S10: Acceleration and Normalized Emittance
S10A: Introduction

If the beam is accelerated longitudinally in a linear focusing channel, the *x*-particle equation of motion (see: S1 and S2) is:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

Analogous equation holds in *y*

Neglects:

- Nonlinear applied focusing fields
- Momentum spread effects

Comments:

- $\gamma_b, \ \beta_b$ are regarded as prescribed functions of *s* set by the acceleration schedule of the machine
- Variations in $\gamma_b, \ \beta_b$ due to acceleration must be included in and/or compensated by adjusting the strength of the optics via optical parameters contained in $\kappa_x, \ \kappa_y$
  - Example: for quadrupole focusing adjust field gradients (see: S2)

Acceleration Factor: Characteristics of

Relativistic Factor

$$\gamma_b\beta_b \simeq \begin{cases} \gamma_b, & \text{Relativistic Limit} \\ \beta_b, & \text{Nonrelativistic Limit} \end{cases}$$

$$\gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}}$$

Beam/Particle Kinetic Energy:

$$\mathcal{E}_b(s) = (\gamma_b - 1)mc^2 = \text{Beam Kinetic Energy}$$

- Function of s specified by Acceleration schedule for transverse dynamics
- See S11 for calculation of $\mathcal{E}_b$ and $\gamma_b\beta_b$ from longitudinal dynamics and J.J. Barnard lectures on Longitudinal Dynamics

Approximate energy gain from average gradient:

$$\mathcal{E}_b \simeq \mathcal{E}_i + G(s - s_i)$$

$\mathcal{E}_i = \text{const} = \text{Initial Energy}$

$G = \text{const} = \text{Average Gradient}$

- Real energy gain will be rapid when going through discreet acceleration gaps

$$\mathcal{E}_b \simeq \begin{cases} \gamma_b mc^2, & \text{Relativistic Limit}, \gamma_b \gg 1 \\ \frac{1}{2}m\beta_b^2c^2, & \text{Nonrelativistic Limit}, |\beta_b| \ll 1 \end{cases}$$

Comments Continued:

- In typical accelerating systems, changes in $\gamma_b\beta_b$ are slow and the fractional changes in the orbit induced by acceleration are small
  - Exception near an injector since the beam is often not yet energetic
- The acceleration term:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} > 0$$

will act to damp particle oscillations (see following slides for motivation)

Even with acceleration, we will find that there is a Courant-Snyder invariant (normalized emittance) that is valid in an analogous context as in the case without acceleration provided phase-space coordinates are chosen to compensate for the damping of particle oscillations

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Identify relativistic factor with average gradient energy gain:

Relativistic Limit: $\gamma_b \gg 1$

$$\gamma_b \simeq \frac{\mathcal{E}_b}{mc^2} = \frac{\mathcal{E}_i}{mc^2} + \frac{G}{mc^2}(s - s_i)$$

$$\Longrightarrow \quad \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\gamma_b'}{\gamma_b} \simeq \frac{1}{\frac{\mathcal{E}_i}{G} + (s - s_i)} \sim \frac{1}{s - s_i}$$

Nonrelativistic Limit: $|\beta_b| \ll 1$

$$\beta_b \simeq \sqrt{2\frac{\mathcal{E}_b}{mc^2}} = \sqrt{2\frac{\mathcal{E}_i}{mc^2} + 2\frac{G}{mc^2}(s - s_i)}$$

$$\Longrightarrow \quad \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\beta_b'}{\beta_b} = \frac{1/2}{\frac{\mathcal{E}_i}{G} + (s - s_i)} \sim \frac{1}{2(s - s_i)}$$

- Expect Relativistic and Nonrelativistic motion to have similar solutions
  - Parameters for each case will often be quite different

/// Aside: Acceleration and Continuous Focusing Orbits with $\kappa_x = k_{\beta 0}^2 =$ const

Assume relativistic motion and negligible space-charge:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\gamma_b'}{\gamma_b} = \frac{1}{\left(\frac{\mathcal{E}_i}{G} - s_i\right) + s} \qquad \frac{\partial \phi}{\partial x} \simeq 0$$

