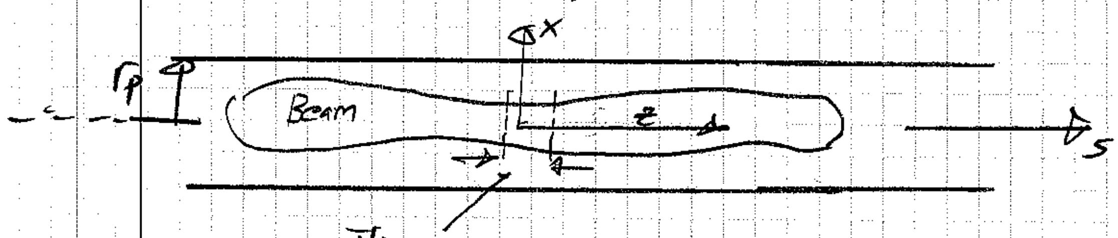
PHY 905
Fundamentals of Accelerator Physics
18. lecture. pdf

Space Charge Physics : A Very Basic Introduction

Assume beam bunch long relative to machine apeture



x-z system moving with beam

$\frac{\partial}{\partial z} \approx 0$ Axial changes in beam weak.

Then we can apply the equation of motion we derived earlier in ??. lecture. pdf

Assume:

- ★ No axial momentum spread: All particles move with axial velocity βc
- ★ One species of charge q and mass m
- ★ No bends
- ★ No nonlinear applied focusing.

$$\boxed{\begin{aligned} &\frac{d^2}{ds^2}x + \kappa_x(s)\,x = \frac{-q}{m\gamma^3\beta^2c^2}\frac{\partial\phi}{\partial x} \\ &\frac{d^2}{ds^2}y + \kappa_y(s)\,y = \frac{-q}{m\gamma^3\beta^2c^2}\frac{\partial\phi}{\partial y} \\ &\text{Poisson Eqn.}\quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0} \qquad \rho = \text{beam charge density} \\ &\qquad + \text{Boundary Conditions on } \phi \end{aligned}}$$

κ_x = lattice function x-plane

κ_y = lattice function y-plane.

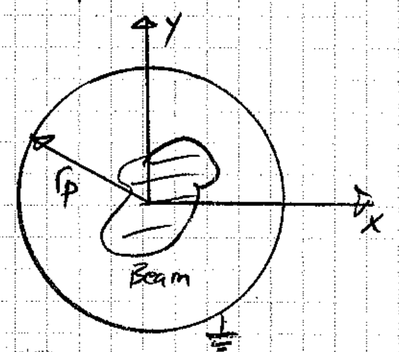
In $\frac{1}{\gamma^3}$ factor;
$\frac{1}{\gamma^2}$ from Bo gen
$\frac{1}{\gamma}$ from kinematics

2/

wher we take:

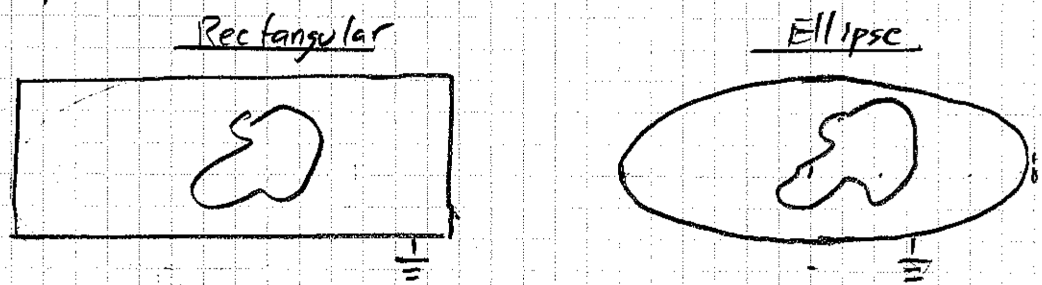
Quadrupole Focusing:	$\kappa_x = -\kappa_y = \dfrac{B'}{(B\rho)}$	
Solenoid Focusing:	$\kappa_x = \kappa_y = \left(\dfrac{B_{z0}}{2(B\rho)}\right)^2$	[In Larmor Frame]
Continuous Focusing	$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const.}$	

Assume the beam pipe is a circular cylinder to be simple:



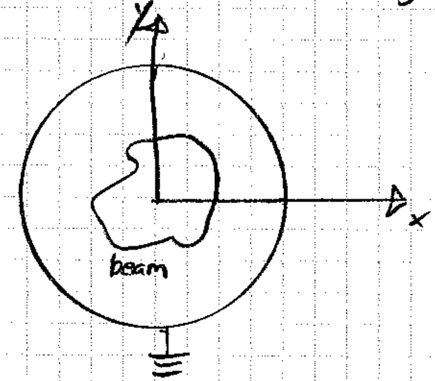
r_p = Pipe Radius.

Other shapes sometimes used:



★ To allow more beam clearance in specific situations like dispersive bends.

