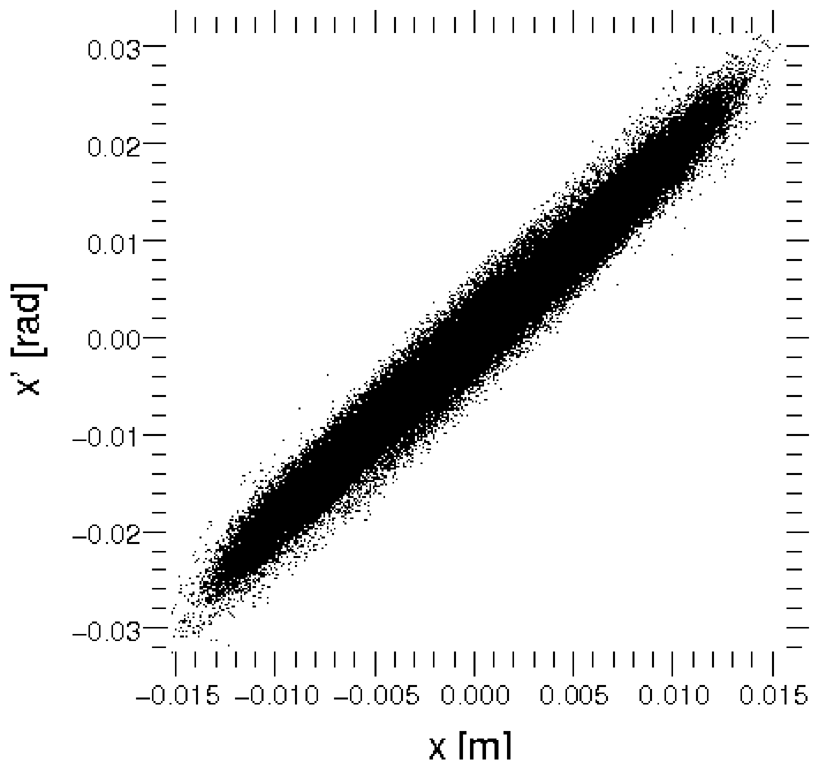
When forces are linear particles tend to move on ellipses of constant area
- Ellipse may elongate/shrink and rotate as beam evolves in lattice

Nonlinear force components distort orbits and cause undesirable effects
- Growth in effective phase-space area reduces focusability





The "effective" phase-space volume of a distribution of beam particles is of fundamental interest



Effective area measure in *x*-*x*' phase-space is the *x*-emittance

$$\text{Statistical "Area"} \sim \pi \varepsilon_x$$

$$\varepsilon_x = 4[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle x x' \rangle_\perp^2]^{1/2}$$

We will find in statistical beam descriptions that:

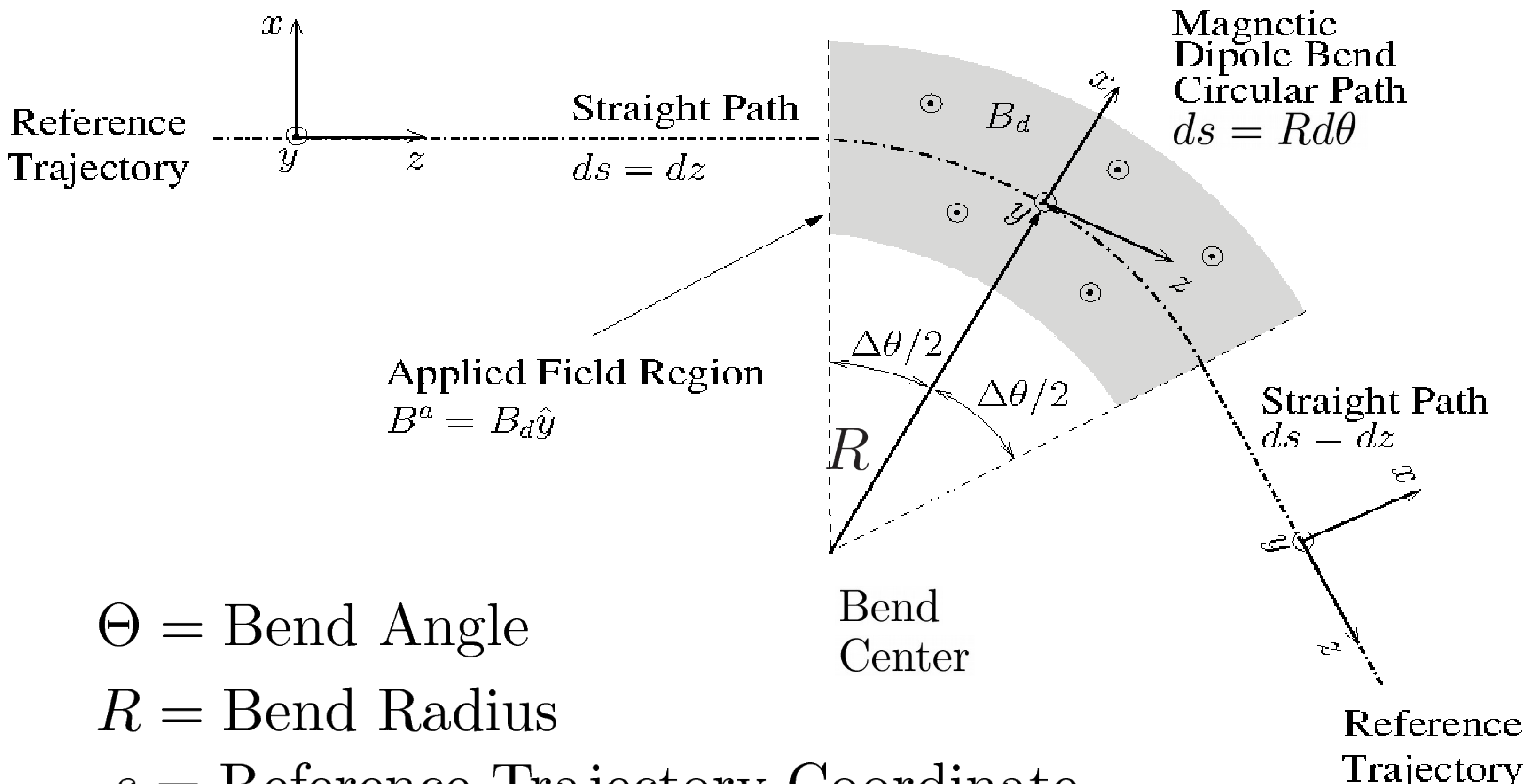
Larger/Smaller beam phase-space areas (Larger/Smaller emittances) $\Longleftrightarrow$ Harder/Easier to focus beam on <u>small</u> final spots

Much of advanced accelerator physics centers on preserving beam quality by understanding and controlling emittance growth due to nonlinear forces arising from both space-charge and the applied focusing. In the remainder of the next few lectures we will review the physics of a single particles moving in linear applied fields with emphasis on transverse effects. Later, we will generalize concepts to include forces from space-charge in this formulation and nonlinear effects from both applied and self-fields.

S1I: Bent Coordinate System and Particle Equations of Motion with Dipole Bends and Axial Momentum Spread

The previous equations of motion can be applied to dipole bends provided the *x,y,z* coordinate system is fixed. It can prove more convenient to employ coordinates that follow the beam in a bend.

- Orthogonal system employed called Frenet-Serret coordinates



Θ = Bend Angle

R = Bend Radius

s = Reference Trajectory Coordinate

In this perspective, dipoles are adjusted given the design momentum of the reference particle to bend the orbit through a radius *R*.

- Bends usually only in one plane (say *x*)
  - Implemented by a dipole applied field: E_x^a or B_y^a
- Easy to apply material analogously for *y*-plane bends, if necessary

Denote:

$$p_0 = m\gamma_b\beta_b c = \text{design momentum}$$

Then a magnetic *x*-bend through a radius *R* is specified by:

$$\mathbf{B}^a = B_y^a \hat{\mathbf{y}} = \text{const in bend}$$

$$\frac{1}{R} = \frac{qB_y^a}{p_0}$$

Analogous formula for Electric Bend will be derived in problem set

The particle rigidity is defined as ($[B\rho]$ read as one symbol called "B-Rho"):

$$[B\rho] \equiv \frac{p_0}{q} = \frac{m\gamma_b\beta_b c}{q}$$

is often applied to express the bend result as:

$$\frac{1}{R} = \frac{B_y^a}{[B\rho]}$$

Comments on bends:

- *R* can be positive or negative depending on sign of $B^a_y/[B\rho]$
- For straight sections, $R \rightarrow \infty$ $\quad$ (or equivalently, $B^a_y = 0$)
- Lattices often made from discrete element dipoles and straight sections with separated function optics
  - Bends can provide "edge focusing"
  - Sometimes elements for bending/focusing are combined
- For a ring, dipoles strengths are tuned with particle rigidity/momentum so the reference orbit makes a closed path lap through the circular machine
  - Dipoles adjusted as particles gain energy to maintain closed path
  - In a Synchrotron dipoles and focusing elements are adjusted together to maintain focusing and bending properties as the particles gain energy. This is the origin of the name "Synchrotron."
- Total bending strength of a ring in Tesla-meters limits the ultimately achievable particle energy/momentum in the ring

For a magnetic field over a path length S , the beam will be bent through an angle:

$$\Theta = \frac{S}{R} = \frac{SB_y^a}{[B\rho]}$$

To make a ring, the bends must deflect the beam through a total angle of 2π :

- Neglect any energy gain changing the rigidity over one lap

$$2\pi = \sum_{i,\mathrm{Dipoles}} \Theta_i = \sum_i \frac{S_i}{R_i} = \sum_i \frac{S_i B_{y,i}^a}{[B\rho]}$$

For a symmetric ring, N dipoles are all the same, giving for the bend field:

- Typically choose parameters for dipole field as high as technology allows for a compact ring

$$B_y^a = 2\pi \frac{[B\rho]}{NS}$$

For a symmetric ring of total circumference C with straight sections of length L between the bends:

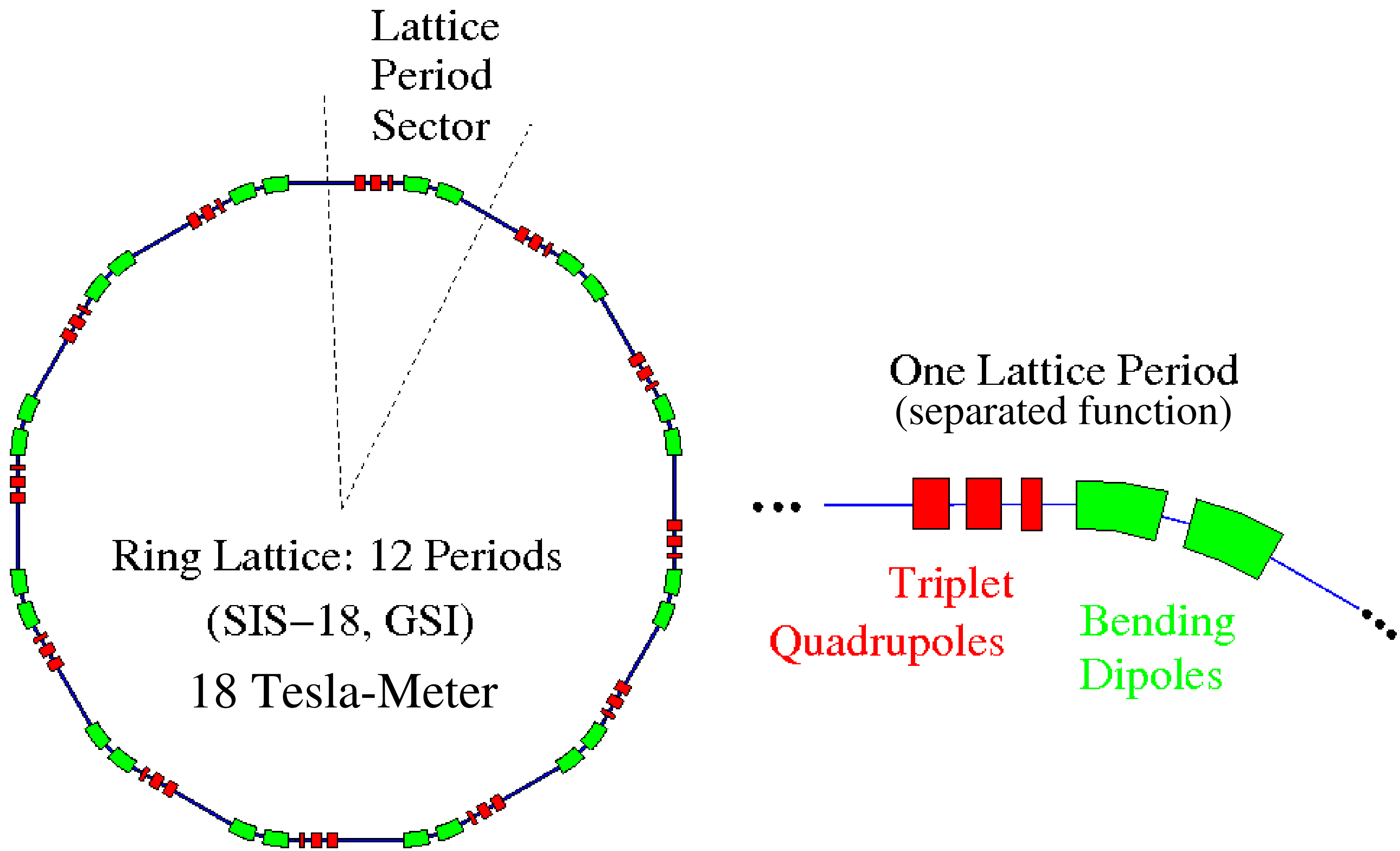
- Features of straight sections typically dictated by needs of focusing, acceleration, and dispersion control