Then the equation of motion reduces to:

$$\boxed{x'' + \frac{1}{\left(\frac{\mathcal{E}_i}{G} - s_i\right) + s} x' + k_{\beta 0}^2 x = 0}$$

This equation is the equation of a Bessel Function of order zero:

$$\frac{d^2x}{d\xi^2} + \frac{1}{\xi}\frac{dx}{d\xi} + x = 0 \qquad \xi = k_{\beta 0} s + k_{\beta 0}\left(\frac{\mathcal{E}_i}{G} - s_i\right)$$

$$\xi' = k_{\beta 0}$$

$$x = C_1 J_0(\xi) + C_2 Y_0(\xi)$$

$$x' = -C_1 k_{\beta 0} J_1(\xi) - C_2 k_{\beta 0} Y_1(\xi)$$

$dJ_0(x)/dx = -J_1(x)$ and same for Y_0

$C_1 =$ const $\quad C_2 =$ const

$J_n =$ Order n Bessel Func (1st kind)

$Y_n =$ Order n Bessel Func (2nd kind)

Solving for the constants in terms of the particle initial conditions:

$$\begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \begin{bmatrix} J_0(\xi_i) & Y_0(\xi_i) \\ -k_{\beta 0} J_1(\xi_i) & -k_{\beta 0} Y_1(\xi_i) \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$x_i \equiv x(s = s_i) \qquad \xi_i \equiv k_{\beta 0}\frac{\mathcal{E}_i}{G} = \xi(s = s_i)$$

$$x_i' \equiv x'(s = s_i)$$

Invert matrix to solve for constants in terms of initial conditions:

$$\Longrightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -k_{\beta 0} Y_1(\xi_i) & -Y_0(\xi_i) \\ k_{\beta 0} J_1(\xi_i) & J_0(\xi_i) \end{bmatrix} \cdot \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$

$$\Delta \equiv k_{\beta 0}[Y_0(\xi_i) J_1(\xi_i) - J_0(\xi_i) Y_1(\xi_i)]$$

Comments:

- Bessel functions behave like damped harmonic oscillators
  - See texts on Mathematical Physics or Applied Mathematics
- Nonrelativistic limit solution is *not* described by a Bessel Function solution
  - Properties of solution will be similar though (similar special function)
  - The coefficient in the damping term $\propto x'$ has a factor of 2 difference, preventing exact Bessel function form

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Using this solution, plot the orbit for (contrived parameters for illustration only):

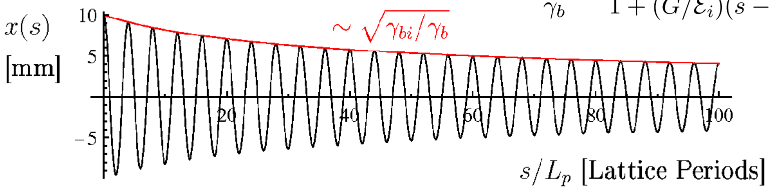
$$k_{\beta 0} = \frac{\sigma_0}{L_p} \qquad \sigma_0 = 90^\circ/\text{Period} \qquad \mathcal{E}_i = 1000\ \text{MeV}$$

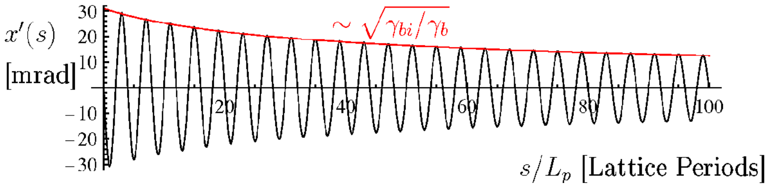
$$L_p = 0.5\ \text{m} \qquad G = 100\ \text{MeV/m}$$

$$x(0) = 10\ \text{mm} \qquad s_i = 0$$

$$x'(0) = 0\ \text{mrad}$$

$$\frac{\gamma_{bi}}{\gamma_b} = \frac{1}{1 + (G/\mathcal{E}_i)(s - s_i)}$$





- Solution shows damping: phase volume scaling $\sim 1/(\gamma_b\beta_b) \simeq 1/\gamma_b$ ///