To find effect of space-charge fields, the Poisson Equation must be solved



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\varepsilon_0}$$

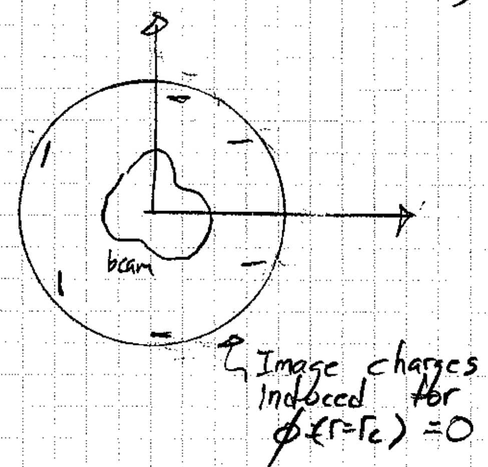
$\phi = 0$ on boundary

$\rho = q n$ = charge density

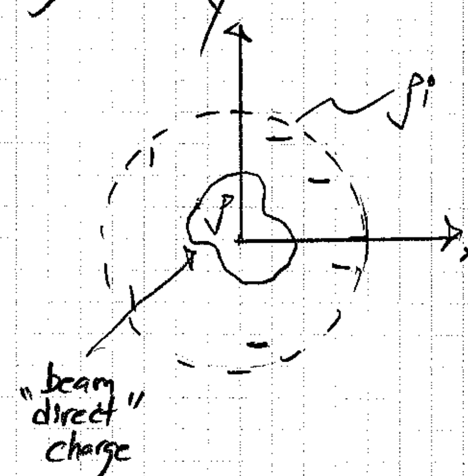
n = number density

continuous density approx (many particles)

Resolve the self-field problem using "method of images"



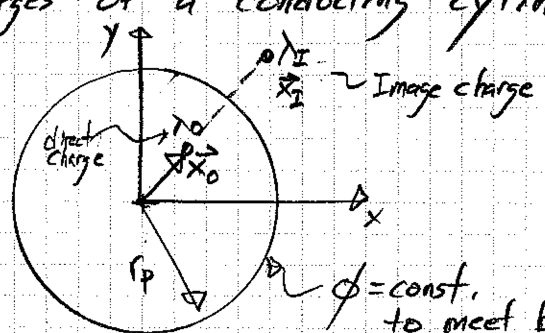
$\Longrightarrow$ Replace



Direct + Image charges in free-space.

* Replacement exact for $r < r_c$.
* Images must be right to meet $\phi(r=r_c)=0$ boundary condition.

Image charges of a conducting cylinder



$\lambda_I = -\lambda_0$ (Image = −Direct)

$$\vec{x}_I = \frac{r_p^2}{|\vec{x}_0|}\vec{x}_0$$

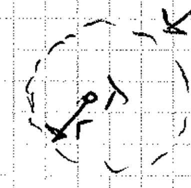
Field of a line charge λ at origin $r = 0$

$$\nabla_\perp \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \rho = \frac{\lambda\,\delta(r)}{2\pi r}$$

Apply Gauss' Law

$$E_r = \frac{\lambda}{2\pi\varepsilon_0 r}$$

2D Cylinder



$$\int \vec{E}_r \cdot \hat{r}\, dA = \frac{\lambda}{\varepsilon_0} = \frac{Q_{in}}{\varepsilon_0}$$

$$2\pi r E_r = \frac{\lambda}{\varepsilon_0}$$

When line charge λ at $\vec{x}_\perp = \tilde{\vec{x}}_\perp$, by symmetry

$$\vec{E}_\perp(\vec{x}_\perp) = \frac{\lambda}{2\pi\varepsilon_0} \frac{\vec{x}_\perp - \tilde{\vec{x}}_\perp}{|\vec{x}_\perp - \tilde{\vec{x}}_\perp|^2}$$

Now apply to our beam + Image problem using linear superposition of beam + image charges in free space.

$$\boxed{-\frac{\partial \phi}{\partial \vec{x}_\perp} = \vec{E}_\perp(\vec{x}) = \underbrace{\frac{1}{2\pi\varepsilon_0}\int_{Pipe} \rho(\tilde{\vec{x}}_\perp) \frac{(\vec{x}_\perp - \tilde{\vec{x}}_\perp)}{|\vec{x}_\perp - \tilde{\vec{x}}_\perp|^2}\, d^2\tilde{x}_\perp}_{\text{Direct}} - \underbrace{\frac{1}{2\pi\varepsilon_0}\int_{Pipe} \rho(\tilde{\vec{x}}_\perp) \frac{\left[\vec{x}_\perp - r_p^2 \tilde{\vec{x}}_\perp / |\tilde{\vec{x}}_\perp|^2\right]}{\left[\vec{x}_\perp - r_p^2 \tilde{\vec{x}}_\perp / |\tilde{\vec{x}}_\perp|^2\right]^2}\, d^2\tilde{x}_\perp}_{\text{Image}}}$$

* ★ Difficult to solve even for simple beam charge distributions $\rho(\vec{x})$
* ★ Non-simple pipes even harder!
  - Ellipse / Rectangle
  - Electric Quadrupole electrodes

