

see Conte and Mackay, Chapter 9
 Wille, Chapter 5
 Wiedemann, § 20.2

RF Cavities

Maxwell's equations in vacuum region:

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

Vector Identity

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} \nabla \times \vec{B}$$

$$\nabla \times (\nabla \times \vec{B}) = \frac{1}{c^2} \frac{\partial^2 \nabla \times \vec{E}}{\partial t^2}$$

$$\begin{aligned} \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} &= 0 \\ \nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} &= 0 \end{aligned}$$

Maxwell Eqn to eliminate

$$\nabla \cdot \vec{E} = -\frac{\partial}{\partial t} \left(\frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right)$$

$$\nabla \cdot \vec{B} = \frac{1}{c} \frac{\partial}{\partial t} \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

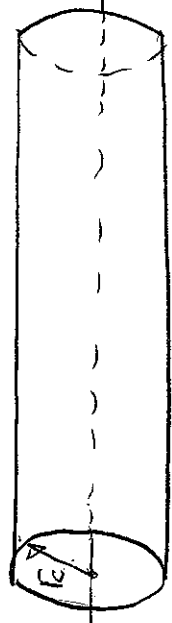
\vec{E}, \vec{B} satisfy wave equations.

1st step!

We will look for EM wave solutions in a perfectly conducting, cylindrical pipe "waveguide".



End View



Side View

$$r_c = \text{Radius Cylinder}$$

Maxwell eqns give boundary conditions on perfect conductor: $\vec{E} \cdot \hat{n}$: Tangential zero, $\vec{B} \cdot \hat{n}$: Normal zero

Search for a solution with $z-t$ traveling wave form with harmonic time (t) and z dependence $\sim e^{-i\omega t}$ time variation, $i = \sqrt{-1}$, take $Re\{ \}$ for physical part.

$$\begin{aligned} E_z &= E_z(r, \theta) \cdot e^{i(\omega t - kz)} \\ E_r &= E_r(r, \theta) \cdot e^{i(\omega t - kz)} \\ B_\theta &= B_\theta(r, \theta) \cdot e^{i(\omega t - kz)} \end{aligned}$$

Angular Frequency ω = const, Axial Wavenumber k = const
 Transverse Magnetic TM form since want longitudinal E_z for acceleration

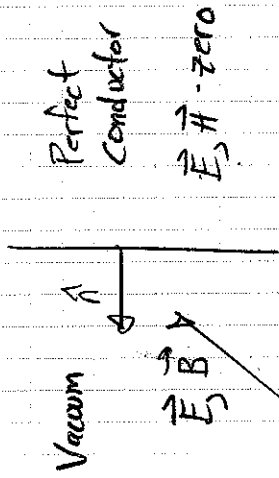
None zero field components, in cylindrical-polar coordinates.

Later will restrict $E_z(r=r_c) = 0$ to meet boundary conditions.

Field Boundary Conditions:

Conducting walls

Apply Maxwell's eqns at boundary of perfect conductor



$$\begin{aligned} \nabla \cdot \vec{D} &= \rho \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \nabla \cdot \vec{B} &= 0 \end{aligned}$$

Integrate over \Rightarrow limiting pill box $\int_{S_1}^{\prime} \vec{D} \cdot \vec{V}$

$$\begin{aligned} \hat{n} \cdot \vec{D} &= \Sigma \\ \hat{n} \times \vec{E} &= 0 \\ \hat{n} \times \vec{H} &= \vec{K} \\ \hat{n} \cdot \vec{B} &= 0 \end{aligned}$$

Σ = Surface Charge Density
 \vec{K} = Surface Current Density

In Vacuum:

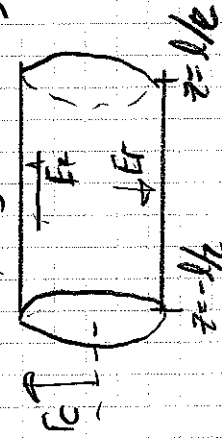
$$\begin{aligned} \vec{D} &= \epsilon_0 \vec{E} \\ \vec{B} &= \mu_0 \vec{H} \end{aligned}$$

So we have for field boundary conditions in the ideal Vacuum / perfect conductor interface:

Exclude $\left\{ \begin{array}{l} \vec{E} |_{\text{tangential}} = 0 \\ \vec{B} |_{\text{normal}} = 0 \end{array} \right.$

Allow $\left\{ \begin{array}{l} \vec{E} |_{\text{normal}} \text{ allowed} \\ \vec{B} |_{\text{tangential}} \text{ allowed} \end{array} \right.$

Implications in Pipe Segment: E_z, E_θ, B_θ allowed



$E_z \rightarrow 0$: pipe edge
 $r = r_c$: pipe edge
 $z = \pm \frac{l}{2}$: pipe ends
 $E_r \rightarrow 0$
 $z = \pm \frac{l}{2}$: pipe ends

surface charge Σ adjusts to allow
 surface current \vec{K} adjusts to allow.

B_θ No restrictions

$$\nabla^2 = \nabla_r^2 + \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial}{\partial t} = i\omega$$

$$\frac{\partial}{\partial z} = -ik$$

Examine only E_z :

$$\nabla^2 E_z - \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2} = 0 \quad \Rightarrow \quad \nabla_r^2 E_z + (\frac{\omega^2}{c^2} - k^2) E_z = 0$$

Look for a solution with harmonic azimuthal variation

$$E_z \sim \cos(n\theta) \quad \text{choose } \theta=0 \text{ to make true.}$$

$$\frac{\partial^2}{\partial r^2} E_z + \frac{1}{r} \frac{\partial}{\partial r} E_z + (k_c^2 - n^2) E_z = 0$$

$$k_c^2 \equiv \frac{\omega^2}{c^2} - k^2$$

Bessel Function Equation.

Recognizing this as Bessel's equation, the general solution is

$$E_z = C_1 J_n(k_c r) + C_2 Y_n(k_c r) \quad C_1, C_2 \text{ constants}$$

$J_n(x)$ = Ordinary n th order Bessel function of 1st kind
 $Y_n(x)$ = Ordinary n th order Bessel function of 2nd kind

$$\lim_{r \rightarrow 0} Y_n(k_c r) \rightarrow \infty \Rightarrow C_2 = 0 \quad \text{for finite (physical) E-Field near } r=0.$$

Putting back in variation in θ, z, t , we have:

$$E_z = E_0 J_n(k_c r) \cos(n\theta) e^{i(\omega t - kz)}$$

$$E_0 = \text{const. (complex)}$$

We can now substitute this back in the Maxwell's eqns to find the form of B_θ and E_r consistent. But first, simplify by further restricting to $n=0$ since for accelerating particles we prefer no azimuthal variation.

Maxwell Eqns

$$\nabla \cdot \vec{E} = 0: \quad \frac{1}{r} \frac{\partial}{\partial r} (r E_r) - i k E_z = 0 \quad (1)$$

$$\nabla \times \vec{B} = \mu_0 \frac{\partial \vec{J}}{\partial t} \quad \hat{r}: \quad i k B_\theta = \frac{\omega}{c^2} E_r \quad (2)$$

$$\hat{z}: \quad \frac{\partial}{\partial r} (r B_\theta) = \frac{i \omega}{c^2} E_z \quad (3)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \frac{\partial E_r}{\partial r} - i k E_r = -i \omega B_\theta \quad (4)$$

$\nabla \cdot \vec{B} = 0$ satisfied.

$$E_r = \frac{-i/k \frac{\partial E_z}{\partial r}}{1 - \omega^2/c^2 k^2} = \frac{i k \frac{\partial E_z}{\partial r}}{k_c^2}$$

$$B_\theta = \frac{-i \omega/c^2 k^2}{1 - \omega^2/c^2 k^2} \frac{\partial E_z}{\partial r} = \frac{i \omega/c^2 \frac{\partial E_z}{\partial r}}{k_c^2}$$

$$k_c^2 = \omega^2/c^2 - k^2$$

Then:

$$E_z = E_0 J_0(k_c r) e^{i(\omega t - k z)}$$

$$E_r = E(r) e^{i(\omega t - k z)}$$

$$B_\theta = B_\theta(r) e^{i(\omega t - k z)}$$

From 2) $B_\theta = \frac{\omega}{c^2 k} E_r$

From 4) $\frac{\partial E_r}{\partial r} = i k E_r - i \omega^2/c^2 E_r$

$$k_c^2 = \omega^2/c^2 - k^2$$

Using $J_0'(x) = -J_1(x)$

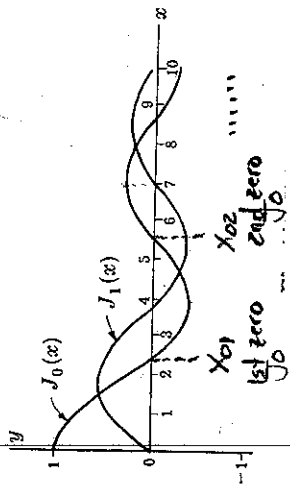
$$E_z = E_0 J_0(k_c r) e^{i(\omega t - k z)}$$

$$E_r = -i E_0 \frac{k_c}{k} J_1(k_c r) e^{i(\omega t - k z)}$$

$$B_\theta = -i \frac{E_0 \omega}{c^2 k_c} J_1(k_c r) e^{i(\omega t - k z)}$$

Finally, need $E_z(r=r_c) = 0$ to satisfy tangential $\vec{E} = 0$ on conducting boundary
 $\Rightarrow J_0(k_c r_c) = 0 \Rightarrow k_c r_c = X_{0j} \quad j=1, 2, 3, \dots$ zero of $J_0(x_{0j}) = 0$

Bessel function:



Wave phase velocity

$$\psi = \omega t - k z = \text{const}$$

$$\dot{\psi} = \omega - k \dot{z} = 0 \Rightarrow \dot{z} = v_{\text{phase}} = \frac{\omega}{k} = \frac{\omega c}{\sqrt{\omega^2 - \omega_c^2}} > c$$

$$v_{\text{phase}} = \frac{c}{\sqrt{1 - \omega_c^2/\omega^2}}$$

Cannot maintain resonance with particle

Note energy propagation speed at group velocity

$$v_{\text{group}} = \frac{d\omega}{dk} = \frac{dc}{(\omega^2 + k^2 c^2)^{1/2}} = \frac{c^2}{\omega k} = \frac{c^2}{v_{\text{phase}}} < c$$

* $v_{\text{group}} < c$ as must be case for physical energy transmission.

Note: $v_{\text{group}} \cdot v_{\text{phase}} = c^2 = \text{const.}$

$$x_{01} \approx 2.405$$

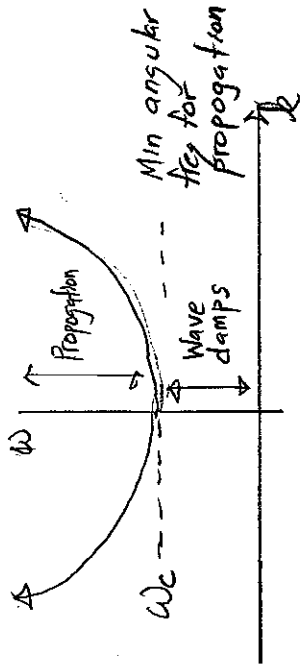
1st zero.

Choose 1st zero to try and get flat field near $r \approx 0$.

$$\Rightarrow k c r_c = \sqrt{\omega^2 - k^2 c^2} r_c = x_{01}$$

Dispersion Relation

$$\omega^2 = \omega_c^2 + k^2 c^2 \quad ; \quad \omega_c \equiv \frac{x_{01} c}{r_c} \quad \text{cutoff freq}$$



Cylindrical Waveguide TM_{01} Modes

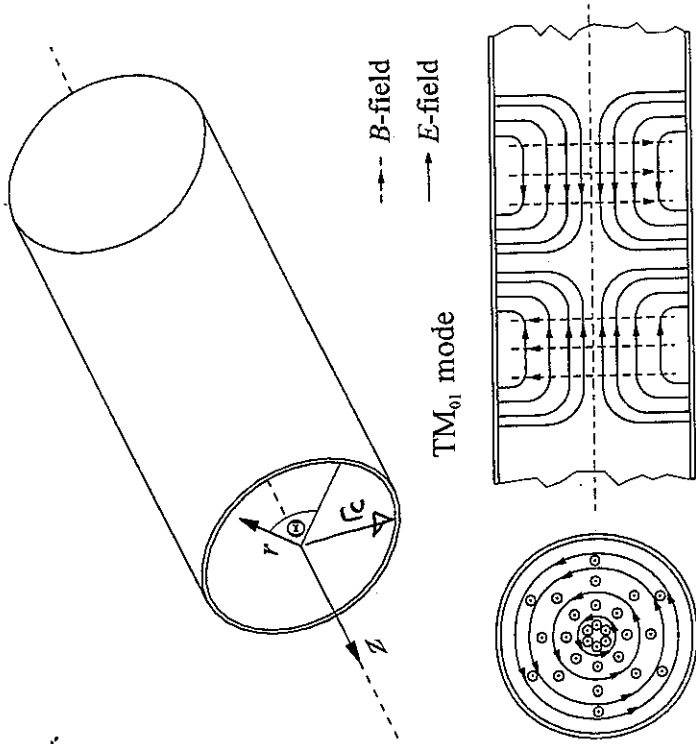


Fig. 5.2 Cylindrical waveguide with TM_{01} wave.

