Physics 905
Fundamentals of Accelerator Physics

Problem Set #6
April 14, 2016 Due April 21, 2016

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Problem #1 Normalized Emittance 20 pts

Consider a distribution of particles evolving according to the particle equation of motion

$$x'' + \frac{(\gamma\beta)'}{(\gamma\beta)} x' + \kappa(s)\, x = 0 , \qquad ' \equiv \frac{d}{ds}$$

Denote an average over the distribution as

$$\langle \cdots \rangle$$

A statistical measure of beam phase-space area is provided by the normalized rms emittance.

$$\varepsilon_{nx} \equiv (\gamma\beta)\left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right]^{1/2}$$

Show that

$$\varepsilon_{nx} = \text{const.}$$

Would you expect ε_{nx} to be conserved if the equation of motion had nonlinear terms?

$$x'' + \frac{(\gamma\beta)'}{(\gamma\beta)} x' + \kappa(s)\, x = F_{nl} \qquad F_{nl} \text{ some function of } x, \text{ not } \propto x.$$

Explain why. Be specific.

Problem #2 Resonances 40 pts.

Consider the driven harmonic oscillator equation for $U(\varphi)$:

$$\frac{d^2U(\varphi)}{d\varphi^2} + \nu_0^2\, U(\varphi) = \overbrace{A\cos(\nu\varphi) + B\sin(\nu\varphi)}^{\text{driving term.}}$$

ν = constant driving frequency.
A, B constant amplitudes.

The general solution for $U(\varphi)$ can be expanded as

$$U(\varphi) = U_h(\varphi) + U_p(\varphi)$$

where U_h is the general solution to the homogeneous equation:

$$\frac{d^2U_h}{d\varphi^2} + \nu_0^2\, U_h = 0$$

$$\Rightarrow U_h = C_1\cos(\nu_0\varphi) + C_2\sin(\nu_0\varphi)$$

C_1, C_2 constants

and U_p is any particular solution to

$$\frac{d^2U_p}{d\varphi^2} + \nu_0^2\, U_p = A\cos(\nu\varphi) + B\sin(\nu\varphi)$$

5 pts a) For $\nu \neq \nu_0$ show that a solution U_p exists proportional to the driving term and find the constant of proportionality.

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5pts b) Use the results of part a) to construct the solution $(\nu \neq \nu_0)$ for $U(\varphi)$ satisfying the initial conditions at $\varphi = 0$:

$$U(\varphi=0) = U_0$$

$$\left.\frac{dU}{d\varphi}\right|_{\varphi=0} = \dot{U}_0 \quad ; \quad \frac{dU}{d\varphi} \equiv \dot{U}$$

10pts c) Set $\nu = \nu_0 + \delta\nu$ and find the leading order form of the solution valid for $|\delta\nu/\nu_0| \ll 1$ and $|\delta\nu\,\varphi| \ll 1$.
What does this limit imply on the amplitude of the particle oscillation as $\nu \to \nu_0$

5pts d) What do these results imply for a general periodic forcing function:

$$\frac{d^2U(\varphi)}{d\varphi^2} + \nu_0^2\,U(\varphi) = f(\varphi) \quad \leftarrow \text{forcing function}$$

$$f(\varphi + 2\pi) = f(\varphi)$$

How does this fit in with the analysis of machine tunes carried out in the class notes?

5pts e) Suppose the drive frequency is exactly equal to the resonant frequency:

$$\frac{d^2U}{d\varphi^2} + \nu_0^2 U = A\cos(\nu_0\varphi) + B\sin(\nu_0\varphi)$$

Show that a particular solution exists (motivated by c))

$$U_p = \frac{A}{2\nu_0}\varphi \sin(\nu_0\varphi) - \frac{B}{2\nu_0}\varphi \sin(\nu_0\varphi)$$

with no approximation. Write down the general solution. Does this agree with c) and should it?

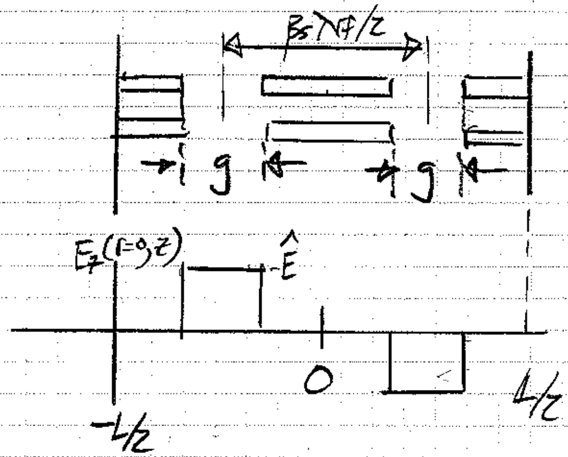
10 pts f) For the case of $\nu \neq \nu_0$, estimate the deviation in $\frac{\delta\nu}{\nu_0}$ to wash out the resonance.

Please keep arguments simple. Recommend looking at 2nd order deviations in $\delta\nu/\nu_0$.

Problem #3 Transit Time Factor 50 pts

Many cavities are multi-gap — Including at FRIB. They can be modeled by the usual Panofsky equation if an appropriate transit time factor T is employed

2 Gap Cavity



$$E_z = E_z(r,z)\cos(\omega t + \phi)$$

Approx E_z – uniform in gaps.

The energy gain of a particle traversing the cavity is

$$\Delta W = q\int_{-L/2}^{L/2} E(0,z)\sin\left(\frac{2\pi z}{\beta\lambda_{rf}} + \phi\right) dz$$

Approximate $\beta \simeq$ const in the cavity.

~~20 pts~~ 15pts a) For this structure derive a transit-time factor T to show that

$$\Delta W = q E_0 L T \cos\phi$$

with $E_0 = \frac{1}{L}\int_{-L/2}^{L/2} |E(0,z)|\, dz$ = Avg. Field over cell (Note absolute value)

$$T = \frac{\sin[\pi g/(\beta\lambda)]}{\pi g/(\beta\lambda)} \sin\left(\frac{\pi \beta_s}{2\beta}\right)$$

30pts b) Assume that the length of each gap is $g = \frac{\beta_s \lambda_{rf}}{8}$. Plot T vs β for the following 4 cases:

1) $\lambda_{rf} \Leftrightarrow$ 80.5 MHz i) $\beta_s = 0.041$ ii) $\beta_s = 0.085$

2) $\lambda_{rf} \Leftrightarrow$ 322 MHz i) $\beta_s = 0.29$ ii) $\beta_s = 0.53$

For each case estimate the approximate range of β for $T > 0.65$ corresponding to efficient RF.

Use any graphics package you want to make plots. Please no hand plots!

5pts

c) Explain why (qualitative only) why this two gap transit time factor shows more variation in β than a 1 gap model. Why can T be zero for some values of β?