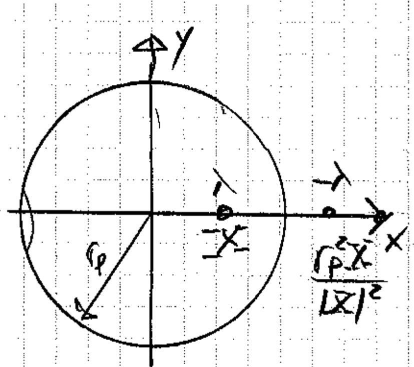
  ⋮

Estimate the strength of the image term: replace beam by line charge λ at the beam center of mass.

$$\lambda = \int_{\text{pipe}} \rho(\vec{x}_\perp)\, d^2x_\perp \quad = \text{line charge of beam}$$

$$\vec{X} = \frac{\int_{\text{pipe}} \vec{x}\, \rho(\vec{x}_\perp)}{\lambda} \quad = \text{"center of mass" of beam}$$

Choose coordinates where X is on axis: $\quad X = \mathrm{sign}(\vec{X})\,|\vec{X}|$



$$\vec{E}_\perp\Big|_{\text{Image}} = \frac{-\lambda}{2\pi\varepsilon_0}\int_{\text{pipe}} \delta(\tilde{\vec{x}}_\perp - X\hat{x})\, \frac{\left[\vec{x}_\perp - r_p^2\, \tilde{\vec{x}}_\perp/|\tilde{\vec{x}}_\perp|^2\right]}{\left[\vec{x}_\perp - r_p^2\, \tilde{\vec{x}}_\perp/|\tilde{\vec{x}}_\perp|^2\right]^2}\, d^2\tilde{x}_\perp$$

evaluate at charge

* Only location field needed to estimate strength.

$$\boxed{\vec{E}_\perp\Big|_{\text{Image}}(\vec{x}_\perp = X\hat{x}) = \frac{\lambda}{2\pi\varepsilon_0 (r_p^2/X - X)}}$$

If $|X|/r_p \ll 1$ and r_p large

* Makes line charge replacement good also:

$$\Rightarrow \boxed{\vec{E}_\perp\Big|_{\text{Image}} \simeq \frac{\lambda X}{2\pi\varepsilon_0 r_p^2}}$$

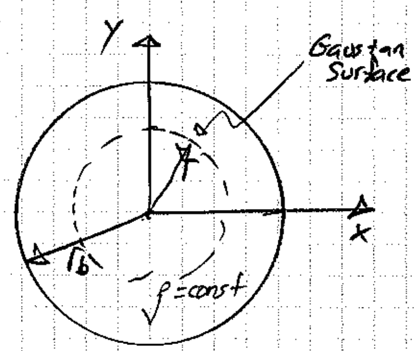
* Weak when $X/r_p \ll 1$.

* Neglect for large pipe. $\Rightarrow$ Choose aperture big to mitigate. Bigger aperture also limits losses but costs more.

Neglecting image charges we have:

$$\boxed{\vec{E}_\perp = -\frac{\partial \phi}{\partial \vec{x}_\perp} = \frac{1}{2\pi\varepsilon_0}\int_{pipe} \rho(\tilde{\vec{x}}_\perp)\,\frac{(\vec{x}_\perp - \tilde{\vec{x}}_\perp)}{|\vec{x}_\perp - \tilde{\vec{x}}_\perp|^2}\, d^2\tilde{x}_\perp}$$

Lets' further approximate the beam as being uniform density within a radius $r = r_b$.



$$\rho = \begin{cases} \dfrac{\lambda}{\pi r_b^2} = \text{const} & r < r_b \\ 0 & r > r_b \end{cases}$$

$$\lambda = \iint_{cylinder} \rho(\vec{x}_\perp)\, d^2x_\perp = \text{line charge.}$$

Apply Gauss' Law to calculate $\underline{\vec{E} \text{ inside the beam}}$ $(r < r_b)$

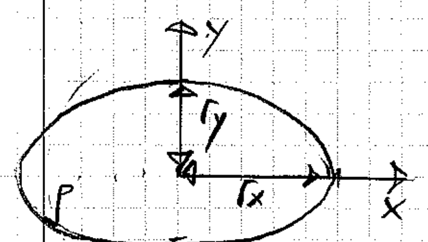
$$\int_{\substack{S\ \text{radius } r \\ r < r_b}} \vec{E}_\perp \cdot \hat{r}\, d^2x = \Big(\underbrace{\frac{\lambda}{\pi r_b^2}}_{\rho}\Big)\pi r^2 = \lambda \frac{r^2}{r_b^2}$$

$$2\pi r\, E_r = \lambda \frac{r^2}{r_b^2}$$

$$\boxed{\begin{aligned} E_r &= \frac{\lambda r}{2\pi r_b^2} & r \le r_b \\ \vec{E}_\perp &= \frac{\lambda r}{2\pi r_b^2}\hat{r} & r \le r_b \end{aligned}}$$

★ Linear field!