W/ile

Nonzero Fields:

$$E_z = E_0 J_0(k_c r) e^{i(\omega t - kz)}$$

$$E_r = -i E_0 \frac{k}{k_c} J_1(k_c r) e^{i(\omega t - kz)}$$

$$B_\theta = -i E_0 \frac{\omega}{c^2 k_c} J_1(k_c r) e^{i(\omega t - kz)}$$

Nomenclature:

$TM =$ Transverse Magnetic
(Longitudinal E_z)

$TM_{n\theta r}$

$n_\theta =$ azimuthal θ -harmonic E_z
 $= 0 \Rightarrow$ None

$n_r =$ Number radial zeros E_z
 $= 1 \Rightarrow$ One at $r = a$
(min needed for BC, with nonzero sol.)

// Side Point: Traveling wave accelerator works by adding disks to waveguide to slow down EM wave phase velocity to maintain particle resonance!

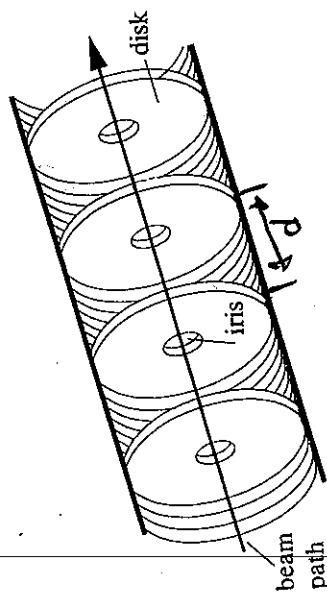
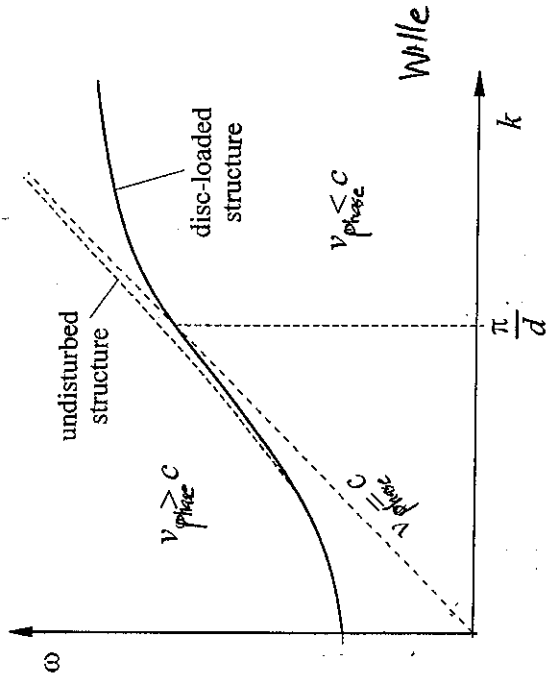


Fig. 2.8. Disk loaded accelerating structure for an electron linear accelerator (schematic) **Wiedemann**



Irises give partial reflections allowing loss free propagation only at RF wavelengths with integer multiples of the iris separation d.

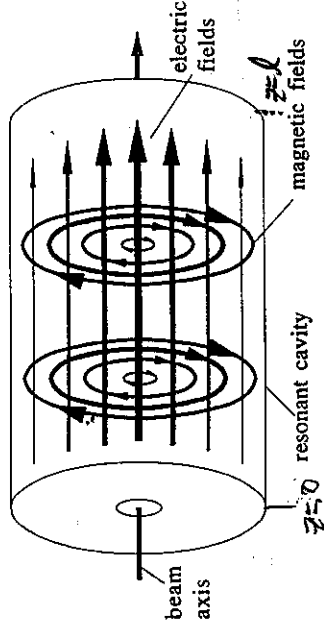
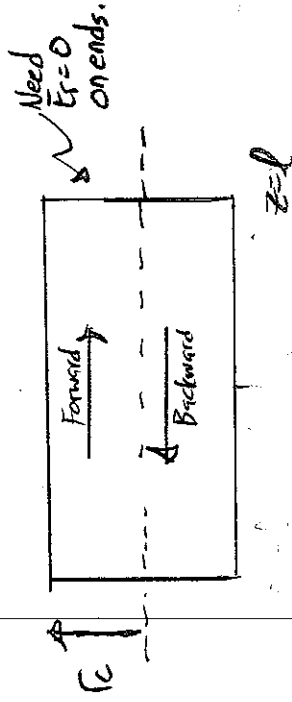
This method is commonly used in e^- accelerators. See Wangler for details.

* water loaded waveguide behaves like (weakly) coupled cavities.

//

So what do we do in our case? Make resonant cavity.

- * Add conducting walls at $z=0, z=l$
- * Superimpose forward and backward waves in cavity to meet boundary conditions and setup standing wave.
- * Time phasing of particles traversing cavity to gain energy and focus.
 - Use formulation developed in earlier notes.



Wiedemann

For cavity: $E_z = E_0 J_0(kr) e^{i(\omega t - kz)} + E_0 J_0(kr) e^{i(\omega t + kz)}$

$\left\{ \begin{array}{l} \text{Forward} \\ \text{Reflected} \\ \text{Backward} \end{array} \right.$

$$E_z \sim E_0 J_0(kr) \cos(kz) e^{i\omega t}$$

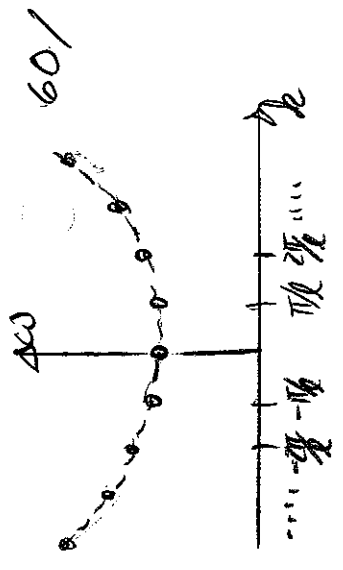
E_0 redefined amplitude.

Some analysis then shows:

$$E_r \sim E_0 \frac{k}{k_c} J_1(kr) \sin(kz) e^{i\omega t}$$

$$\Rightarrow k_z l = n\pi \quad n = 0, 1, 2, \dots$$

for $E_r|_{z=l} = 0$



Thus, for pill box cavity $k = n_z \pi / l$

$$\omega^2 = \omega_c^2 + k^2 c^2 = \omega_c^2 + \left(\frac{n_z \pi c}{l} \right)^2$$

$n_z = 0, 1, \dots$

$$\omega_c = \frac{\chi_{01} c}{r_c}$$

Choose simplest solution with $n_z = 0$

$$TM_{00} n_r n_z = TM_{010} \text{ mode.}$$

Longitudinal index

$$\omega = \omega_c$$

$$k_c = \frac{\omega_c}{c} = \frac{\chi_{01}}{r_c} \quad \chi_{01} \approx 2.405$$

Only discrete values of ω possible, for standing wave

Giving

$$E_z = E_0 J_0(k_c r) \cos(\omega t + \phi)$$

$$B_\theta = -\frac{E_0}{c} J_1(k_c r) \sin(\omega t + \phi)$$

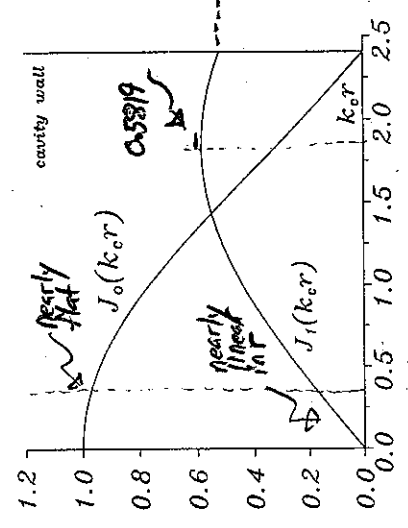
TM_{010} cavity fields

Used phase choices for previous convention ($t=0$ at $z=0$ center of cavity)

Used $k_c = \frac{\omega}{c}$ for $k=0$ in B_θ + reflection terms with consistent phase choices

Comments:

- * All other field terms zero. $E_r = 0$ due to $k=0$
- * Finite beam aperture at ends will allow $E_r \neq 0$, for this mode.



- * Beam will only fill a small fraction of $r_c \Rightarrow k_c r / c \ll 1$
- $J_0(k_c r) \approx 1$ Nearly uniform E_r
- $J_1(k_c r) \approx \frac{k_c r}{2}$; $B_\theta \propto r \Rightarrow$ Linear focus optic. (usually limited impact)

$$k_c = \frac{\chi_{01}}{r_c} \approx 2.405 \frac{1}{r_c}$$

$$J_1(\chi_{01}) \approx 0.519$$

Wiedemann

Notes

Max B_0 at $k_e r = 1.891$ where $J_1(k_e r) = J_1(1.891) \approx 0.5819$

Max $E_r = E_0$ at $r = 0$ where $J_0(0) = 1$

Therefore: $\frac{CB_{Max}}{E_{Max}} = \frac{J_1(1.891)}{J_0(0)} = \frac{0.5819}{1} = 0.5819$

This number can have implications for the cavity field stress/breakdown.

$E_{Max} = E_0$ as large as possible for strong acceleration.

However, larger E_{Max} can trigger breakdown issues and larger

$E_{Max} \Rightarrow$ larger B_{Max} (on cavity ends) which can also induce

a quench for superconducting cavities. Realistic cavities shaped to try to limit these issues.

Full box cavity resonant frequency:

$$\omega = 2\pi f = \omega_c = \frac{X_0 c}{l_c} \quad X_0 \approx 2.405$$

$$f = \frac{2.405 c}{2\pi l_c} \quad \text{Cavity Frequency}$$

Some numbers:

Cavity Freq f	Cavity Diameter $2l_c$
1 MHz	240 m
10 MHz	24 m
50 MHz	5 m
100 MHz	2.5 m
1 GHz	25 cm
3 GHz	8 cm

Higher frequencies desired to limit size of cavities and control cost.

DORIS Storage Ring Cavity
German Electron Synchrotron
Lab DESY

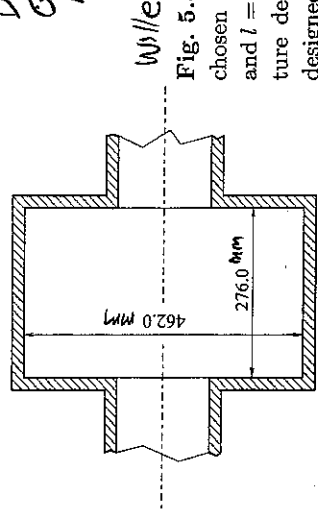
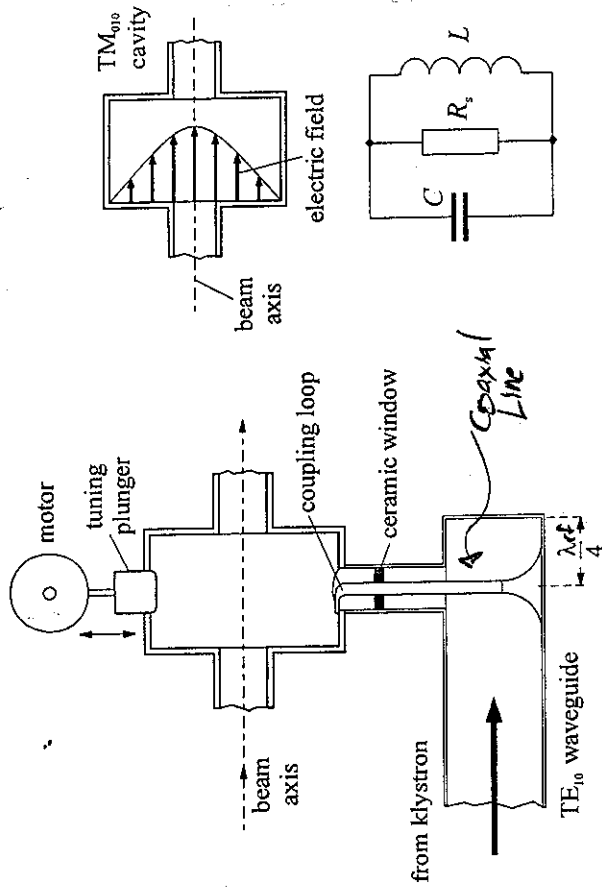


Fig. 5.3 Example of a single-cell cavity. It is chosen to have the dimensions $D = 462$ mm and $l = 276$ mm used in the accelerating structure developed for the storage ring DORIS, designed for a resonant frequency of 500 MHz.