S10B: Transformation to Normal Form

"Guess" transformation to apply motivated by conjugate variable arguments (see: J.J. Barnard, Intro. Lectures)

$$\boxed{\tilde{x} \equiv \sqrt{\gamma_b\beta_b}\, x}$$

Here we reuse tilde variables to denote a transformed quantity we choose to look like something familiar from simpler contexts

Then:

$$x = \frac{1}{\sqrt{\gamma_b\beta_b}}\tilde{x}$$

$$x' = \frac{1}{\sqrt{\gamma_b\beta_b}}\tilde{x}' - \frac{1}{2}\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)^{3/2}}\tilde{x}$$

$$x'' = \frac{1}{\sqrt{\gamma_b\beta_b}}\tilde{x}'' - \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)^{3/2}}\tilde{x}' + \left[\frac{3}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^{5/2}} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)^{3/2}}\right]\tilde{x}$$

The inverse phase-space transforms will also be useful later:

$$\tilde{x} = \sqrt{\gamma_b\beta_b}\, x$$

$$\tilde{x}' = \sqrt{\gamma_b\beta_b}\, x' + \frac{1}{2}\frac{(\gamma_b\beta_b)'}{\sqrt{\gamma_b\beta_b)}} x$$

Applying these results, the particle *x*- equation of motion with acceleration becomes:

$$\boxed{\tilde{x}'' + \left[\kappa_x + \frac{1}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^2} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)}\right]\tilde{x} = -\frac{q}{m\gamma_b^2\beta_b c^2}\frac{\partial\phi}{\partial\tilde{x}}}$$

Note:

- Factor of $\gamma_b\beta_b$ difference from untransformed expression in the space-charge coupling coefficient

It is instructive to also transform the Possion equation associated with the space-charge term:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0}$$

Transform:

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial\tilde{x}}{\partial x}\frac{\partial}{\partial\tilde{x}}\right)\left(\frac{\partial\tilde{x}}{\partial x}\frac{\partial}{\partial\tilde{x}}\right) = \gamma_b\beta_b\frac{\partial^2}{\partial\tilde{x}^2}$$

$$\frac{\partial^2}{\partial y^2} = \left(\frac{\partial\tilde{y}}{\partial y}\frac{\partial}{\partial\tilde{y}}\right)\left(\frac{\partial\tilde{y}}{\partial y}\frac{\partial}{\partial\tilde{y}}\right) = \gamma_b\beta_b\frac{\partial^2}{\partial\tilde{y}^2}$$

Using these results, Poisson's equation becomes:

$$\left(\frac{\partial^2}{\partial\tilde{x}^2} + \frac{\partial^2}{\partial\tilde{y}^2}\right)\phi = -\frac{\rho}{\gamma_b\beta_b\epsilon_0}$$

Or defining a transformed potential $\tilde{\phi}$

$$\boxed{\begin{aligned}&\tilde{\phi} = \gamma_b\beta_b\phi\\&\left(\frac{\partial^2}{\partial\tilde{x}^2} + \frac{\partial^2}{\partial\tilde{y}^2}\right)\tilde{\phi} = -\frac{\rho}{\epsilon_0}\end{aligned}}$$

Applying these results, the *x*-equation of motion with acceleration becomes:

$$\boxed{\tilde{x}'' + \left[\kappa_x + \frac{1}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^2} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)}\right]\tilde{x} = -\frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial\tilde{\phi}}{\partial\tilde{x}}}$$

- Usual form of the space-charge coefficient with $\gamma_b^3\beta_b^2$ rather than $\gamma_b^2\beta_b$ is restored when expressed in terms of the transformed potential $\tilde{\phi}$

An additional step can be taken to further stress the correspondence between the transformed system with acceleration and the untransformed system in the absence of acceleration.