If we had a uniform density elliptical beam, some analysis (far from trivial) shows that the field within the beam is still linear with



$$\rho = \frac{\lambda}{\pi r_x r_y} \qquad r_x, r_y \text{ half-widths}$$

$$\boxed{\begin{aligned} E_x &= -\frac{\partial \phi}{\partial x} = \frac{\lambda}{\pi\varepsilon_0}\frac{x}{(r_x + r_y) r_x} \\ E_y &= -\frac{\partial \phi}{\partial y} = \frac{\lambda}{\pi\varepsilon_0}\frac{y}{(r_x + r_y) r_y} \end{aligned}} \qquad \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1$$

★ Reduces to round beam case with $r_x = r_y = r_b$

Let's now apply the equations of motion for a particle oscillating within the a uniform density beam:

* Take $\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$ — simple + consistent with round beam (same focus in x, y)
* $r = \sqrt{x^2+y^2} < r_b \Rightarrow$ inside beam — where $E_x = E_r \frac{x}{r}$, $E_y = E_r \frac{y}{r}$

$E_r = \frac{\lambda r}{2\pi r_b^2}$ in beam

Then our particle equations of motion are

$$\frac{d^2}{ds^2}x + \kappa_x(s)\, x = \frac{-q}{m\gamma^3\beta^2c^2}\frac{\partial \phi}{\partial x}$$
$$\frac{d^2 y}{ds^2} + \kappa_y(s)\, y = \frac{-q}{m\gamma^3\beta^2c^2}\frac{\partial \phi}{\partial y}$$

$\Rightarrow$

$$\boxed{\begin{aligned}\frac{d^2}{ds^2}x + k_{\beta 0}^2 x &= \frac{-q\lambda}{2\pi\epsilon_0 r_b^2 m\gamma^3\beta^2c^2}\,x \\ \frac{d^2}{ds^2}y + k_{\beta 0}^2 y &= \frac{-q\lambda}{2\pi\epsilon_0 r_b^2 m\gamma^3\beta^2c^2}\,y\end{aligned}}$$ in beam

Define:

$$\boxed{Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma^3\beta^2c^2} = \begin{array}{l}\text{Dimensionless}\\ \text{Perveance}\end{array} = \text{const}}$$

Typically small number:
$Q \sim 10^{-3} - 10^{-4}$ near injector
$\sim 10^{-6}$ downstream if not intense beam accelerator.

Can show in our simple situation in nonrelativistic limit

$$Q = \frac{q\Delta\phi}{\mathcal{E}}$$

$\mathcal{E} = \frac{1}{2}m\beta^2c^2$ = kinetic energy

$\Delta\phi = \phi(r=0) - \phi(r=r_b)$

$q\Delta\phi$ = potential energy accross beam

Then equations are

$$\boxed{\begin{aligned}\frac{d^2}{ds^2}x + k_{\beta 0}^2 x - \frac{Q}{r_b^2}x &= 0 \\ \frac{d^2}{ds^2}y + k_{\beta 0}^2 y - \frac{Q}{r_b^2}y &= 0\end{aligned}}$$ $r < r_b$

Equations of motion in beam $(r < r_b)$

$$\boxed{\begin{aligned} \frac{d^2}{ds^2}x + k_{\beta 0}^2 x - \frac{Q}{r_b^2}x &= 0 \\ \frac{d^2}{ds^2}y + k_{\beta 0}^2 y - \frac{Q}{r_b^2}y &= 0 \end{aligned}}$$

$\Rightarrow$ both same form

$$\boxed{\begin{aligned} \frac{d^2 x}{ds^2} + k_\beta^2 x &= 0 \\ k_\beta^2 &= k_{\beta 0}^2 - \frac{Q}{r_b^2} \end{aligned}}$$

Recall we can write:

$$k_{\beta 0} = \frac{\sigma_0}{L_p}$$

$\sigma_0 =$ "undepressed" phase advance per lattice period

$L_p =$ Lattice period.

By analogy we take:

$$k_\beta = \frac{\sigma}{L_p} = \sqrt{k_{\beta 0}^2 - \frac{Q}{r_b^2}} = \sqrt{\frac{\sigma_0^2}{L_p^2} - \frac{Q}{r_b^2}}$$

$\sigma =$ "depressed" phase advance

$$\boxed{0 \leq \frac{k_\beta}{k_{\beta 0}} = \frac{\sigma}{\sigma_0} \leq 1}$$

$Q = k_{\beta 0}^2 r_b^2$
Space-charge cancels applied focus.
(cold beam)

$Q = 0$
Single particle dynamics.
(warm beam)

$0 \leq \frac{\sigma}{\sigma_0} \leq 1$

provides a normalized measure of space-charge strength measured by fraction of applied focusing canceled.