Cavities must be connected to an RF source such as a klystron. Typical connection sketched below.

- Waveguide carries TE₁₀ mode from klystron.
- Waveguide terminated near RF cavity
- Coaxial cable pickup ~ $\lambda/4$ from waveguide termination (~ E max location)
- Connections shaped to inhibit reflections/losses.
- Ceramic window separates waveguide/coaxial cable (normal pressure) from cavity (high vacuum) without impeding RF wave.
- Cavity window technology demanding for high power/voltages?
- RF wave coupled to TM₀₁₀ symmetry of cavity by a loop.

Many details to do optimally: just a brief outline here:



Wille
 Fig. 5.4 Design of a single-cell-accelerating structure using the TM₀₁₀ mode. The exact resonant frequency is adjusted using a tuning plunger. The resonator is excited by an inductive coupling loop.

A stable standing wave will exist in cavity only if the resonance condition of the TM₀₁₀ mode is precisely satisfied.

Following an identification of cavity equivalent circuit parameters, will show that

$$Q = \frac{\omega_{res}}{\Delta\omega} = \frac{R_s}{|Z|} \gg 1 \Rightarrow \Delta\omega \text{ small}$$

ω_{res} = resonant cavity ω
 $\Delta\omega$ = Frequency bandwidth for 1% power.

Cavity Stored Energy: Pillbox Cavity TM₀₁₀ mode

At any given instant in time t the energy stored in an RF cavity is:

$$U = \frac{\epsilon_0}{2} \int_{\text{cavity}} \vec{E}^2 d^3x + \frac{1}{2\mu_0} \int_{\text{cavity}} \vec{B}^2 d^3x = \text{Stored EM Energy}$$

Use Pillbox cavity fields and take $\omega t + \phi = 0$; $U = \text{const}$ so can take any time.

* This choice \Rightarrow all energy in E-field.

$$E_z = E_0 J_0(k_c r) \cos(\omega t + \phi) = E_0 J_0(k_c r)$$

$$B_\theta = -\frac{E_0}{c} J_1(k_c r) \sin(\omega t + \phi) = 0$$

$$k_c = \frac{x_{01}}{lc}$$

and

$$U = \frac{\epsilon_0}{2} \int_{\text{cavity}} E_z^2 d^3x = \frac{\epsilon_0 (2\pi)^2 E_0^2}{2} \int_0^c [J_0(k_c r)]^2 r dr$$

Using

$$\int_0^l t J_n(x_{0j} t) J_n(x_{0k} t) dt = \frac{1}{2} [J_n'(x_{0j})]^2 \delta_{jk}$$

$$\int_0^l dt J_0(x_{01} t)^2 = \frac{1}{2} [J_0'(x_{01})]^2 = \frac{1}{2} [J_1'(x_{01})]^2$$

We have

$$\int_0^{lc} [J_0(\frac{x_{01} r}{lc})]^2 r dr = lc^2 \int_0^1 [J_0(x_{01} t)]^2 t dt = \frac{lc^2}{2} [J_1'(x_{01})]^2$$

$$U = \frac{\epsilon_0 E_0^2}{2} \pi c^2 l [J_1'(x_{01})]^2$$

\Rightarrow

$$U \approx 0.423 \epsilon_0 E_0^2 c^2 l$$

$$J_1'(x_{01}) \approx J_1'(2.405) \approx 0.51911$$

$$\int_{\text{cavity}} dz = l = \text{length cavity}$$

$$\int_{\text{cavity}} d\theta = 2\pi$$

$$J_0'(t) = -J_1(t)$$

Ref: Pick favorite EM Book.

Cavity Dissipation: PML box Cavity

No perfect conductors exist, but conductivity can be high:

Copper $\frac{1}{\delta} \approx 1.7 \times 10^{-8} \Omega \cdot m$

For a good but imperfect conductor, the fields penetrate the conductor in a thin surface layer where they fall off rapidly beyond a "skin depth" δ for fields varying at harmonic frequency ω :

$$\delta = \sqrt{\frac{2}{\sigma \mu \omega}}$$

Copper @ 100 MHz $\Rightarrow \delta \approx 10^{-6} m \sim 1 \mu m$

Because of skin depth AC and DC resistances are not equal.

$$R_{surf} = \frac{1}{\sigma \delta} = \sqrt{\frac{\omega \mu}{2 \sigma}}$$

$\propto [RF \text{ frequency}]^{1/2}$

AC and DC resistance varies.
Copper @ 100 MHz
 $R_{surf} \sim \text{milli-Ohm}$.

Electromagnetic theory texts show that the time averaged power loss to the walls over the RF cycle is given by:

$$\langle P_{loss} \rangle_{RF} = \frac{1}{T_{RF}} \int_0^{T_{RF}} P_{loss} dt = \frac{R_{surf}}{2} \int |\vec{H}_t|^2 ds$$

$\vec{H}_t = \hat{n} \times \vec{H} = \text{Tangential Component } \vec{H} \text{ at conductor.}$
 $\sim e^{-i\omega t}$ vary

Interpretation $H_t \rightarrow$ surface current.
Integrate loss over cavity surface.

Apply this loss formula to RF pillbox cavity:

$$\langle P_{loss} \rangle_A = \frac{R_{surf}}{2} \int |\vec{H}_z|^2 ds$$

will be contributions from



$$\langle P_{loss} \rangle_A = \frac{R_{surf}}{2} \left\{ \begin{array}{l} \text{some} \\ \text{each} \\ \text{end} \end{array} \right. 2 \times 2\pi \left(\frac{E_0}{\rho_0 c} \right)^2 \int_0^c [J_1 \left(\frac{x_{01} r}{r_c} \right)]^2 r dr + \pi c l \left(\frac{E_0}{\rho_0 c} \right)^2 [J_1(x_{01})]^2 \left. \begin{array}{l} \text{side} \\ \text{side} \end{array} \right\}$$

Algebra + Integral Tables

$$\langle P_{loss} \rangle = \pi r_c (c+l) R_{surf} \left(\frac{E_0}{\rho_0 c} \right)^2 [J_1(x_{01})]^2$$

$$\approx 0.847 r_c (c+l) R_{surf} \left(\frac{E_0}{\rho_0 c} \right)^2$$

Typical result:

- * Loss depends on surface resistance (R_{surf}), peak field (E_0) and geometric parameters.
- * Need Low R_{surf} for low losses.

Scaling of R_{surf} :

Normal Conducting:

$$R_{surf} = \sqrt{\frac{\rho_{00}}{2\sigma}} \propto f^{1/2}$$

Copper at $f_A \sim 100$ MHz
 $R_{surf} \sim \text{milli-Ohm}$

Superconducting Niobium

$$R_{surf} = 9 \times 10^{-5} \frac{\mu_0^2 (G/Hz)}{T(0K)} \exp\left(-2 \frac{T_c}{T}\right) \text{ Ohms} + R_{residual}$$

$T_c = 9.2$ K Critical Temp.
 $R_{surf} \sim 10^{-5} \times (R_{surf} \text{ of Copper})$

Ret. Wangler
 From imperfections

Residual = Residual Resistance
 $\sim 10^{-9} - 10^{-8} \Omega$ typical

Quality Factor

Define in full generality (any cavity):

$$\text{Quality Factor} = Q = 2\pi \frac{U}{\langle P_{\text{loss}} \rangle_{1T}} = 2\pi \times \frac{\text{Energy Stored}}{\text{Energy Dissipated in RF Cycle}}$$

$$Q = \frac{\omega U}{\langle P_{\text{loss}} \rangle_{1T}}$$

$$Q = \frac{2\pi U}{1T \langle P_{\text{loss}} \rangle_{1T}} = \frac{\omega U}{\langle P_{\text{loss}} \rangle_{1T}}$$

Pillbox Cavity Q

Using previous results for pillbox cavity

$$Q = \omega \left[\frac{\epsilon_0 E_0^2 \pi r_c^2 l}{2} \left[\bar{J}_1(x_{01}) \right]^2 \right] \frac{1}{\left[\frac{1}{2} \pi r_c (r_c + l) R_{\text{surf}} \left(\frac{E_0}{\rho_0 c} \right)^2 \left[\bar{J}_1(x_{01}) \right]^2 \right]} \quad \text{"U"} \quad \text{"} \langle P_{\text{loss}} \rangle_{1T}$$

$$= \omega \frac{(\epsilon_0 \rho_0 c^2) \rho_0 r_c^2 l}{2 R_{\text{surf}}} \frac{r_c l}{r_c (r_c + l)}$$

$$= \frac{\omega}{c} \frac{c \rho_0 r_c l}{2 R_{\text{surf}} r_c + l}$$

$$Q = \frac{x_{01} \sqrt{\rho_0 \epsilon_0}}{2 R_{\text{surf}}} \frac{1}{1 + r_c/l}$$

Pillbox Cavity

Want very high Q for cavity

$\Rightarrow R_s$ low : good conductor or superconductor

NC Example: DESY DORIS pillbox CU cavity

SC Example: FRIB Quarter Wave SRF Cavity

But at resonant frequency

$$\frac{\omega}{c} = \frac{x_{01}}{r_c}$$

$$x_{01} \approx 2.405$$

$$c^2 = \frac{1}{\mu_0 \epsilon_0}$$

$$x_{01} \approx 2.405$$

$$Q \approx 38,000 \quad \text{c} \approx 500 \text{ MHz}$$

$$Q \approx 10^9 - 10^{10} \quad \text{range.}$$

High-Q corresponds to:

- ★ Low heat generation
- ★ High efficiency
- ★ High stability

∴ to variations in RF drive and beam loading

To understand the stability in oscillatory mode with energy U will change as: point, suppose an isolated cavity has stored energy U_0 in oscillatory mode with angular frequency ω . If the drive is removed the energy U will change as:

$$\frac{dU}{dt} = -\langle P_{loss} \rangle_{rt} = -\frac{\omega U}{Q} \quad \text{since } Q \equiv \frac{\omega U}{\langle P_{loss} \rangle_{rt}}$$

This has solution:

$$U(t) = U_0 \cdot e^{-\omega t / Q} \Rightarrow \text{slow decay for } Q \text{ large, giving good stability}$$

A commonly used Figure of merit of an RF acceleration system is the so-called shunt impedance See Wangler Sec. 2.65

$$V_0 = E_0 L = \text{Effective cavity voltage}$$

$$\text{Shunt Impedance: } R_s \equiv \frac{V_0^2}{\langle P_{loss} \rangle_{rt}}$$

Note Ohm's Law: $V = IR$
 $P = VI = \frac{V^2}{R}$

Cautions: Sometimes defined as $R_s = \frac{V_0^2}{2 \langle P_{loss} \rangle_{rt}}$ due to interpretation of harmonic averaging factors.

Large shunt impedance \Rightarrow

Large acceleration potential relative to cavity dissipation, for economical acceleration.

But due to transit time factor, the accel potential V_0 is not fully imparted to particles. Therefore, define an "effective shunt impedance" to take this into account using synchronous phase $\phi_s = 0$ (Max accel.)

$$\Delta W = q(E_0 L) T \cos \phi_s$$

$$\Rightarrow \Delta W_{Max} = q V_0 T \quad T = \text{Transit Time}$$

Effective Shunt Impedance

$$R_{seff} = \frac{(V_0 T)^2}{\langle P_{loss} \rangle_A} = \left(\frac{V_0}{\langle P_{loss} \rangle_A} \right)^2 T^2 = R_s T^2$$

Sometimes these are analyzed per axial length L for long systems:

$$\frac{R_{seff}}{L} = \frac{E_0^2 T^2}{L \langle P_{loss} \rangle_A} = \frac{(E_0 T)^2}{\langle P_{loss} \rangle_A / L}$$

Typically given in MR/meter

Another figure of merit is "R over Q":

$$R_{over} = \frac{R}{Q} = \frac{R_{seff}}{Q} = \frac{(V_0 T)^2 / \langle P_{loss} \rangle_A}{\omega U} = \frac{(V_0 T)^2}{\omega U}$$

- ★ Measures efficiency acceleration per unit stored energy at specific frequency RF
- ★ Function only of cavity geometry, - Independent of surface properties of power loss.