$$C = NS + NL$$

Example: Typical separated function lattice in a Synchrotron

Focus Elements in Red

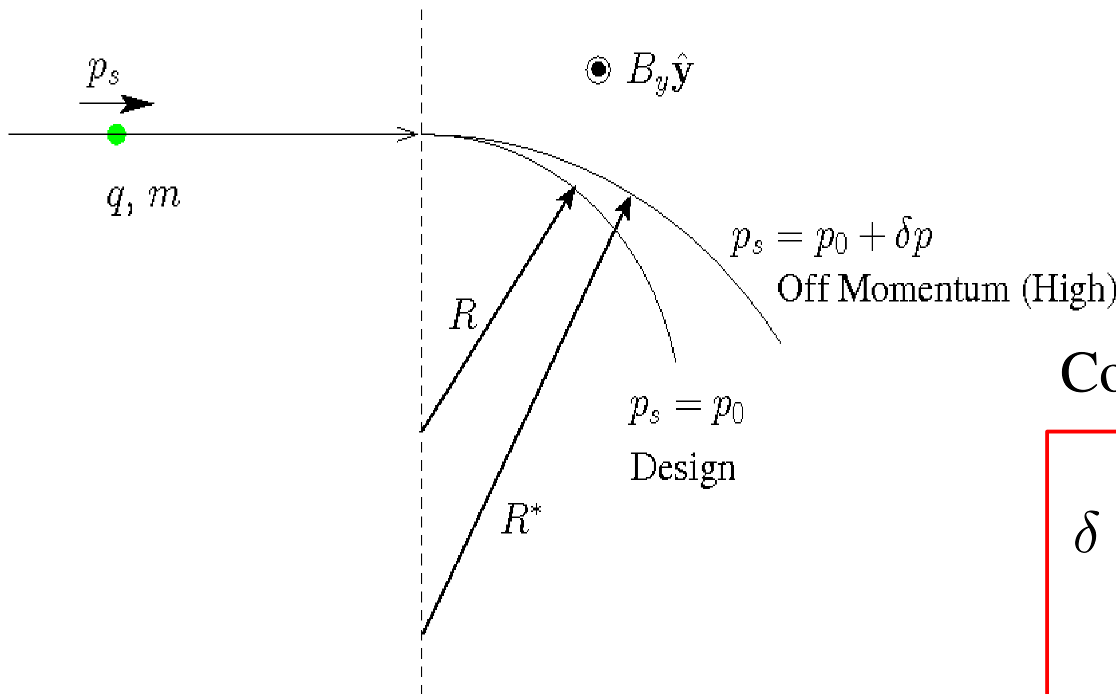
Bending Elements in Green



For "off-momentum" errors:

$$p_s = p_0 + \delta p$$
$$p_0 = m\gamma_b\beta_b c = \text{design momentum}$$
$$\delta p = \text{off- momentum}$$

This will modify the particle equations of motion, particularly in cases where there are bends since particles with different momenta will be bent at different radii



Common notation:

$$\delta \equiv \frac{\delta p}{p_0} = \text{Fractional Momentum Error}$$

- Not usual to have acceleration in bends
  - Dipole bends and quadrupole focusing are sometimes combined

Derivatives in accelerator Frenet-Serret Coordinates

Summarize results only needed to transform the Maxwell equations, write field derivatives, etc.