Equation of motion

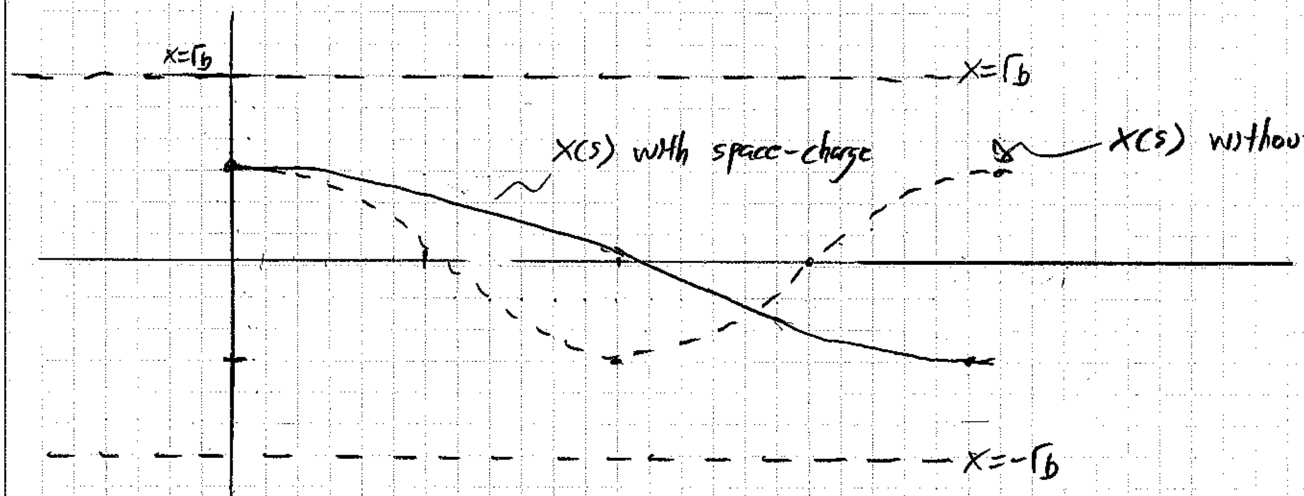
$$\frac{d}{ds^2} x(s) + k_\beta^2 x(s) = 0 \qquad k_\beta^2 = k_{\beta 0}^2 - \frac{Q}{r_b^2} = \left(\frac{\sigma}{\sigma_0}\right)^2 k_{\beta 0}^2$$

has solution

$$\boxed{\begin{aligned} x(s) &= x_i^0 \cos[k_\beta(s-s_i)] + \frac{x_i'}{k_\beta} \sin[k_\beta(s-s_i)] \\ x'(s) &= -\frac{x_i}{k_\beta} \sin[k_\beta(s-s_i)] + x_i' \cos[k_\beta(s-s_i)] \end{aligned}}$$

$s = s_i$ = initial axial coord.
x_i^0 = initial coord.
x_i' = initial angle

Graphically/ $x_i' = 0 \quad x_i = r_b/2 \qquad \sigma/\sigma_0 = 0.5$



★ Space-charge slows down particle oscillations.
$k_\beta \to 0$ at space-charge limit: No oscillation

★ A self-consistent continuous distribution called the Kapchinsky–Vladimirsky/ distribution exists which generates a bundle of particles consistent with the envelope
— Works for an elliptical beam too: See Lund USPAS notes.

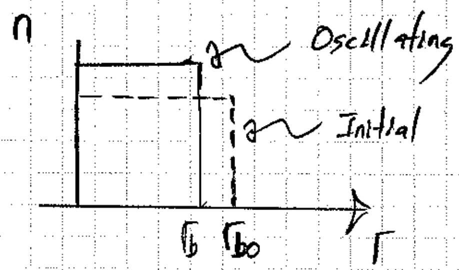
Comments continued:

- ★ Space charge can shift tune in rings driving into nearby resonance lines.
- ★ For nonuniform beam situation is <u>much</u> more complicated. – Density profile evolves.
  See Lund and Barnard, "Beam Physics with Intense Space-Charge" USPAS
- ★ $$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma^3 \beta^2 c^2} \sim \frac{1}{\gamma^3 \beta^2} \quad \text{if } \lambda = \text{const.}$$
  $\Rightarrow$ Space charge matters most for low energy beam near injector, and will weaken with acceleration <u>unless</u> λ is increased by longitudinal beam compression.
- ★ situation outlined with space charge depressing oscillation frequencies remains true for a uniform density elliptical beam allowing us to apply basic concepts outlined to quadrupole focusing of beams with intense space-charge.

Envelope Modes: Introduction to collective beam behavior

Space charge also results in many collective waves and instabilities which must be understood to reliably operate an accelerator with intense space charge.

evolving density $\Rightarrow$ ripples produce nonlinear force.



lowest order collective mode:
self-similarly oscillate uniform density
$\Rightarrow$ "envelope mode"

Let's examine this situation using <u>statistical averages</u> over the beam

$$\underbrace{\langle \{\cdots\} \rangle}_{\text{some quantity}} = \underbrace{\frac{\sum_i \{\cdots\}_i}{N}}_{\text{Average over } n \text{ particles}} = \frac{\int\int \{\cdots\} f(\vec{x}_\perp, \vec{x}'_\perp, s)\, d^2x_\perp\, d^2x'_\perp}{\int\int_{\text{phase-space}} f(\vec{x}_\perp, \vec{x}'_\perp, s)\, d^2x_\perp\, d^2x'_\perp}$$

continuum approx.