Energy imparted to beam particles must also come from RF cavity fields.

Instantaneous

Power Delivered by Beam

$$P_B = (\# \text{ Particles}) \cdot \Delta W = \frac{I_{\text{beam}} \Delta W}{\tau}$$

I_{beam} = beam electrical current.

The total average power delivered will be

$$\langle P_{\text{Total}} \rangle_{\text{TA}} = \langle P_{\text{loss}} \rangle_{\text{TA}} + \langle P_B \rangle_{\text{TA}}$$

Take

$$\langle P_B \rangle_{\text{TA}} = \tau_{\text{fill}} \frac{\langle I_{\text{beam}} \rangle_{\text{TA}} \Delta W}{\tau}$$

$$\langle I_{\text{beam}} \rangle_{\text{TA}} = \frac{Q_{\text{bunch}}}{\tau_{\text{TA}}}$$

$$Q_{\text{bunch}} / \tau = N_{\text{bunch}} = \# \text{ particles in bunch}$$

τ_{fill} = Bucket fill fraction in machine pulse



$\tau_{\text{fill}} = 1$ All buckets filled

$\tau_{\text{fill}} = \frac{1}{2}$ Half buckets filled

$$\langle P_{\text{Total}} \rangle_{\text{TA}} = \langle P_{\text{loss}} \rangle_{\text{TA}} + \tau_{\text{fill}} \frac{N_{\text{bunch}} \Delta W}{\tau_{\text{TA}}}$$

The efficiency of the accelerating structure can be

$$E = \frac{\langle P_B \rangle_A}{\langle P_{Total} \rangle_A}$$

Efficiency

For "wall-plug" efficiency must account for other losses:

- * RF Generation
- * Focusing + Bending magnet dissipation
- * Front end

...

More efficient accelerators opens the door for more applications:

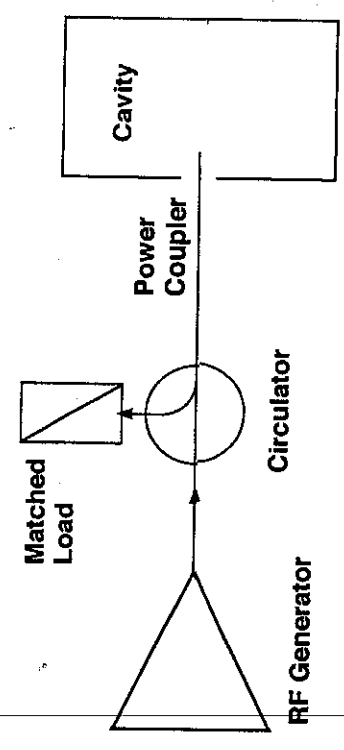
- * Material processing
- * Energy Production : Subcritical reactors, Actinide Burning, Fusion drivers
- ...

Generally want more beam current for high efficiency and this can make/accelerator physics much more difficult due to beam space-charge effects.

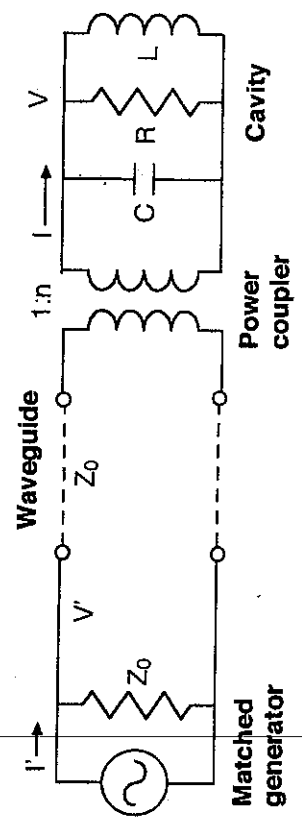
Equivalent Circuit for RF Cavity

Motivated by the qualitative correspondence to circuit parameters for the RF cavity the response of the system is idealized in terms of an equivalent circuit.

Equivalent Circuit



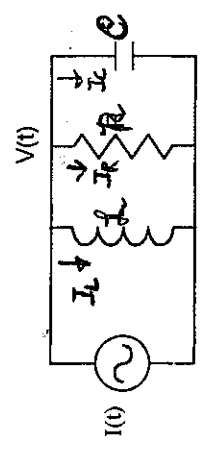
(a)



(b)

Figure 5.3 (a) Block diagram of RF system components and (b) the equivalent circuit.

Cavity Component (idealized)



Wangler

$$I(t) = I_L + I_R + I_C$$

$$= \frac{V(t)}{L} + \frac{V}{R} + C \frac{dV}{dt}$$

$$I = \frac{V}{L} + \frac{V}{R} + C \frac{dV}{dt}$$

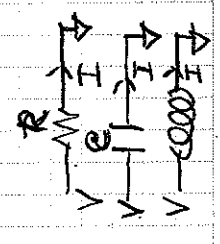
$$V(t) = \text{RF cavity Voltage} \sim E_0 \cdot L$$

Recall:

Resistor:

Capacitors:

Inductor:



$$V = IR$$

$$I = C \frac{dV}{dt}$$

$$V = L \frac{dI}{dt}$$

Driving current $I(t)$ produces voltage $V(t)$

$V(t) \stackrel{\text{fast}}{\sim} V_0 e^{i\omega t} \Leftrightarrow$ Axial accelerating voltage $V = E_0 L$ of cavity, with harmonic variation.

$\frac{1}{2} e V_0^2 = U \Leftrightarrow$ Energy U stored in the cavity. Sets capacitance C

$\langle P_{\text{loss}} \rangle_A = \frac{1}{2} \frac{V_0^2}{R} \Leftrightarrow$ Power lost in cavity. Sets resistance R

Express equation as

$$\ddot{V} + \underbrace{\frac{1}{RC}}_{\text{Damping}} \dot{V} + \underbrace{\frac{1}{LC}}_{\text{Restore}} V = \frac{I}{C}$$

$$\ddot{V} + \frac{\omega_{\text{res}}}{Q} \dot{V} + \omega_{\text{res}}^2 V = \frac{I}{C}$$

$\omega_{\text{res}} = \frac{1}{\sqrt{LC}}$ = Resonant Freq \Leftrightarrow Set ω to get correct angular freq.

$$\frac{1}{RC} = \frac{\omega_{\text{res}}}{Q} \Rightarrow Q = R \sqrt{\frac{C}{L}} = \omega_{\text{res}} \frac{U}{\langle P_{\text{loss}} \rangle_A} = \omega_{\text{res}} RC$$

$$Q = \omega_{\text{res}} \frac{U}{\langle P_{\text{loss}} \rangle_A} = \omega_{\text{res}} RC$$

Search for a harmonic steady-state solution

$$I(t) = I_0 e^{i\omega t}$$

$\omega = \text{const}$ angular freq. (need not satisfy $\omega = \omega_{res}$)
 $I_0 = \text{const}$

Analysis shows that

$$V(t) = \frac{R I_0 e^{i(\omega t + \phi)}}{\sqrt{1 + Q^2 \left(\frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega} \right)^2}}$$

$$\phi = -\tan^{-1} \left[Q \left(\frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega} \right) \right]$$

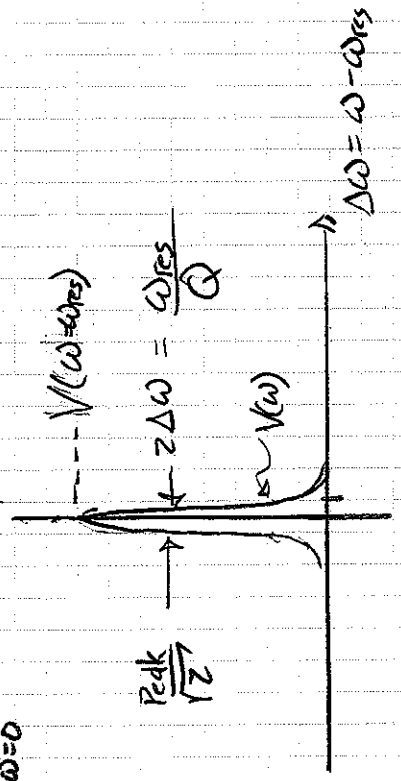
Denote $\Delta\omega = \omega - \omega_{res}$

Then

$$Q \left[\frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega} \right] = Q \left[1 + \frac{\Delta\omega}{\omega_{res}} - \frac{1}{1 + \frac{\Delta\omega}{\omega_{res}}} \right] \approx 2Q \frac{\Delta\omega}{\omega_{res}}$$

The frequency shift $\Delta\omega$ to reduce the voltage amplitude to $1/\sqrt{2}$ the value (i.e., the 1/2 power value) relative to on resonance is:

$$V(t) \Big|_{\substack{\omega = \omega_{res} \\ \Delta\omega = 0}} = R I_0 e^{i\omega t} = R I_0 e^{i\omega_{res} t + \phi} \equiv V(\omega) e^{i\omega t}$$



High Q means very sharply tuned resonant frequency.

Frequency scalings in RF Cavity figures of Merit

Wangler 2.7

One of the most important parameters to choose in design is the cavity frequency.

$$\omega = \frac{2\pi}{T_{FA}} = 2\pi f_{FA}$$

Take:

$E_0 = \text{const}$
 $\Delta W = \text{const}$ } Fixed independent of f_{FA} and fix length

Scale all other cavity dimensions with RF wavelength $\lambda_{FA} = c / f_{FA} = \frac{c}{f_{FA}}$

Then

Transit Time T independent of f

Cavity surface Area $\sim r_c^2 \sim \frac{1}{f_{FA}^2}$

Cavity Volume $\sim r_c^3 \sim \frac{1}{f_{FA}^3}$

Cavity Stored Energy $\sim \frac{1}{f_{FA}^2}$

Surface Resistance

$R_{surf} \sim \begin{cases} f_{FA}^{1/2} & \text{Normal Cond (NC)} \\ f_{FA}^2 & \text{Superconducting (SC)} \end{cases}$

\sim Neglect residual Resistance (good approx at large f_{FA})

Avg. Power Loss $\langle P_{loss} \rangle_{FA} \sim R_s \frac{|B|^2}{\rho_0} S \sim \begin{cases} f_{FA}^{-1/2} & \text{NC} \\ f_{FA}^{-2} & \text{SC} \end{cases}$

Quality Factor $Q = \frac{\omega U}{\langle P_{loss} \rangle_{FA}} \sim \begin{cases} f_{FA}^{-1/2} & \text{NC} \\ f_{FA}^{-2} & \text{SC} \end{cases}$

Effective Shunt "Impedance"

$$R_{s, \text{eff}} = \frac{(V_0 T)^2}{\langle P_{\text{loss}} \rangle_{\text{RF}}} \sim \frac{1}{\langle P_{\text{loss}} \rangle_{\text{RF}}} \sim \begin{cases} f_{\text{RF}}^{1/2} & \text{NC} \\ f_{\text{RF}}^{-1} & \text{SC} \end{cases}$$

* Effective shunt impedance per unit axial length scales same.

R over Q

$$\frac{R}{Q} \equiv \frac{R_{s, \text{eff}}}{Q} \sim \begin{cases} f_{\text{RF}} & \text{NC} \\ f_{\text{RF}} & \text{SC} \end{cases}$$

* R over Q scales same for NC and SC since it should be independent of surface properties.

Phase-space Area Bucket that can be accelerated

$$\sim \frac{1}{f_{\text{RF}}^{1/2}}$$

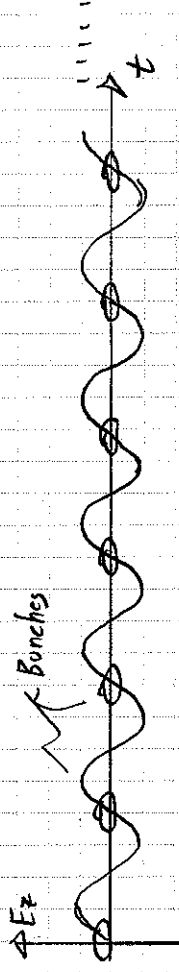
* Higher frequency will lead to lower longitudinal "acceptance" for phase space area that can be accelerated by bucket.

- "Matching" important too if frequency transitions.