- Reference: Chao and Tigner, *Handbook of Accelerator Physics and Engineering*

$$\Psi(x,y,s) \qquad = \text{Scalar}$$

$$\mathbf{V}(x,y,s) = V_x(x,y,s)\hat{\mathbf{x}} + V_y(x,y,s)\hat{\mathbf{y}} + V_s(x,y,s)\hat{\mathbf{s}} \qquad = \text{Vector}$$

Vector Dot and Cross-Products: $(\mathbf{V}_1,\ \mathbf{V}_2\ \ \text{Two Vectors})$

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1s}V_{2s}$$

$$\mathbf{V}_1 \times \mathbf{V}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{s}} \\ V_{1x} & V_{1y} & V_{1s} \\ V_{2x} & V_{2y} & V_{2s} \end{vmatrix}$$

$$= (V_{1x}V_{2s} - V_{1s}V_{2x})\hat{\mathbf{x}} + (V_{1s}V_{2x} - V_{1x}V_{2s})\hat{\mathbf{y}} + (V_{1x}V_{2y} - V_{1y}V_{2x})\hat{\mathbf{s}}$$

Elements:

$$d^2x_\perp = dxdy$$

$$d^3x_\perp = \left(1 + \frac{x}{R}\right) dxdyds$$

$$d\vec{\ell} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{s}}\left(1 + \frac{x}{R}\right) ds$$

Gradient:

$$\nabla\Psi = \hat{\mathbf{x}}\frac{\partial\Psi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Psi}{\partial y}' + \hat{\mathbf{s}}\frac{1}{1+x/R}\frac{\partial\Psi}{\partial s}$$

Divergence:

$$\nabla\cdot\mathbf{V} = \frac{1}{1+x/R}\frac{\partial}{\partial x}\left[(1+x/R)V_x\right] + \frac{\partial V_y}{\partial y} + \frac{1}{1+x/R}\frac{\partial V_s}{\partial s}$$

Curl:

$$\nabla\times\mathbf{V} = \hat{\mathbf{x}}\left(\frac{\partial V_s}{\partial y} - \frac{1}{1+x/R}\frac{\partial V_y}{\partial s}\right) + \hat{\mathbf{y}}\frac{1}{1+x/R}\left(\frac{\partial V_x}{\partial s} - \frac{\partial}{\partial x}\left[(1+x/R)V_s\right]\right)$$
$$+ \hat{\mathbf{s}}(1+x/R)\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Laplacian:

$$\nabla^2\Psi = \frac{1}{1+x/R}\frac{\partial}{\partial x}\left[\left(1+\frac{x}{R}\right)\frac{\partial\Psi}{\partial x}\right] + \frac{\partial^2\Psi}{\partial y^2} + \frac{1}{1+x/R}\frac{\partial}{\partial s}\left[\frac{1}{1+x/R}\frac{\partial\Psi}{\partial s}\right]$$

Transverse particle equations of motion including bends and "off-momentum" effects

- See texts such as Edwards and Syphers for guidance on derivation steps
- Full derivation is beyond needs/scope of this class

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \left[\frac{1}{R^2(s)}\frac{1-\delta}{1+\delta}\right]x = \frac{\delta}{1+\delta}\frac{1}{R(s)} + \frac{q}{m\gamma_b\beta_b^2c^2}\frac{E_x^a}{(1+\delta)^2}$$

$$- \frac{q}{m\gamma_b\beta_b c}\frac{B_y^a}{1+\delta} + \frac{q}{m\gamma_b\beta_b c}\frac{B_s^a}{1+\delta}y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{1}{1+\delta}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' = \frac{q}{m\gamma_b\beta_b^2c^2}\frac{E_y^a}{(1+\delta)^2} + \frac{q}{m\gamma_b\beta_b c}\frac{B_x^a}{1+\delta}$$

$$- \frac{q}{m\gamma_b\beta_b c}\frac{B_s^a}{1+\delta}x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{1}{1+\delta}\frac{\partial\phi}{\partial y}$$

$$p_0 = m\gamma_b\beta_b c = \text{Design Momentum}$$

$$\delta \equiv \frac{\delta p}{p_0} = \text{Fractional Momentum Error}$$

$$\frac{1}{R(s)} = \frac{B_y^a(s)|_{\text{Dipole}}}{[B\rho]} \qquad [B\rho] = \frac{p_0}{q}$$