$f(\vec{x}_\perp, \vec{x}'_\perp, s) =$ beam distribution

$\rho(\vec{x}_\perp, s) = q\int f(\vec{x}, \vec{x}'_\perp, s)\, d^2x'_\perp$ = charge density

Relate the beam edge to a statistical average over the uniform density beam:

$$\boxed{r_b = 2\langle x^2\rangle^{1/2}}$$

Proof:

$$\langle x^2\rangle = \frac{\iint_{\text{phase-space}} x^2 f \, d^2x\, d^2x'}{\iint_{\text{phase space}} f \, d^2x\, d^2x'}$$

Take norm $\int_{\text{angles}} f\, d^2x' = n(x,y,s) = \text{density}$

$$\langle x^2\rangle = \frac{\int_{\text{round beam}} x^2 n \, d^2x}{\int_{\text{round beam}} n\, d^2x} = \frac{\int_0^{2\pi} \cos^2\theta\, d\theta \int_0^{r_b} r^2 \cancel{n}\, r\, dr}{\cancel{n}\, \pi r_b^2}$$

$$= \frac{\pi r^4 \big|_0^{r_b}}{4\pi r_b^2} = \frac{r_b^2}{4}$$

$$\Rightarrow \quad r_b = 2\langle x^2\rangle^{1/2}$$

Next differentiate this equation for r_b twice to obtain an evolution equation for r_b

$$\frac{d}{ds} r_b = \frac{d}{ds} 2\langle x^2\rangle^{1/2} = \frac{2\langle xx'\rangle}{\langle x^2\rangle^{1/2}}$$

$$\frac{d^2}{ds^2} r_b = \frac{2\langle x'^2\rangle + 2\langle xx''\rangle}{\langle x^2\rangle^{1/2}} - \frac{2\langle xx'\rangle^2}{\langle x^2\rangle^{3/2}} \qquad ' = \frac{d}{ds}$$

Or

$$\boxed{\frac{d^2}{ds^2} r_b - \frac{2\langle xx''\rangle}{\langle x^2\rangle^{1/2}} - \frac{2\left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right]}{\langle x^2\rangle^{3/2}} = 0}$$

Next, apply the uniform density beam equation of motion.

$$x'' + k_{\beta 0}^2 x - \frac{Q}{r_b^2} x = 0$$

to simplify

$$\langle x x'' \rangle = -k_{\beta 0}^2 \langle x^2 \rangle + \frac{Q}{r_b^2} \langle x^2 \rangle$$

$$\Rightarrow \quad \frac{d^2}{ds^2} r_b + 2 k_{\beta 0}^2 \frac{\langle x^2 \rangle}{\langle x^2 \rangle^{1/2}} - \frac{2Q}{r_b^2} \langle x^2 \rangle^{1/2} - \frac{4\left[\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2 \right]}{\langle x^2 \rangle^{3/2}} = 0$$

Use:

$$r_b = 2 \langle x^2 \rangle^{1/2}$$

$$\varepsilon_{x,rms} = \left[\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2 \right]^{1/2} = \text{rms emittance.}$$

Statistical measure beam phase-space area.

rms Envelope equation

$$\Rightarrow \quad \boxed{\frac{d^2}{ds^2} r_b + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{16 \varepsilon_{x,rms}^2}{r_b^3} = 0}$$

More advanced studies show that

$\varepsilon_{x,rms} = \text{const}$ for linear space-charge forces

$\varepsilon_{x,rms}$ varies little for most "reasonable" distributions in linear transport channels.

For $r_b = \text{const}$, $\varepsilon_{x,rms} = \text{const}$

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} = \frac{16 \varepsilon_{x,rms}^2}{r_b^3} \quad \Rightarrow \quad \boxed{k_\beta^2 = k_{\beta 0}^2 - \frac{Q}{r_b^2} = \frac{16 \varepsilon_{x,rms}^2}{r_b^2}}$$

This implies when

$$\frac{\sigma}{\sigma_0} = \frac{k_{\beta}}{k_{\beta 0}} \rightarrow 0 \qquad \text{that} \qquad \varepsilon_{x,rms} \rightarrow 0$$

beam phase-space area is zero in "cold" limit with full space-charge depression.

In the envelope equation

$$\boxed{\frac{d^2}{ds^2} r_b + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{16\,\varepsilon_{x,rms}^2}{r_b^3} = 0}$$

"Inertial" Term — Applied Linear Focus — Space Charge Defocusing — Thermal Defocusing from beam phase-space area.

★ Envelope response is highly nonlinear: solve numerically from initial condition.

★ <u>As beam expands</u> r_b increases

$$\boxed{\frac{Q}{r_b} \;\; \text{space charge term} \;\; \overset{\text{eventually}}{\gg} \;\; \frac{16\,\varepsilon_{x,rms}^2}{r_b^3} \;\; \text{emittance term}}$$

Space-charge matters more as beam expands radially with fixed phase-space area.