Comment: If linac has frequency transitions only harmonics and sub-harmonics are possible for a wave train of RF buckets. In certain cases only certain fractions of buckets will be filled.

RF Bunch Structures

All Buckets Filled Continuous Wave



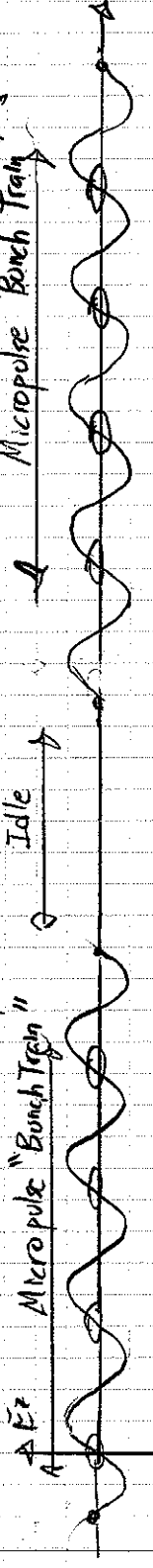
- * Highest intensity on target
- * Max use of RF

50% Buckets Filled Continuous Wave



- * Skip any # buckets to reduce intensity on target.

2 Micro-pulses With all Buckets Filled



"Bunch Train" also called "Batch" in synchrotrons

- * Trains of bunches for consistency with sources etc.

+ Many Variants.

Many reasons for various micro-pulse structures.

- * RF Structure limits in Power (more idle time)
- * Source limitations of particles
- * Frequency changes
- * Target limitations

...

More on Cavities

RF Cavities very diverse topic. Can teach whole course on just aspects of technology.

Beam tube on pillbox cavity adds complication:

- * Want field concentrated on gap for larger transit-time factor.
- * Opening large enough to get beam in and out of cavity \Rightarrow E_r generated.
- * Peak E may no typically no longer on-axis!

SC Cavities
High E_{peak} \Rightarrow Field emission e⁻'s + decreased efficiency + damage possible

NC Cavities
High E_{peak} \Rightarrow Electric Breakdown + cavity damage + loss of E_{acc}

E_{acc} = Accelerating E-field

E_{peak} \sim 2-3 x E_{acc}

Figure of Merit = $\frac{E_{peak}}{E_{acc}}$

- * Resonant cavity angular freq ω more sensitive to cavity dimensions.
- * Large B₀ on outer walls of cavity can quench if superconducting critical magnetic field exceeded. The critical field depends on temperature.

B_{critical} \sim 0.2 Tesla for 2-420K Niobium

Impurities reduce:

B_{max} \sim 0.1 Tesla typical

For pill box cavity $\frac{B_{max}}{E_{max}} = \frac{C B_{max}}{E_{acc}} = 0.5919$

but this value can increase on draft-tubes, nose cones, etc.

Electron Field Emission

Limits SC Cavity E_{Max} ; Wangler 5.10

e^- emitted from surface in strong E field. \Rightarrow strike cavity after gaining energy and generate heat + X-rays when stopping.
lowers Q

Fowler-Nordheim Law:

$$\text{Current Density} \propto \frac{E_{peak}}{\Phi} \exp\left(-\frac{a\Phi^{3/2}}{E_{peak}}\right)$$

Φ = work function
 $\approx 4.3\text{eV}$ for Niobium
 E_{peak} = peak electric field on surface.
 $a = \text{const.}$

$$E_{peak} \sim 250 \times (E_{Max \text{ of Cavity}} \text{ on surface})$$

Due to surface roughness.

Very important for superconducting surfaces to be clean and smooth.

RF Electric Breakdown

Limits NC Cavity E_{Max} ; Wangler 5.11

It is found empirically by Kilpatrick (Rev. Sci. Inst., 28, 824 (1957)).

* for a given freq the peak E field on the surface before breakdown given by

$$f_f (\text{MHz}) = 1.64 E_{Max}^2 \frac{-8.5 E_{Max}}{E_{Max} \text{ in MV/m}} \Rightarrow \text{Plot}$$

* Somewhat conservative, often take $E_{Max} = B (E_{Max \text{ from Kilpatrick}})$
 $B = \text{bravery factor } 1-2 \text{ typical}$

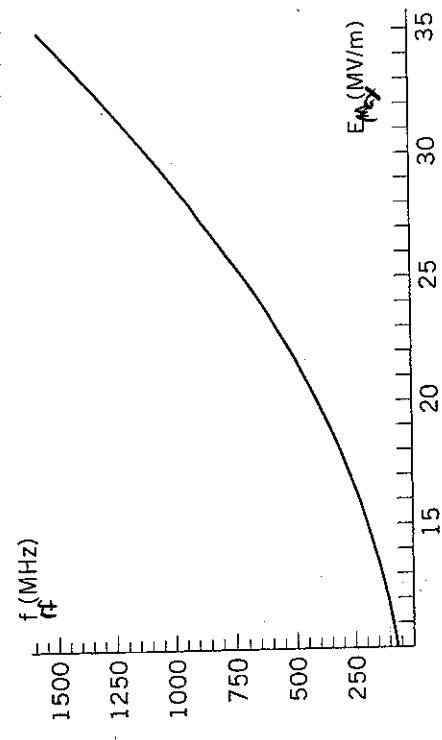
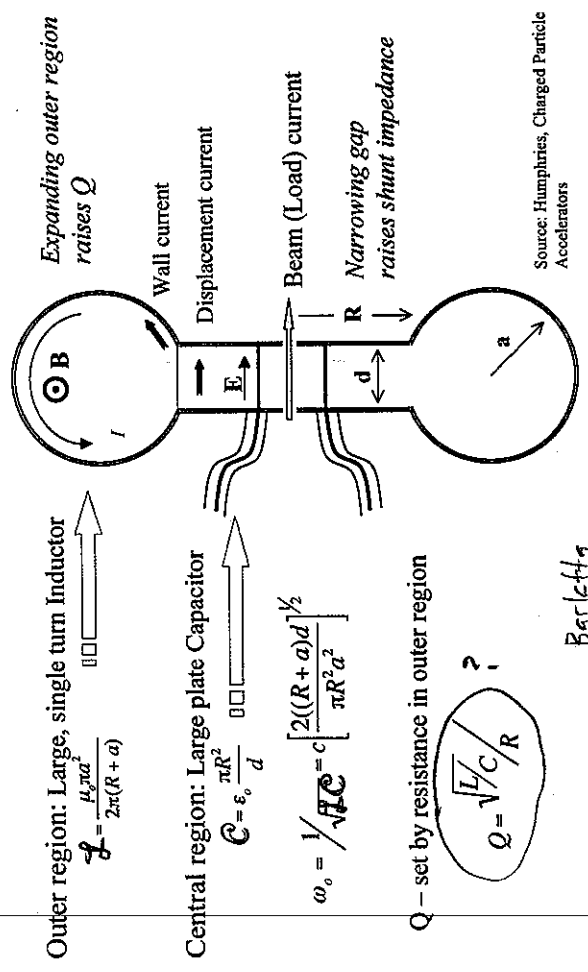


Figure 5.14 Kilpatrick formula from Eq. 15.109. *

Idealized Pillbox cavity is distorted to better optimize.

MIT Translate circuit model to a cavity model:
 Directly driven, re-entrant RF cavity



Outer region: Large, single turn inductor
 $L = \frac{\mu_0 \pi a^2}{2\pi(R+a)}$

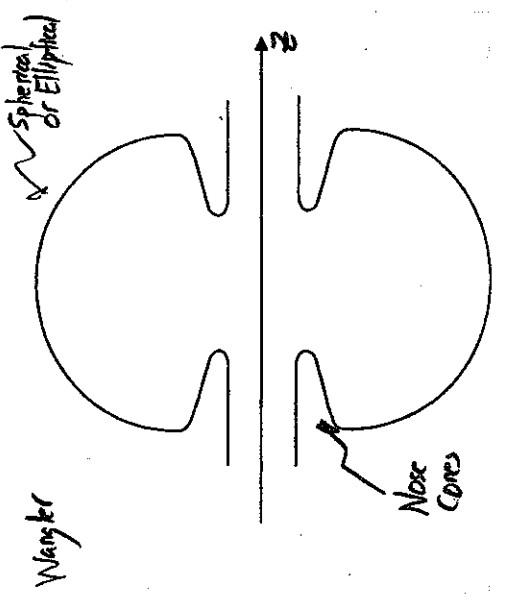
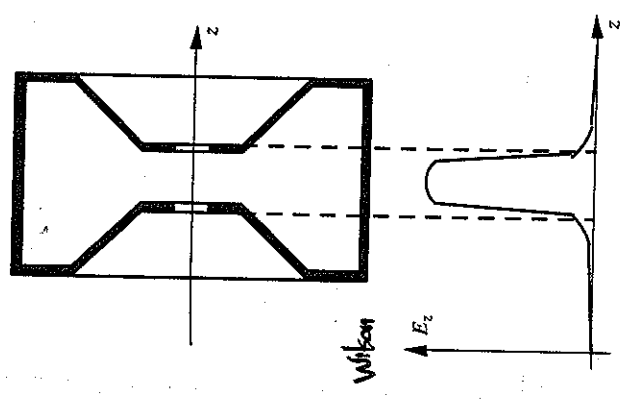
Central region: Large plate Capacitor
 $C = \frac{\pi R^2}{\epsilon_0 d}$

$$\omega_0 = \frac{1}{\sqrt{LC}} = c \left[\frac{2((R+a)d)^{1/2}}{\pi R^2 a^2} \right]^{1/2}$$

Q - set by resistance in outer region
 $Q = \sqrt{L/C/R}$

Source: Humphries, Charged Particle Accelerators

Barlett



Elliptical Cavity

$E_{acc} \sim V_0$
 $U = \frac{1}{2} C V_0^2$
 $\langle P_{loss} \rangle = \frac{1}{2} V_0 / R$
 $\omega_{res} = \frac{1}{\sqrt{LC}}$
 $Q = \omega_{res} \frac{U}{\langle P_{loss} \rangle}$

- want:
- Small gap d for efficient accel.
 - Transit time factor large
 - Raise effective shunt impedance R
 - Expand outer region, raises Q

Coupled Cavities Groups of adjacent RF cavities are coupled together to maintain relative RF Phase control

- * Common for high β particle acceleration
 - Simplifies RF drive
 - Saves cost
 - Many possible geometries
- * Coupling can be through beam apertures or slots, or sometimes special coupling cavities
 - Coupling cavities sometimes off axis or minimal length to save space
- * Usually transverse focusing placed between banks of cavities.

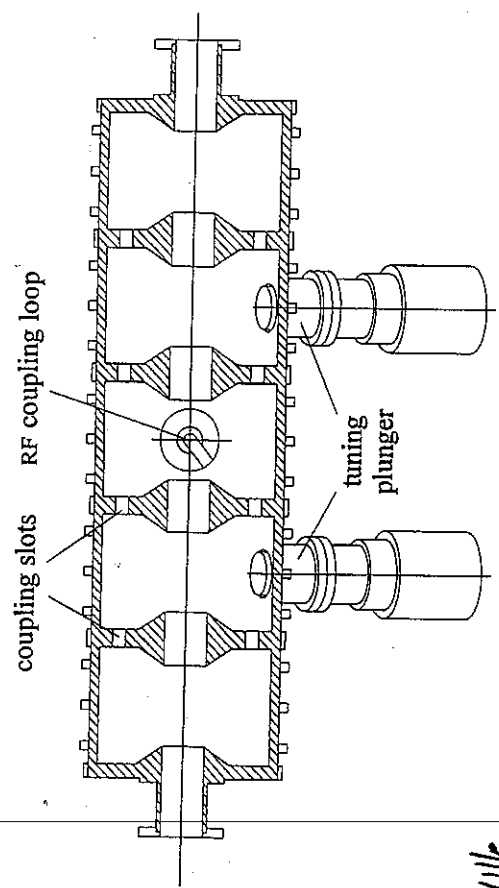
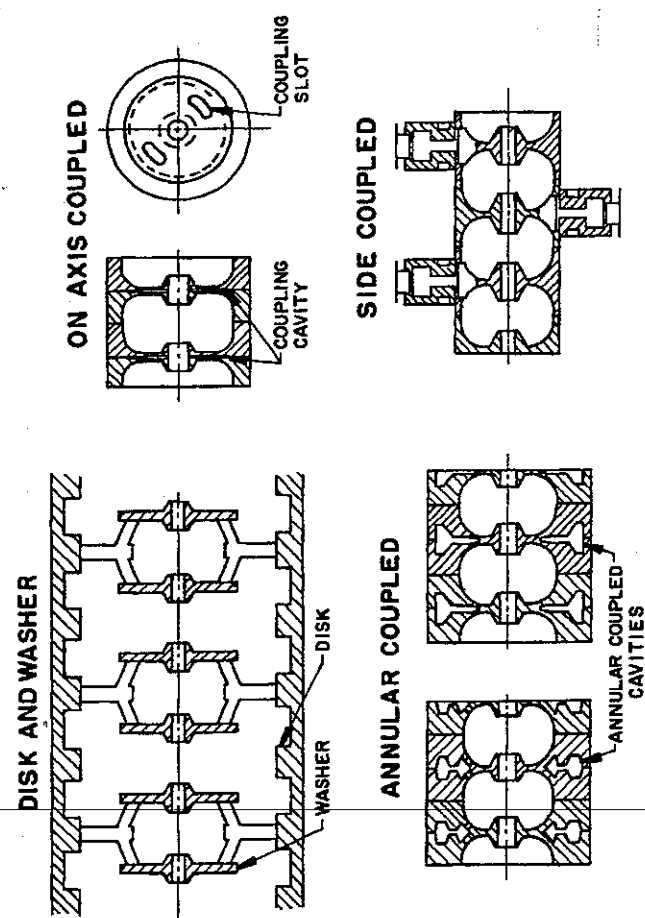


Fig. 5.5 Layout of a five-cell accelerating structure. The power feed is coupled to the middle cell and two tuning plungers are sufficient for the entire structure.