Denote an effective focusing strength:

$$\boxed{\tilde{\kappa}_x \equiv \kappa_x + \frac{1}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^2} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)}}$$

$\tilde{\kappa}_x$ incorporates acceleration terms beyond $\gamma_b,\ \beta_b$ factors already included in the definition of κ_x (see: S2):

$$\kappa_x = \begin{cases} \frac{qG}{m\gamma_b\beta_b^2c^2}, & G = -\partial E_x^a/\partial x = \partial E_y^a/\partial y = \text{Electric Quad. Grad.} \\ \frac{qG}{m\gamma_b\beta_b c}, & G = \partial B_x^a/\partial y = \partial B_y^a/\partial x = \text{Magnetic Quad. Grad.} \\ \frac{qB_{z0}}{4m\gamma_b^2\beta_b^2c^2}, & B_{z0} = \text{Solenoidal Magnetic Field} \end{cases}$$

The transformed equation of motion with acceleration then becomes:

$$\boxed{\tilde{x}'' + \tilde{\kappa}_x\tilde{x} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\tilde{\phi}}{\partial\tilde{x}}}$$

The transformed equation with acceleration has the same form as the equation in the absence of acceleration. If space-charge is negligible ($\partial\phi/\partial\mathbf{x}_\perp \simeq 0$) we have:

Accelerating System		Non-Accelerating System
$\tilde{x}'' + \tilde{\kappa}_x\tilde{x} = 0$	$\Longrightarrow$	$x'' + \kappa_x x = 0$

Therefore, *all previous analysis* on phase-amplitude methods and Courant-Snyder invariants associated with Hill's equation in *x-x'* phase-space can be immediately applied to $\tilde{x} - \tilde{x}'$ phase-space for an accelerating beam

$$\boxed{\begin{aligned}&\left(\frac{\tilde{x}}{\tilde{w}_x}\right)^2 + (\tilde{w}_x\tilde{x}' - \tilde{w}_x'\tilde{x})^2 = \tilde{\epsilon} = \text{const}\\&\tilde{w}_x'' + \tilde{\kappa}_x\tilde{w}_x - \frac{1}{\tilde{w}_x^3} = 0\\&\tilde{w}_x(s + L_p) = \tilde{w}_x(s)\\&\pi\tilde{\epsilon} = \text{Area traced by orbit} = \text{const}\\&\quad\text{in } \tilde{x}\text{-}\tilde{x}' \text{ phase-space}\end{aligned}}$$

- Focusing field strengths need to be adjusted to maintain periodicity of $\tilde{\kappa}_x$ in the presence of acceleration
  - Not possible to do exactly, but can be approximate for weak acceleration

S10C: Phase Space Relation Between Transformed and UnTransformed Systems

It is instructive to relate the transformed phase-space area in tilde variables to the usual *x*-*x*' phase area:

$$d\tilde{x} \otimes d\tilde{x}' = |J| dx \otimes dx'$$

where *J* is the Jacobian:

$$J \equiv \det \begin{bmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}}{\partial x'} \\ \frac{\partial \tilde{x}'}{\partial x} & \frac{\partial \tilde{x}'}{\partial x'} \end{bmatrix}$$

$$= \det \begin{bmatrix} \sqrt{\gamma_b \beta_b} & 0 \\ \frac{1}{2}\frac{(\gamma_b\beta_b)'}{\sqrt{\gamma_b\beta_b}} & \sqrt{\gamma_b\beta_b} \end{bmatrix} = \gamma_b \beta_b$$

Inverse transforms derived in S10B:

$$\tilde{x} = \sqrt{\gamma_b \beta_b} x$$

$$\tilde{x}' = \sqrt{\gamma_b \beta_b} x' + \frac{1}{2}\frac{(\gamma_b\beta_b)'}{\sqrt{(\gamma_b\beta_b)}} x$$