★ As <u>beam is focused</u> more strongly and r_b decreases

$$\boxed{\frac{16\,\varepsilon_{x,rms}^2}{r_b^3} \;\; \text{emittance term} \;\; \overset{\text{eventually}}{\gg} \;\; \frac{Q}{r_b} \;\; \text{space-charge term}}$$

★ Can show space-charge term Q/r_b has <u>same form</u> regardless of distribution of charge if $\rho = \rho(r,s)$. Will prove on final exam. Implies envelope equation more general than ~~for~~ uniform beam only.

Try to better understand small-amplitude oscillations about equilibrium "matched" beam

$$\boxed{\begin{array}{l} r_b = r_{b0} = \text{const} \Rightarrow \text{"Matched" Equilibrium} \\ k_{\beta 0}^2 r_{b0} - \dfrac{Q}{r_{b0}} - \dfrac{16 \varepsilon_{x,rms}^2}{r_{b0}^3} = 0 \quad (*) \end{array}}$$

(*) constrains parameters of matched core in "force balance"

Set:

$$\boxed{r_b = r_{b0} + \delta r_b \quad ; \quad |\delta r_b| / r_{b0} << 1}$$

"Breathing" Oscillation.

$$\rightarrow \frac{d^2}{ds^2} \delta r_b + k_{\beta 0}^2 (r_{b0} + \delta r_b) - \frac{Q}{r_{b0} + \delta r_b} - \frac{16 \varepsilon_{x,rms}^2}{(r_{b0} + \delta r_b)^3} = 0$$

Linearize:

$$\frac{d^2}{ds^2} \delta r_b + k_{\beta 0}^2 r_{b0} + k_{\beta 0}^2 \delta r_b - \frac{Q}{r_{b0}} \left[1 - \frac{\delta r_b}{r_{b0}} \right] - \frac{16 \varepsilon_{x,rms}^2}{r_{b0}^3} \left[1 - \frac{3 \delta r_b}{r_{b0}} \right] = 0$$

$$\frac{d^2}{ds^2} \delta r_b + \underbrace{\left[k_{\beta 0}^2 r_{b0} - \frac{Q}{r_{b0}} - \frac{16 \varepsilon_{x,rms}^2}{r_{b0}^3} \right]}_{= 0 \text{ from } *} + k_{\beta 0}^2 \delta r_b + \underbrace{\frac{Q}{r_{b0}^2}}_{= k_{\beta 0}^2 - k_\beta^2} \delta r_b + 3 \cdot \underbrace{\frac{16 \varepsilon_{x,rms}^2}{r_{b0}^4}}_{= k_\beta^2} \delta r_b = 0$$

$$k_\beta^2 = k_{\beta 0}^2 - \frac{Q}{r_{b0}^2} = \frac{16 \varepsilon_{x,rms}^2}{r_{b0}^4} \quad \text{from } *$$

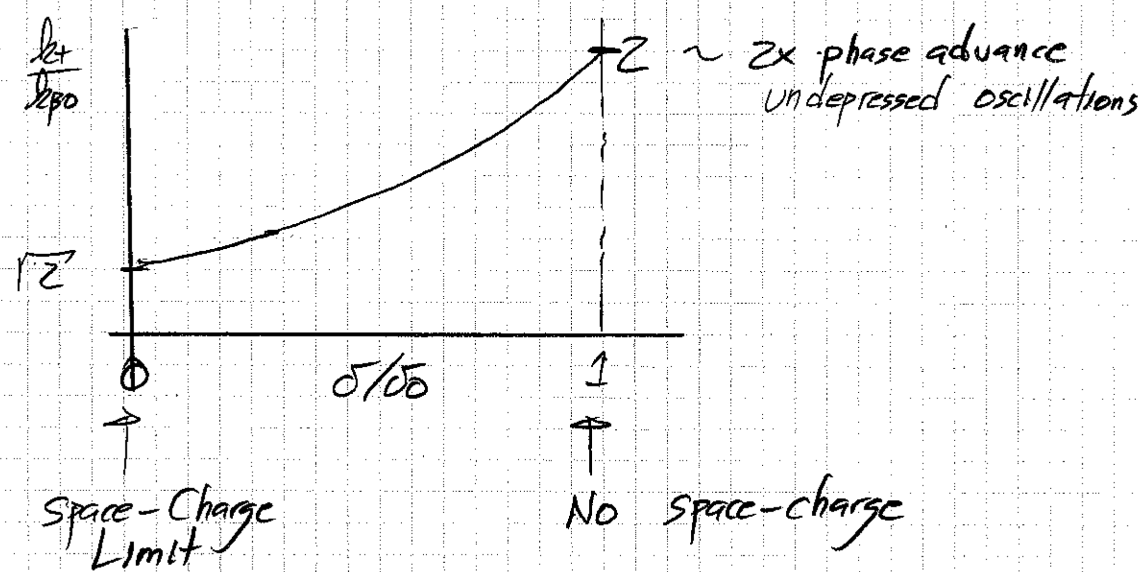
$$\frac{d^2}{ds^2} \delta r_b + k_{\beta 0}^2 \delta r_b + (k_{\beta 0}^2 - k_\beta^2) \delta r_b + 3 k_\beta^2 \delta r_b = 0$$

$$\boxed{\frac{d^2}{ds^2} \delta r_b + (2 k_{\beta 0}^2 + 2 k_\beta^2) \delta r_b = 0} \Rightarrow \boxed{\frac{d^2}{ds^2} \delta r_b + k_+^2 \delta r_b = 0}$$