Four examples of coupled-cavity linacs.

Elliptical: 5-cell Bank

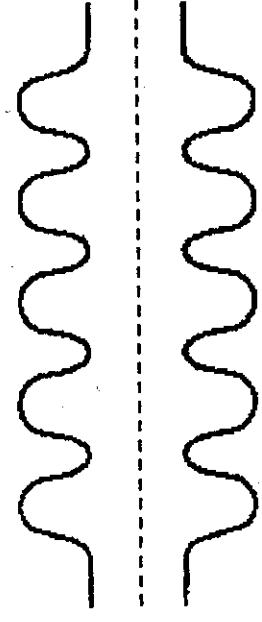
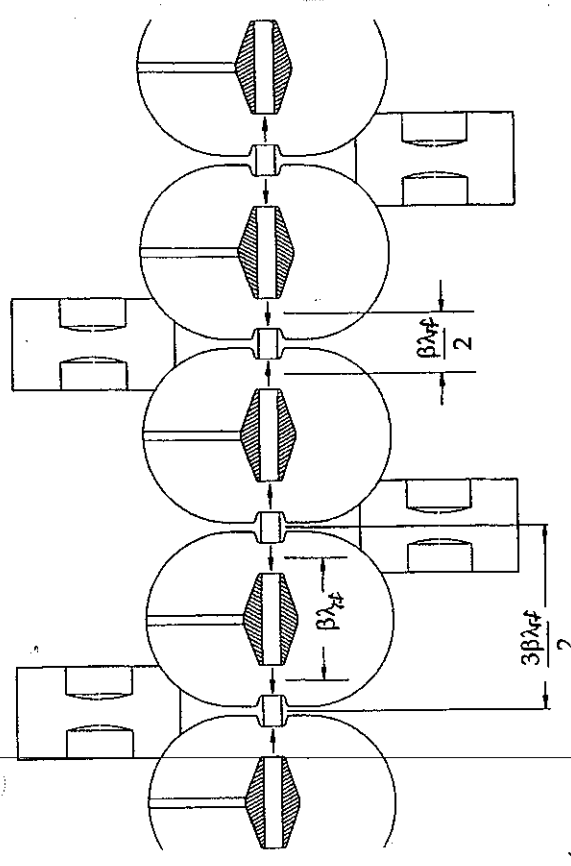
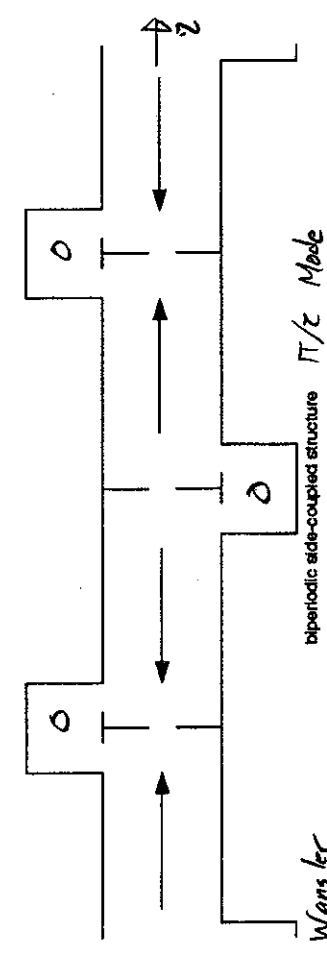
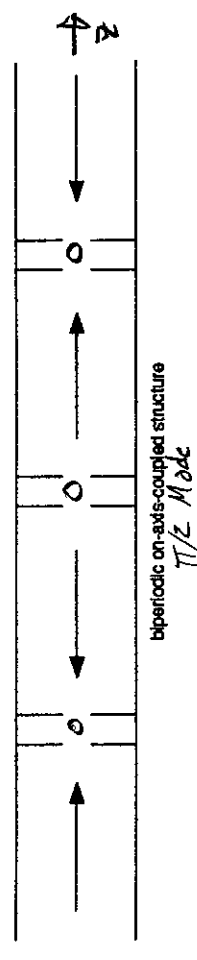
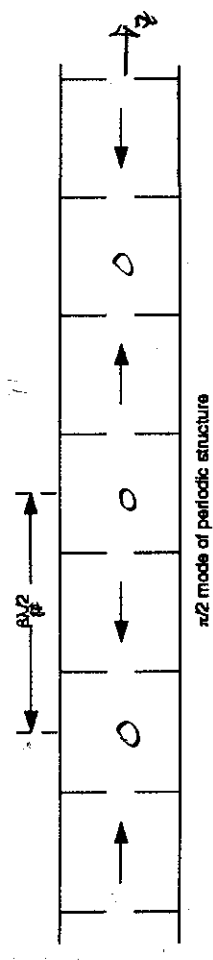


Figure 2.5 Cross section of a $\beta = 0.82$ elliptical cavity designed for a superconducting proton linac. The cross section for each cell consists of an outer circular arc, an ellipse at the iris, and a connecting straight line.

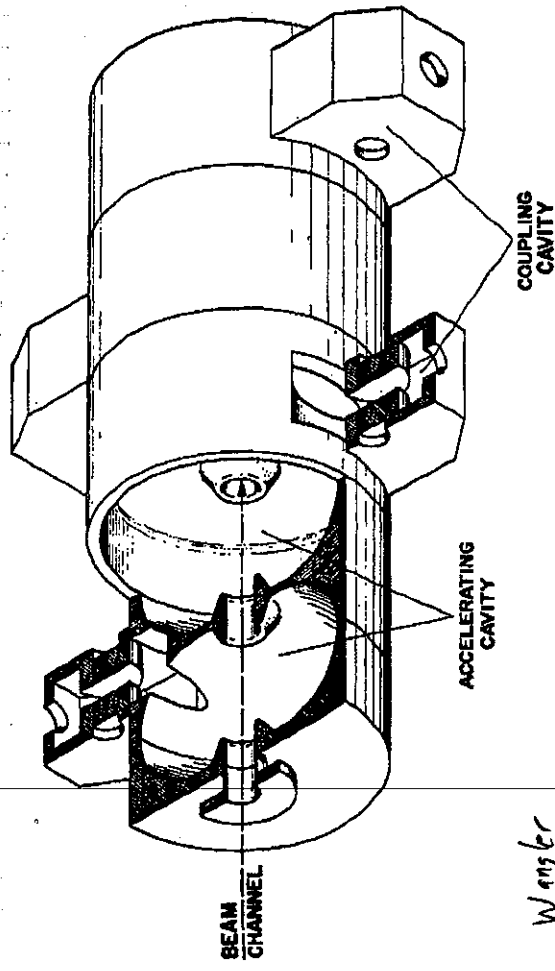
Phase relations between E-fields in cavities can vary.



Wangler
Figure 12.2 Coupled-cavity drift-tube linac (CCDTL) structure with a single drift tube in each accelerating cavity.



Wangler
Figure 4.15 $\pi/2$ -like-mode operation of a cavity resonator chain. From top to bottom are shown a periodic structure in $\pi/2$ mode, a biperiodic on-axis coupled-cavity structure in $\pi/2$ mode, and a biperiodic side-coupled cavity in $\pi/2$ mode.



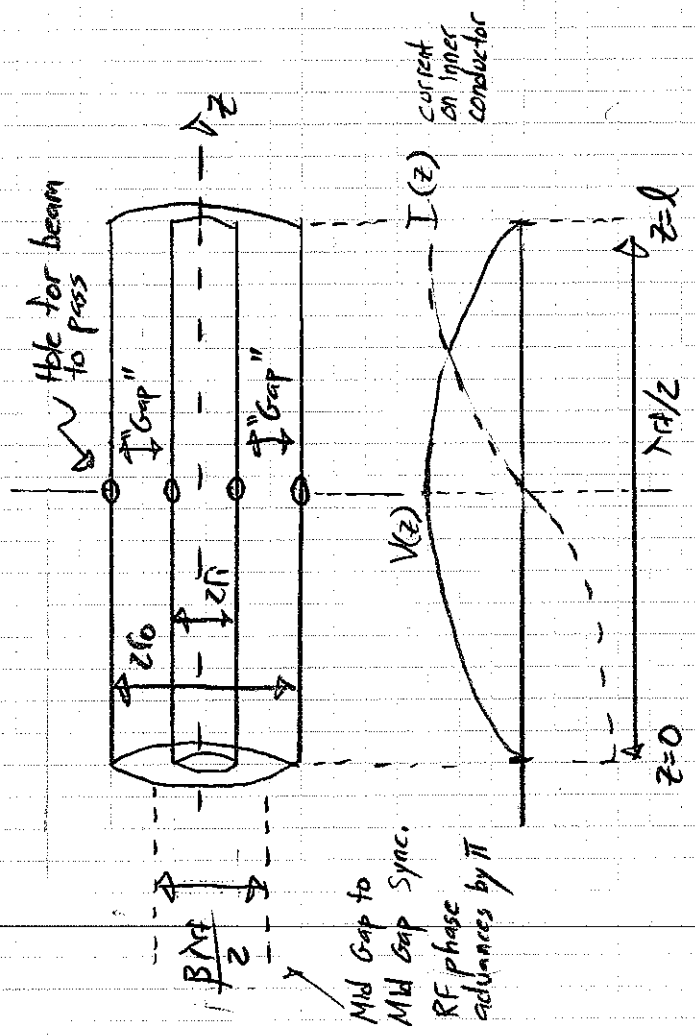
Wangler
Figure 4.11 Side-coupled linac structure as an example of a coupled-cavity linac structure. The cavities on the beam axis are the accelerating cavities. The cavities on the side are nominally unexcited and stabilize the accelerating-cavity fields against perturbations from fabrication errors and beam loading.

Low frequency Half and Quarter Wave RF Structures

For low freq. ion acceleration with cavities operating with $f_{res} \approx 100$ MHz, cavities based on coaxial resonators are employed.

* Used in FRIB, $1/4$ and $1/2$ wave SRF cavities.

Basic Idea: Half Wave Structures



Will show on a homework problem that an EM standing wave solution exists with

$$E_r = -2\sqrt{\frac{\rho_0}{\epsilon_0}} \cdot \frac{I_0}{2\pi r} \sin\left(\frac{\pi r z}{l}\right) \sin(\omega t + \phi)$$

$$B_\theta = \frac{\rho_0 I}{\pi r} \cos\left(\frac{\pi r z}{l}\right) \cos(\omega t + \phi)$$

$p = 1, 2, 3, \dots$

$p = 1 \Rightarrow$ Half-Wave

$$\omega = \frac{\pi c}{l}$$

$I_0 =$ Amplitude of traveling wave current components on inner conductor.

$$V = \int_{r_i}^{r_o} E_r dr = \text{Accel. Voltage.}$$

- * Beam holes at $z = l/2$ where voltage is maximum
- * Beam moves on radial path sees no field when inside inner conductor (like drift tube).
- * RF phase advances by π when traversing the inner conductor so that the particle can be accelerated on both entrance and exit sides.
- * Conductor radii chosen for max energy gain on each side
- * Effectively forms 2 gap cavity. 2 gap Transverse factor of HV problems applies.