Comments:

- Design bends only in *x* and B_y^a, E_x^a contain *no* dipole terms (design orbit)
  - Dipole components set via the design bend radius $R(s)$
- Equations contain only low-order terms in momentum spread δ

Comments continued:

- Equations are often applied linearized in δ
- Achromatic focusing lattices are often designed using equations with momentum spread to obtain focal points independent of δ to some order
  x and *y* equations differ significantly due to bends modifying the *x*-equation when *R*(*s*) is finite
- It will be shown in the problems that for electric bends:

$$\frac{1}{R(s)} = \frac{E_x^a(s)}{\beta_b c [B\rho]}$$

- Applied fields for focusing: $\mathbf{E}_\perp^a, \ \mathbf{B}_\perp^a, \ B_s^a$
  must be expressed in the bent *x*,*y*,*s* system of the reference orbit
  - Includes error fields in dipoles
- Self fields may also need to be solved taking into account bend terms
  - Often can be neglected in Poisson's Equation

$$\left\{ \frac{1}{1+x/R} \frac{\partial}{\partial x} \left[\left(1 + \frac{x}{R} \right) \frac{\partial}{\partial x} \right] + \frac{\partial^2}{\partial y^2} + \frac{1}{1+x/R} \frac{\partial}{\partial s} \left[\frac{1}{1+x/R} \frac{\partial}{\partial s} \right] \right\} \phi = -\frac{\rho}{\epsilon_0}$$

if $R \rightarrow \infty$

reduces to familiar:

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial s^2} \right\} \phi = -\frac{\rho}{\epsilon_0}$$

Appendix A: Gamma and Beta Factor Conversions

It is frequently the case that functions of the relativistic gamma and beta factors are converted to superficially different appearing forms when analyzing transverse particle dynamics in order to more cleanly express results. Here we summarize useful formulas in that come up when comparing various forms of equations. Derivatives are taken wrt the axial coordinate *s* but also apply wrt time *t*

Results summarized here can be immediately applied in the paraxial approximation by taking:

$$v = |\mathbf{v}| \simeq v_b = \beta_b c \qquad \Longrightarrow \qquad \begin{aligned} \beta &\simeq \beta_b \\ \gamma &\simeq \gamma_b \end{aligned}$$

Assume that the beam is forward going with $\beta \geq 0$:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{1}{\gamma}\sqrt{\gamma^2 - 1}$$

$$\gamma^2 = \frac{1}{1-\beta^2} \qquad \beta^2 = 1 - 1/\gamma^2$$

A commonly occurring acceleration factor can be expressed in several ways:

- Depending on choice used, equations can look quite different!

$$\frac{(\gamma\beta)'}{(\gamma\beta)} = \frac{\gamma'}{\gamma} + \frac{\beta'}{\beta} = \frac{\gamma'}{\gamma\beta^2}$$

Axial derivative factors can be converted using:

$$\gamma' = \frac{\beta\beta'}{(1-\beta^2)^{3/2}} \qquad \beta' = \frac{\gamma'}{\gamma^2\sqrt{\gamma^2-1}}$$

Energy factors:

$$\mathcal{E}_{\rm tot} = \gamma mc^2 = \mathcal{E} + mc^2$$

$$\gamma\beta = \sqrt{\left(\frac{\mathcal{E}}{mc^2}\right)^2 + 2\left(\frac{\mathcal{E}}{mc^2}\right)}$$

Rigidity:

$$[B\rho] = \frac{p}{q} = \frac{\gamma mv}{q} = \frac{mc}{q}\gamma\beta = \frac{mc}{q}\sqrt{\left(\frac{\mathcal{E}}{mc^2}\right)^2 + 2\left(\frac{\mathcal{E}}{mc^2}\right)}$$