Thus:

$$\boxed{d\tilde{x} \otimes d\tilde{x}' = \gamma_b \beta_b \; dx \otimes dx'}$$

Based on this area transform, if we define the (instantaneous) phase space area of the orbit trance in *x*-*x*' to be $\pi\epsilon_x$ "regular emittance", then this emittance is related to the "normalized emittance" $\tilde{\epsilon}_x$ in $\tilde{x} - \tilde{x}'$ phase-space by:

$$\boxed{\begin{aligned} \tilde{\epsilon}_x &= \gamma_b \beta_b \epsilon_x \\ &\equiv \text{Normalized Emittance} \equiv \epsilon_{nx} \end{aligned}}$$

- Factor $\gamma_b\beta_b$ compensates for acceleration induced damping in particle orbits
- Normalized emittance is very important in design of lattices to transport accelerating beams
  - Designs usually made assuming conservation of normalized emittance
- Same result that J.J. Barnard motivated in the Intro. Lectures using alternative methods

S11: Calculation of Acceleration Induced Changes in gamma and beta

S11A: Introduction

The transverse particle equation of motion with acceleration was derived in a Cartesian system by approximating (see: S1):

$$\frac{d}{dt}\left(m\gamma \frac{d\mathbf{x}_\perp}{dt}\right) \simeq \quad q\mathbf{E}^a_\perp + q\beta_b c \hat{\mathbf{z}} \times \mathbf{B}^a_\perp + qB^a_z \mathbf{v}_\perp \times \hat{\mathbf{z}} \quad - q\frac{1}{\gamma_b^2}\frac{\partial \phi}{\mathbf{x}_\perp}$$

using

$$m\frac{d}{dt}\left(\gamma\frac{d\mathbf{x}_\perp}{dt}\right) \simeq m\gamma_b\beta_b^2 c^2 \left[\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp\right]$$

to obtain:

$$\boxed{\begin{aligned} \mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp &= \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}^a_\perp + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}} \times \mathbf{B}^a_\perp + \frac{qB^a_z}{m\gamma_b\beta_b c}\mathbf{x}'_\perp \times \hat{\mathbf{z}} \\ &\quad - \frac{q}{\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial \mathbf{x}_\perp}\phi \end{aligned}}$$

To integrate this equation, we need the variation of β_b and $\gamma_b = 1/\sqrt{1-\beta_b^2}$ as a function of s. For completeness here, we briefly outline how this can be done by analyzing longitudinal equations of motion. More details can be found in JJ Barnard lectures on Longitudinal Dynamics.

S11B: Solution of Longitudinal Equation of Motion

Changes in $\gamma_b\beta_b$ are calculated from the longitudinal particle equation of motion:

- See equation at end of S1D

$$\boxed{\underset{\text{Term 1}}{\frac{d}{dt}\left(m\gamma\frac{dz}{dt}\right)} \simeq \underset{\text{Term 2}}{qE_z^a} - \underset{\text{Term 3}}{q(v_xB_y^a - v_yB_x^a)} - \underset{\text{Neglect Rel to Term 2}}{q\frac{\partial\phi}{\partial z}}}$$

Using steps similar to those in S1, we approximate terms:

Term 1: $\frac{d}{dt}\left(\gamma\frac{dz}{dt}\right) \simeq c^2\beta_b(\gamma_b\beta_b)'$ $\qquad \frac{dz}{dt} = v_z \simeq \beta_b c \qquad \gamma \simeq \gamma_b$

$\frac{d}{dt} \simeq \beta_b c\frac{d}{ds}$

Term 2: $\frac{q}{m}E_z^a \simeq -\frac{q}{m}\left.\frac{\partial\phi^a}{\partial s}\right|_{x=y=0}$

ϕ^a is a quasi-static approximation accelerating potential (see next pages)

Term 3: $-q(v_xB_y^a - v_yB_x^a) = -q\left(\frac{dx}{dt}B_y^a - \frac{dy}{dt}B_x^a\right) \simeq 0$

- Transverse magnetic fields typically only weakly change particle energy and terms can be neglected relative to others