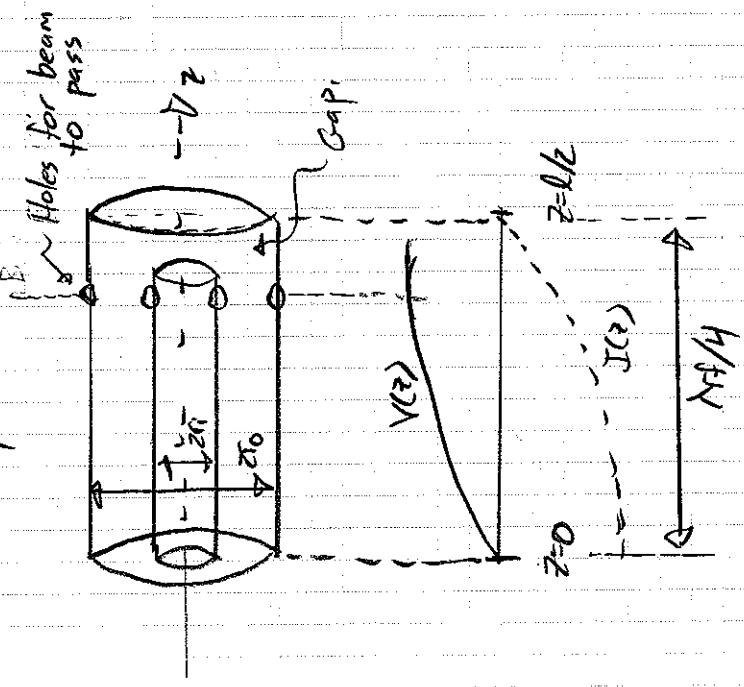
Will also show in homework problems for the $\frac{1}{2}$ -wave coaxial resonator;

$$U = \frac{\rho_0 L I_0^2 \ln(\rho_0/\rho_i)}{2\pi} \quad \text{RF Energy Stored}$$

$$Q = \frac{P_{\text{in}}}{R_{\text{surf}}} \sqrt{\frac{\rho_0}{\epsilon_0}} \frac{\ln(\rho_0/\rho_i)}{L \left[\frac{1}{\rho_i} + \frac{1}{\rho_0} \right] + 4 \ln(\rho_0/\rho_i)} \quad \text{Quality Factor}$$

Quarter Wave Structure

Essentially split the half-wave structure divided in two with a capacitive termination at the division point.



* Has a lesser degree of symmetry and fields will be distorted more than in the half-wave resonator.

Design formulas including the contribution to the fields from the capacitive gap termination can be found in

Moreno, Microwave Transmission Design Aids, Dover, NY 1948, pp. 227-230.

Both Quarter and Half-Wave structures produce more compact low freq. cavities;
 * Save RF power
 * Cheaper Superconducting (less materials less losses to cool...)

Coupling to RF Cavities

See Wille, "The Physics of Particle Accelerators", Chapter 5
 Wilson, "An Introduction to Particle Accelerators", Chapter 5
 Wangler, "RF Linear Accelerators", Chapter 5

Beyond scope to discuss in this class.

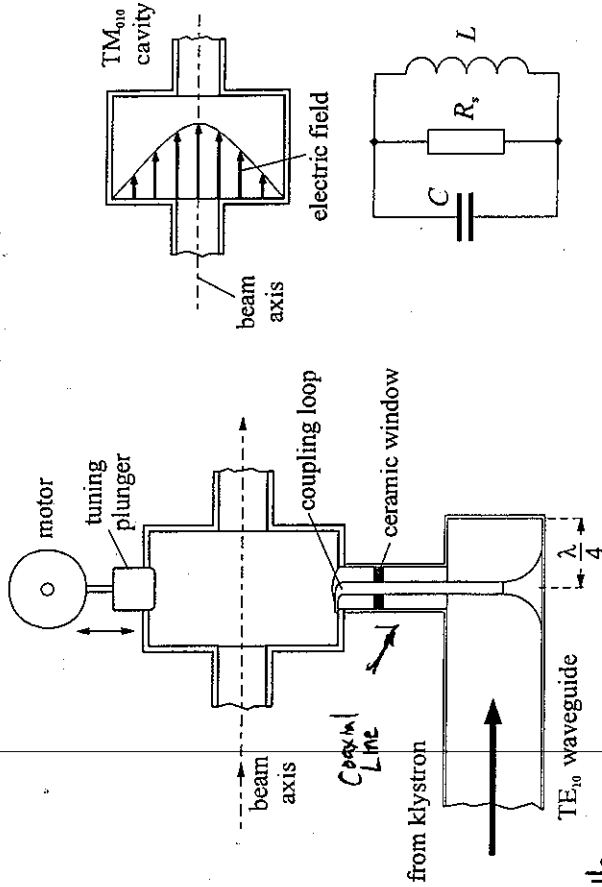
Many ways to couple RF power to resonant cavities.

Most common may be with a loop to couple with magnetic field of EM TM₀₁₀ type standing wave.

Place where magnetic field high in outer radial extent of cavity
 * Field created by loop should have component in common with B₀ of TM₀₁₀ type mode (or whatever mode) desired to excite.

Coupling of klystron to waveguide + coaxial cable also an issue. Much to consider.

Magnetic Coupling Loop at end of Coaxial Transmission Cable



Wille

Fig. 5.4 Design of a single-cell accelerating structure using the TM₀₁₀ mode. The exact resonant frequency is adjusted using a tuning plunger. The resonator is excited by an inductive coupling loop.

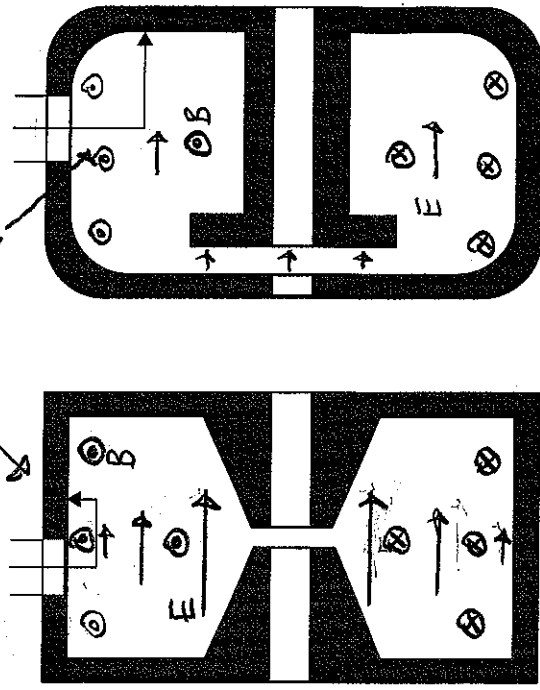


Fig. 10.15 Two examples of loop coupling.
 Wilson

TM₀₁₀ type mode

Common Methods Coupling

- 1) Magnetic Loop at end of coaxial transmission line connected to cavity
- 2) Hole or Aperture in cavity wall connected to a wave guide
- 3) Electric Coupling Probe or Antennas using the central conductor of a coaxial transmission line.

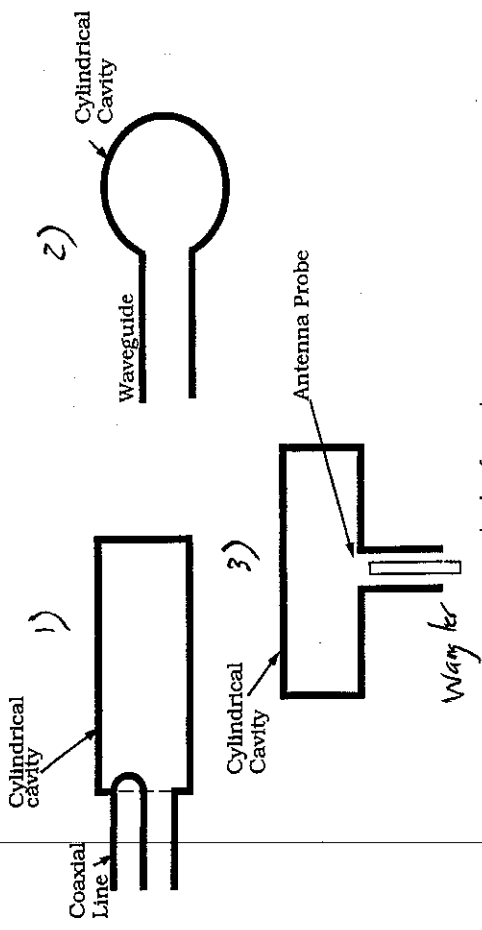


Figure 5.2 Methods of coupling to cavities.

RF Sources

See Wille, "The Physics of Particle Accelerators" Chapter 5
 Wilson, "An Introduction to Particle Accelerators" Chapter 5

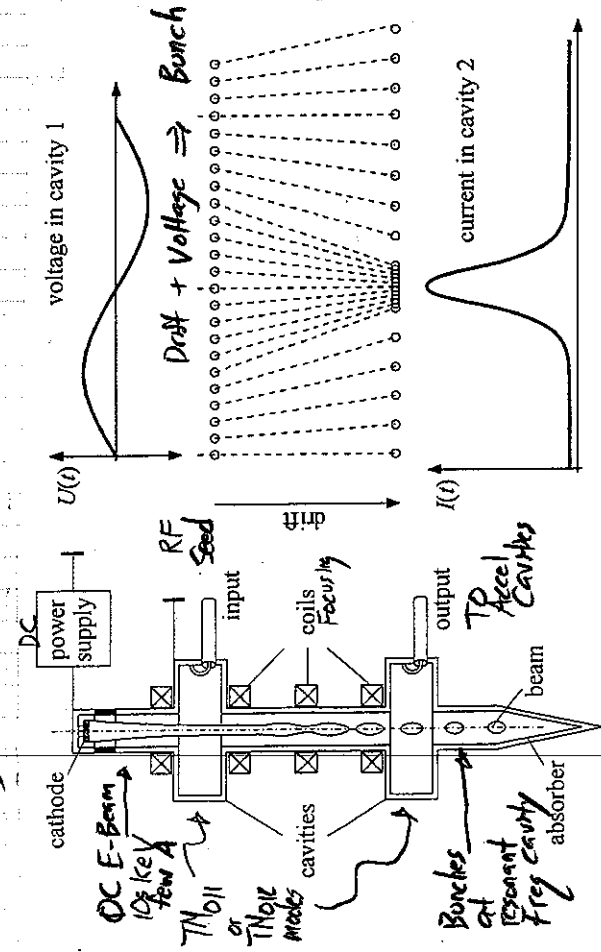
Harmonically varying RF power needed for accelerating structures ranging from a few kW to MW power levels. Pulses may be short, long, or continuous wave (CW).

- 1) Triode / Tetrode: few MHz \rightarrow few 100 MHz ; high power broad band
- 2) Klystron: few 100 MHz +
- 3) Also: Traveling Wave Tubes, Magnetrans, Cross-Field Amplifiers, Gyrotrons,

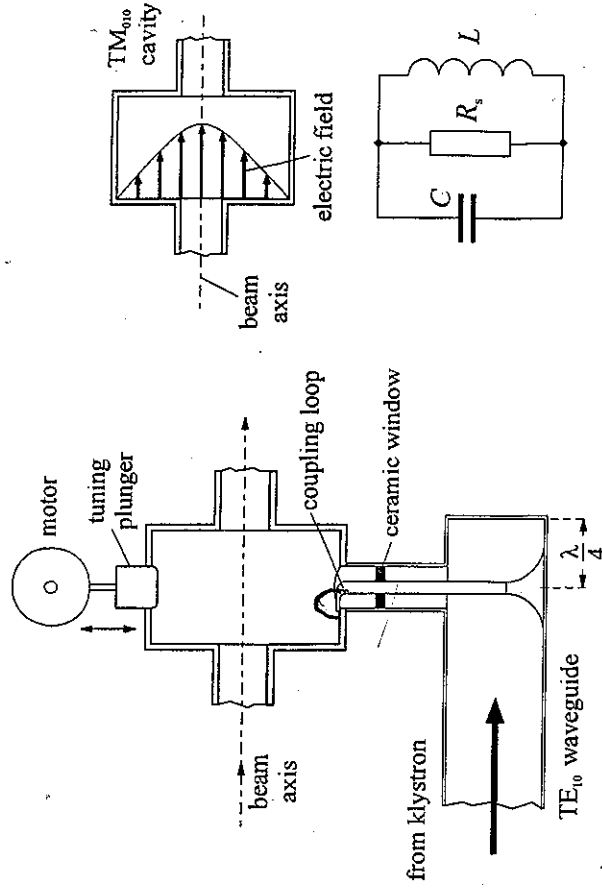
Klystron

Drift long enough to bunch
 using TM_{011} or TM_{012}

Most common for Accel. Applications



Wille Fig. 5.11. The classical microwave klystron, operating in the ten centimetre region.