* simple harmonic oscillator equation with restoring wavenumber

$$k_+ = \sqrt{2 k_{\beta 0}^2 + 2 k_\beta^2}$$

$$\frac{k_+}{k_{\beta 0}} = \sqrt{2\frac{k_{\beta 0}^2}{k_{\beta 0}^2} + 2\frac{k_\beta^2}{k_{\beta 0}^2}} = \sqrt{2 + 2(\sigma/\sigma_0)^2}$$



★ Dispersion curve gives freq. of lowest collective mode response.

★ If frequency breathing mode resonates with errors etc, trouble can result.

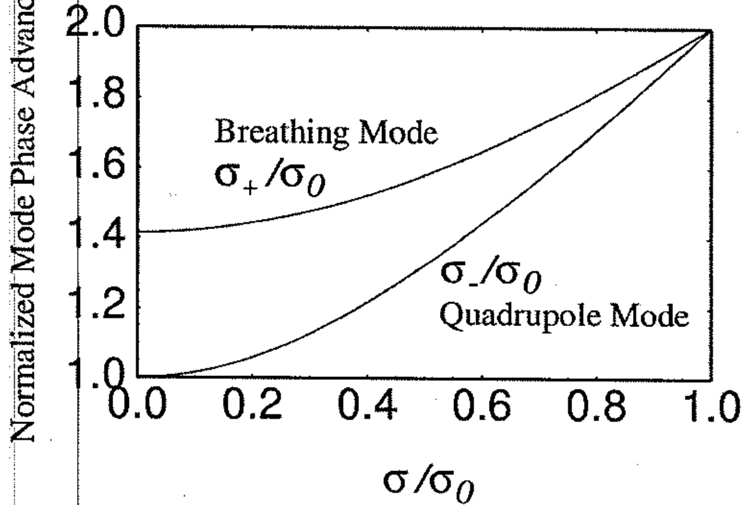
- Mode can be destabalized
- Find it is key in generating beam halo. ← large amplitude tenuous distribution of particles outside the core.

What if different perturbations along x- and y-
⇒ 2 Modes: Breathing + Quadupole
High Freq Low freq.

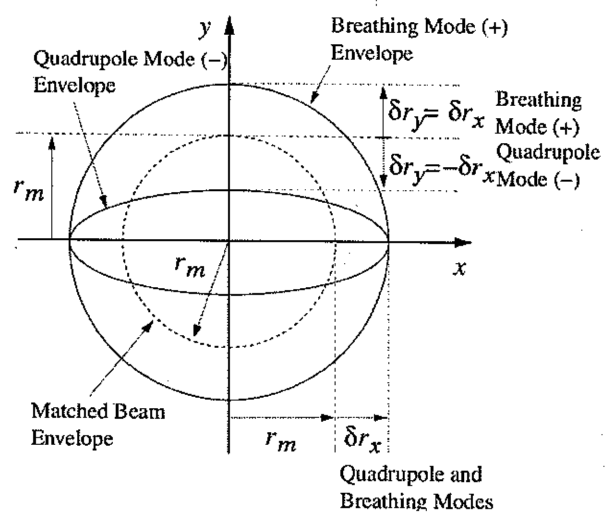
Properties of continuous focusing homogeneous solution: Normal Modes

From Lund, USPAS

Mode Phase Advances



Mode Projections



$$\sigma_+ \equiv \sqrt{2\sigma_0^2 + 2\sigma^2}$$

$$\sigma_- \equiv \sqrt{\sigma_0^2 + 3\sigma^2}$$

Breathing Mode: $\delta r_+ \equiv \dfrac{\delta r_x + \delta r_y}{2}$

Quadrupole Mode: $\delta r_- \equiv \dfrac{\delta r_x - \delta r_y}{2}$

Huge amount more possible:

When $\kappa_x(s)$, $\kappa_y(s)$ periodic functions $\Rightarrow$ different matched beams with many more oscillation modes: some destabilize.

★ See Lund, USPAS, "Transverse Centroid and Envelope Descriptions of Beam Evolution"

Many, many more topics on beam space-charge. If interested, consider taking:

Lund and Barnard, "Beam Physics with Intense Space Charge"
US Particle Accelerator School
Summer, 2017 (MSU Session).

A limited subset of material covering topics immediately progressing from materials in PHY 905 are included in the course web site:

18.supplement.ted.pdf Lectures on: "Transverse Equilibrium Distributions"

18.supplement.tce.pdf Lectures on: "Transverse Centroid and Envelope Descriptions of Beam Evolution"