Physics 905

Fundamentals of Accelerator Physics

Problem Set # 4

March 31, 2016 Due April 7, 2016

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Problem #1

Transfer Matrix Elements Expressed in Phase-Amplitude Form

Show that the principal functions of the transfer matrix solution of the particle orbit 20 pts

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s|s_i) \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix} = \begin{pmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{pmatrix} \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

are:

$$C(s|s_i) = \frac{w(s)}{w_i} \cos \Delta\psi(s) - w_i' w(s) \sin \Delta\psi(s)$$

$$S(s|s_i) = w_i w(s) \sin \Delta\psi(s)$$

$$C'(s|s_i) = \left(\frac{w'(s)}{w_i} - \frac{w_i'}{w(s)} \right) \cos \Delta\psi(s) - \left(\frac{1}{w_i w(s)} + w_i' w'(s) \right) \sin \Delta\psi(s)$$

$$S'(s|s_i) = \frac{w_i}{w(s)} \cos \Delta\psi(s) + w_i w'(s) \sin \Delta\psi(s)$$

$$\Delta\psi(s) = \psi - \psi_i = \int_{s_i}^{s} \frac{d\bar{s}}{w^2(\bar{s})}$$

$$w_i = w(s = s_i)$$
$$w_i' = w'(s = s_i)$$

Hint use:

$$x = A_i w \cos\psi$$
$$x' = A_i w' \cos\psi - \frac{A_i}{w} \sin\psi$$

$$\psi = \psi_i + \Delta\psi$$

Problem #2

Critical Points: Courant-Snyder Ellipse   30 pts

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In class we derived the single-particle Courant-Snyder Invariant:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon = \text{const.}$$

where:

$$\beta(s) = w^2(s)$$

$$\alpha(s) = -w(s)w'(s)$$

$$\gamma(s) = \frac{1}{w^2(s)} + w'(s)^2 = \frac{1+\alpha^2(s)}{\beta(s)}$$

Derive the critical values of the ellipse indicated on the figure below:

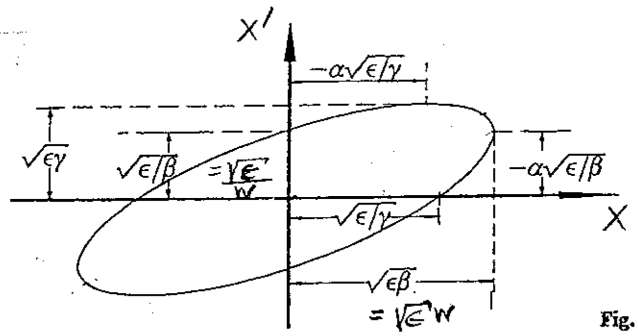

Fig. 5.22. Phase space ellipse

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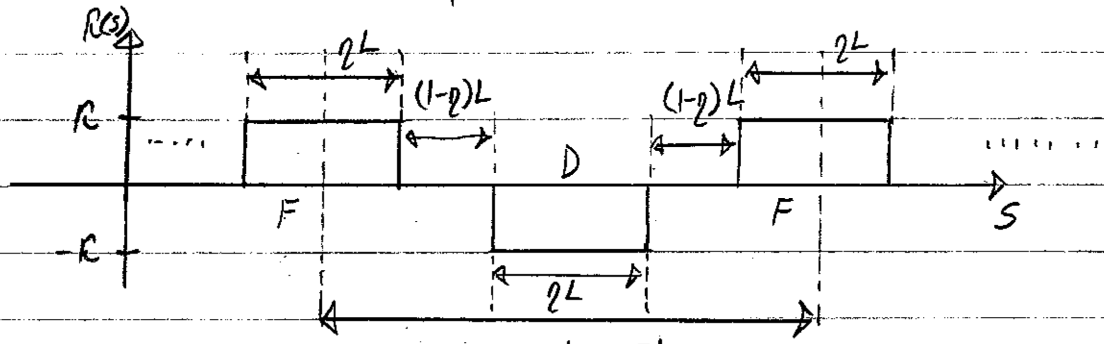
From Wiedemann

Hint: to avoid messy algebra, take a differential of the constraint equation $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \text{const}$ and use this result to find turning points.

$$\Rightarrow \quad 2\gamma x\,dx + 2\alpha x\,dx' + 2\alpha x'\,dx + 2\beta x'\,dx' = 0$$

Problem #3 Phase Advance 40pts

Consider a "FODO" periodic lattice:



$L_p \equiv 2L$
Lattice period

$L_p = 2L$ = lattice period

ηL = Quadrupole lengths

$(1-\eta)L$ = drift lengths

η = Quadrupole occupancy $0 < \eta \leq 1$

κ = Quadrupole strength

5 pts a) Write the transfer matrices $\bar{M}(s|s_i)$ for each section of the periodic lattice in terms of $\theta \equiv \sqrt{\kappa}\,\eta L$, d, and η. (Use results from class.

$\bar{M}_F$: Transfer through Focus Quadrupole.

$\bar{M}_O$: " " Drift

$\bar{M}_D$: " " Defocus Quadrupole

$\bar{M}_O$: " " Drift.

5 pts b) Write the transfer matrix $\bar{M}(s_i+L_p, s_i)$ through one lattice period starting from the left side of a focus quadrupole. No need to fully expand!

20pts c) Show that the phase advance σ_0 of a particle through this lattice period

$$\cos\sigma_0 = \frac{1}{2}\,\mathrm{Trace}\; M(s_i+L_p|s_i)$$

can be expressed as:

$$\cos\sigma_0 = \cos\theta\cosh\theta + \frac{(1-\eta)}{\eta}\,\theta\,(\cos\theta\sinh\theta - \sin\theta\cosh\theta)$$
$$-\frac{1}{2}\frac{(1-\eta)^2}{\eta^2}\,\theta^2\sin\theta\sinh\theta$$

Hint: only calculate simplify elements of M that you need.

3pts d) Will it matter where the lattice period is started in the calculation of σ_0 in part c)? Why?

5pts e) For $\theta \ll 1$ (thin lens limit), show that

$$\cos\sigma_0 \simeq 1 - \frac{1}{2}\left(1-\frac{2}{3}\eta\right)\frac{\theta^4}{\eta^2}$$

f) If $\sigma_0 \ll 1$, and $\eta \ll 1$, show that

$$\sigma_0 \simeq \eta|\hat{\kappa}|L^2$$

2pts g) If one wanted to model a "FODO" focusing lattice by a continuous focusing channel with $\kappa(s) = k_{\beta 0}^2 = \mathrm{const.}$, how could one choose $k_{\beta 0}^2$ based on part f) ?