Wille Fig. 5.4 Design of a single-cell accelerating structure using the TM_{010} mode. The exact resonant frequency is adjusted using a tuning plunger. The resonator is excited by an inductive coupling loop.

Power delivered by klystron

e⁻ beam source large!

$I_{beam} \sim 10 A$ typical
 $V \sim 10^5$ kV Source Voltage typical

$$P_{Klystron} = \eta V I_{beam}$$

$$\eta = \text{Efficiency} \quad 45\% \rightarrow 65\% \text{ typical}$$

~ 1.2 MW per tube now achieved in CW operation.

Real klystrons may use several resonators.

Many variants including relativistic klystrons using higher (MeV) energy e⁻ beams.

Numerous topics on RF sources, coupling, measurements, engineering.
 Many texts exist on topic. Often other books on Engineering.
 Additional important topics:

- * Microwave coupling to cavities / waveguides
- * Slater perturbation theorem - band pull of small metal structure used to measure cavity frequencies.
- * Cavity tuning: usually via detuning

The US Particle Accelerator School regularly offers courses on

Microwave Sources

Microwave Measurements and Beam Instrumentation

Microwave Linear Accelerators

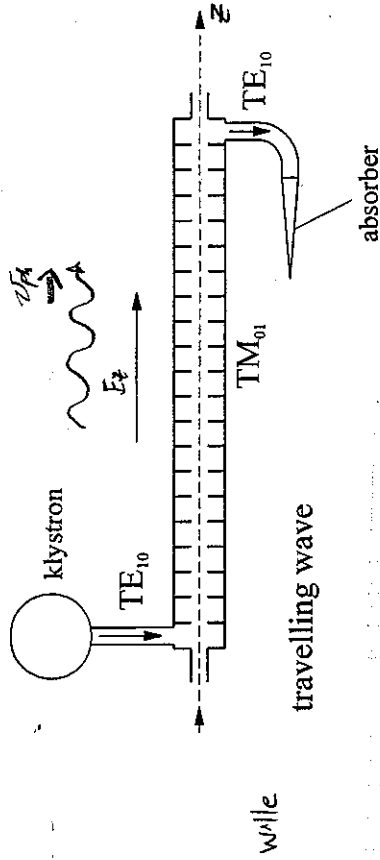
as part of the core curriculum. These courses overview the topic from an accelerator perspective.

Traveling Wave RF Acceleration

See for more info:
Wangler, RF Linear Accelerators, Chapter 3
Wille, The Physics of Particle Accelerators, Chapter 3.

Rather than exchanging energy with an RF standing wave in a fixed cavity, a traveling RF wave can be setup to resonate with a particle to accelerate it.

Consider an EM wave propagating along \hat{z} in a waveguide structure:



$$E_z(\vec{z}, t) = E(z) \cos[\omega t - \int_0^z k(z) dz + \phi]$$

$$k(z) = \frac{\omega}{v_p(z)}$$

$$v_p = \text{phase velocity of wave.}$$

For efficient acceleration of a particle with axial velocity v_z , want

$$v_z \approx v_p$$

for efficient acceleration.

Consider a particle of charge q with $v_z = v_{ph}$ at each instant in time. Then the particle arrives at position z at time

$$t(z) = \int_0^z \frac{dz}{v_z(z)}$$

So the E_z at the particle is:

$$E_z = E(z) \cos[\omega t - \int_0^z k(z) dz + \phi]$$

$$= E(z) \cos\left[\omega \int_0^z \frac{dz}{v_z(z)} - \omega \int_0^z \frac{dz}{v_0} + \phi\right] = E(z) \cos \phi$$

In the traveling wave approach, this particle is called the synchronous particle and $\phi = \phi_s$. The synchronous particle will gain kinetic energy.

$$\Delta W_s = q \int_0^z E(z) \cos \phi_s dz$$

As previously discussed, $v_p > c$ for a simple cylindrical pipe waveguide. But periodic structures can be placed in the waveguide to produce partial reflections to reduce the phase velocity to satisfy $v_p \approx c$. The structure can be analyzed as a periodic array of coupled resonant cavities. Various techniques exist to analyze this situation and gain intuition; See Wangler Chapter 3.

Iris Loaded Waveguide for $v_p < c$

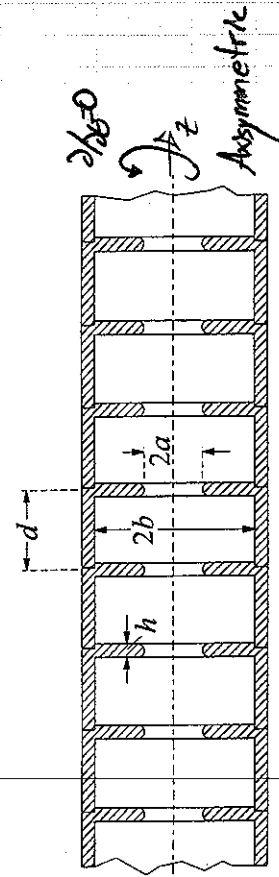


Fig. 5.6 Cross-section through a typical linac structure. The phase velocity of the RF wave is reduced to the particle velocity by the insertion of irises.

Modified Dispersion Curve

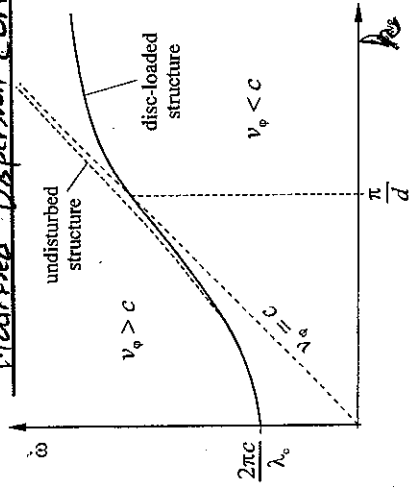


Fig. 5.7 Dispersion curve for a cylindrical waveguide, with and without irises. The frequency ω is plotted as a function of the wavenumber k_z of the waveguide.

Wille

Analogously to cavities in a linac various wave symmetries are compatible with acceleration!

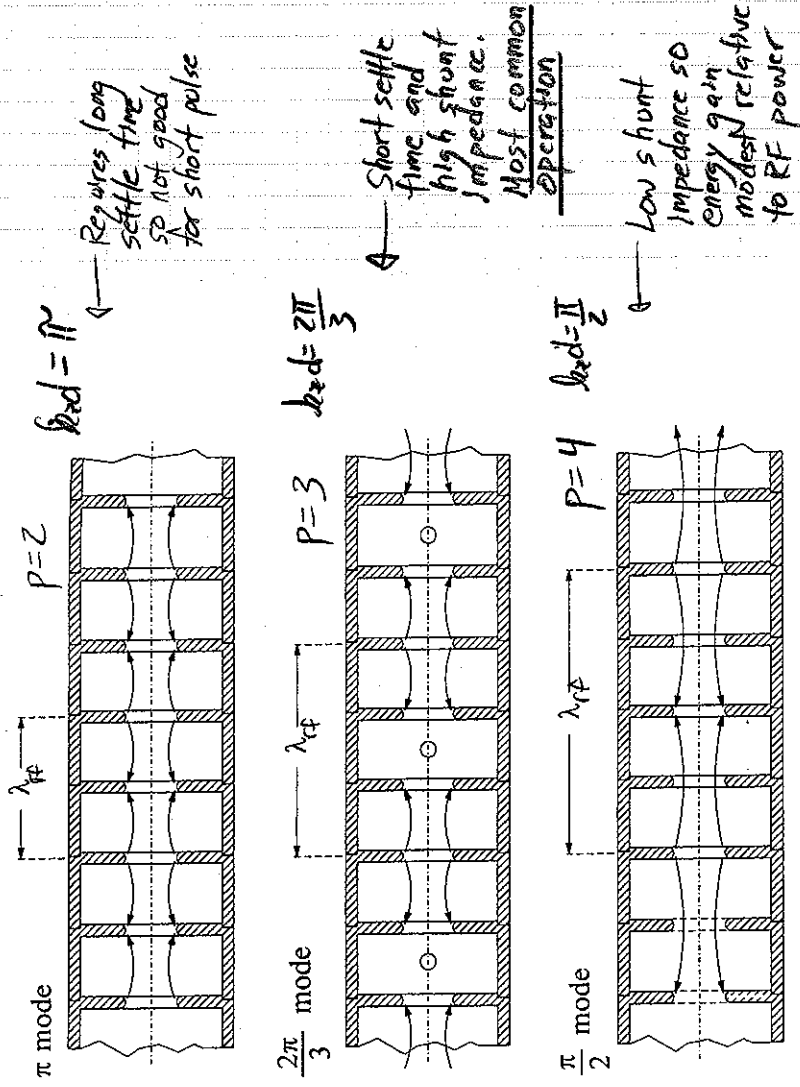
In smooth waveguides a) & above cutoff allowed.

For disc loaded waveguide only

$\lambda_{rf} = p d$ $p = 1, 2, 3, \dots$
 $d = \text{iris separation}$

allowed.

$\Rightarrow \frac{2\pi}{p} = \frac{2\pi}{\lambda_{rf}} d = k_z d$



Wille

Fig. 5.10 The field configurations of the three most important modes in linac structures.

Analysis shows the max energy gain of a particle : see will

- * Max \Rightarrow Synchronous condition : $2\beta = 2\beta_p$
- * Note: Traveling wave structures often used for relativistic electrons with $\beta \approx 1$, so there is no need for phase focusing and operation is for max acceleration. The only need is for the bunch to fit into the bucket.

$$\Delta W = K \cdot \sqrt{P_{RF}} \cdot l \left(\frac{R_s}{L} \right)$$

Max Kinetic Energy Gain

K = correction factor ≈ 0.8 typical
 P_{RF} = Supplied RF Power
 l = Interaction length
 $\left(\frac{R_s}{L} \right)$ = Shunt Impedance per meter of structure.

Empirical formula applied that works well.

$$\left(\frac{R_s}{L} \right) = 5.12 \times 10^8 \cdot \frac{\beta(1-\beta)^2}{p + 2.61\beta(1-\beta)} \left(\frac{\sin D/C}{D/C} \right)^2$$

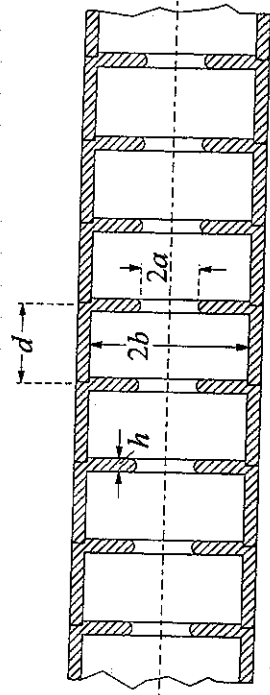
$$\beta = v/c$$

$$\rho = \frac{q}{P}$$

$$P = \# \text{irises per wave length}$$

$$D = \frac{2\pi}{P}(1-\rho)$$

SLAC Structure



- $2b = 82.5 \text{ mm}$
- $2a = 22.6 \text{ mm}$
- $h = 5.5 \text{ mm}$
- $d = 35.0 \text{ mm}$

For the SLAC
travelling wave
linacs:

$\beta \approx 1$ Relativistic e^-
 $P = 3$ $\frac{2\pi}{3}$ Mode

$$\left(\frac{R_s}{L}\right) = 53 \times 10^6 \frac{\Omega}{\text{meter}}$$

$$P_{RF} = 35 \text{ MW}$$

$$l = 3 \text{ meter}$$

$$\Delta W = k \sqrt{P_{RF} l \left(\frac{R_s}{L}\right)} = 59.7 \text{ MV}$$

$$\Rightarrow \frac{\Delta W}{l} = 19.9 \frac{\text{MV}}{\text{meter}}$$

GeV/s

Typical Values achieved for travelling wave structures. $15 \frac{\text{MV}}{\text{m}} \rightarrow 20 \frac{\text{MV}}{\text{m}}$

Values $\sim 100 \frac{\text{MV}}{\text{m}}$ possible for short structures.

Typically only short pulses possible to avoid heat damage to normal conducting structure.

Want very high gradients for any future linear e^+e^- collider?