The longitudinal particle equation of motion for $\gamma_b,\ \beta_b$ then reduces to:

$$\boxed{\beta_b(\gamma_b\beta_b)' \simeq -\frac{q}{mc^2}\left.\frac{\partial\phi^a}{\partial s}\right|_{x=y=0}}$$

Some algebra then shows that:

$$\gamma_b' = \left(\frac{1}{\sqrt{1-\beta_b^2}}\right)' = \gamma_b^3\beta_b\beta_b'$$

$$\begin{aligned}\Longrightarrow \quad \beta_b(\gamma_b\beta_b)' &= \beta_b^2\gamma_b' + \gamma_b\beta_b\beta_b' \\ &= (1+\gamma_b^2\beta_b^2)\gamma_b\beta_b\beta_b' = \gamma_b^3\beta_b\beta_b' \\ &= \gamma_b'\end{aligned}$$

Giving:

$$\boxed{\gamma_b' = -\frac{q}{mc^2}\left.\frac{\partial\phi^a}{\partial s}\right|_{x=y=0}}$$

Which can then be integrated to obtain:

$$\boxed{\gamma_b = -\frac{q}{mc^2}\phi^a(r=0, z=s) + \text{const}}$$

We denote the on-axis accelerating potential as:

$$\boxed{V(s) \equiv \phi^a(x=y=0, z=s)}$$

- Can represent RF or induction accelerating gap fields
  See: J.J. Barnard lectures for more details

Using this and setting $\gamma_b(s=s_i) = \gamma_{bi}$ gives for the gain in axial kinetic energy $\mathcal{E}_b$ and corresponding changes in $\gamma_b,\ \beta_b$ factors:

$$\boxed{\begin{aligned}&\mathcal{E}_b = (\gamma_b - 1)mc^2 = q[V(s_i) - V(s)] + \mathcal{E}_{bi} \\ &\quad \gamma_b = 1 + \mathcal{E}_{bi}/(mc^2) \qquad\qquad \mathcal{E}_{bi} = (\gamma_{bi} - 1)mc^2 \\ &\quad \beta_b = \sqrt{1 - 1/\gamma_b^2}\end{aligned}}$$

These equations can be solved for the consistent variation of $\gamma_b(s),\ \beta_b(s)$ to integrate the transverse equations of motion:

$$\mathbf{x}_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}_\perp' = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}_\perp^a + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}}\times\mathbf{B}_\perp^a + \frac{qB_z^a}{m\gamma_b\beta_b c}\mathbf{x}_\perp'\times\hat{\mathbf{z}} - \frac{q}{\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}_\perp}\phi$$

Nonrelativistic limit results

In the nonrelativistic limit:

$$\gamma_b \simeq 1 + \frac{1}{2}\beta_b^2 \qquad \beta_b^2 \ll 1 \qquad \mathcal{E}_b = (\gamma_b - 1)mc^2 \simeq \frac{1}{2}m\beta_b^2c^2$$

and the previous (relativistic valid) energy gain formulas reduce to:

$$\boxed{\begin{aligned}&\mathcal{E}_b \simeq \frac{1}{2}m\beta_b^2c^2 = q[V(s_i) - V(s)] + \mathcal{E}_{bi} \\ &\gamma_b \simeq 1 \qquad\qquad\qquad\qquad \mathcal{E}_{bi} = \frac{1}{2}m\beta_b^2c^2 \\ &\beta_b = \sqrt{\frac{2\mathcal{E}_b}{mc^2}}\end{aligned}}$$

Using this result, in the nonrelativistic limit we can take in the transverse particle equation of motion:

$$\boxed{\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\beta_b'}{\beta_b} = \frac{1}{2}\frac{\mathcal{E}_b'}{\mathcal{E}_b} = -\frac{1}{2}\frac{qV'(s)}{q[V(s_i) - V(s)] + \mathcal{E}_{bi}}}$$

S11C: Longitudinal Solution via Energy Gain

An alternative analysis of the particle energy gain carried out in S11B can be illuminating. In this case we start from the exact Lorentz force equation with time as the independent variable for a particle moving in the full electromagnetic field:

$$\boxed{\begin{aligned}&\frac{d\mathbf{p}}{dt} = q\mathbf{E} + q\vec{\beta}c \times \mathbf{B}\\ &\mathbf{p} \equiv \gamma m\vec{\beta}c \qquad \gamma \equiv 1/\sqrt{1-\vec{\beta}\cdot\vec{\beta}}\end{aligned}}$$

Dotting $mc\vec{\beta}$ into this equation:

$$mc\vec{\beta}\cdot\frac{d}{dt}(c\gamma\vec{\beta}) = qc\vec{\beta}\cdot\mathbf{E} + qc\vec{\beta}\cdot\overset{0}{\cancel{[c\vec{\beta}\times\mathbf{B}]}}$$

$$\vec{\beta}\cdot\vec{\beta}\dot{\gamma} + \gamma\vec{\beta}\cdot\dot{\vec{\beta}} = \frac{q}{mc}\vec{\beta}\cdot\mathbf{E}$$

and

$$\gamma \equiv (1-\vec{\beta}\cdot\vec{\beta})^{-1/2}$$

$$\vec{\beta}\cdot\vec{\beta} = 1 - 1/\gamma^2$$

$$\Longrightarrow \vec{\beta}\cdot\dot{\vec{\beta}} = \dot{\gamma}/\gamma^3$$

Inserting these factors:

$$(1-1/\gamma^2)\dot{\gamma} + \dot{\gamma}/\gamma^2 = \frac{q}{mc^2}\vec{\beta}\cdot\mathbf{E}$$

or:

$$\boxed{\dot{\gamma} = \frac{q}{mc}\vec{\beta}\cdot\mathbf{E}}$$

Equivalently:

$$\boxed{\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\left[(\gamma-1)mc^2\right] = qc\vec{\beta}\cdot\mathbf{E}}$$

- Only the electric field changes the kinetic energy of a particle

Taking: $\frac{d}{dt} = c\beta_z\frac{d}{ds}$ $\quad \beta_z \simeq \beta \simeq \beta_b$ $\quad \gamma \simeq \gamma_b$ $\quad \mathcal{E} \simeq \mathcal{E}_b = (\gamma_b - 1)mc^2$

and approximating the axial electric field by the applied component then obtains

$$\boxed{\frac{d}{ds}\mathcal{E}_b \simeq \frac{dt}{ds}\frac{d}{dt}\left[(\gamma-1)mc^2\right] \simeq qE_z^a}$$

which is the longitudinal equation of motion analyzed in S11B.

S11D: Quasistatic Potential Expansion

In the quasistatic approximation, the accelerating potential can be expanded in the axisymmetric limit as:

- See: J.J. Barnard, Intro Lectures; and Reiser, *Theory and Design of Charged Particle Beams*, (1994, 2008) Sec. 3.3.

$$\boxed{\phi^a = V(z) - \frac{1}{4}\frac{\partial^2}{\partial z^2}V(z)(x^2+y^2) + \frac{1}{64}\frac{\partial^4}{\partial z^4}V(z)(x^2+y^2)^2 + \cdots}$$

The longitudinal acceleration also result in a transverse focusing field

$$\boxed{\begin{aligned}&\mathbf{E}^a_\perp = \mathbf{E}^a_\perp|_{\text{foc}} - \frac{\partial\phi^a}{\partial\mathbf{x}_\perp}\\ &\mathbf{E}^a_\perp|_{\text{foc}} = \text{Fields from Any Applied Focusing Optics}\\ &-\frac{\partial\phi^a}{\partial\mathbf{x}_\perp} \simeq \frac{1}{2}\frac{\partial^2}{\partial z^2}V(z)\mathbf{x}_\perp = \text{Focusing Field from Acceleration}\end{aligned}}$$

- Results can be used to cast acceleration terms in more convenient forms. See J.J. Barnard, Intro. Lectures for more details.
- Einzel lens focusing exploits accel/de-acell cycle to make AG focusing