04.sup Equations of Motion and Applied Fields*

Prof. Steven M. Lund
Physics and Astronomy Department
Facility for Rare Isotope Beams (FRIB)
Michigan State University (MSU)

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Steven M. Lund and Yue Hao

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S2: Transverse Particle Equations of Motion in Linear Applied Focusing Channels

S2A: Introduction

Write out transverse particle equations of motion in explicit component form:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' = \frac{q}{m\gamma_b\beta_b^2c^2}E_x^a - \frac{q}{m\gamma_b\beta_b c}B_y^a + \frac{q}{m\gamma_b\beta_b c}B_z^a y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' = \frac{q}{m\gamma_b\beta_b^2c^2}E_y^a + \frac{q}{m\gamma_b\beta_b c}B_x^a - \frac{q}{m\gamma_b\beta_b c}B_z^a x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

Equations previously derived under assumptions:

- No bends (fixed *x-y-z* coordinate system with no local bends)
- Paraxial equations ($x'^2, y'^2 \ll 1$)
- No dispersive effects (β_b same all particles), acceleration allowed ($\beta_b \neq \text{const}$)
- Electrostatic and leading-order (in β_b) self-magnetic interactions

The applied focusing fields

$$\text{Electric:} \quad E_x^a, \; E_y^a$$
$$\text{Magnetic:} \quad B_x^a, \; B_y^a, \; B_z^a$$

must be specified as a function of s and the transverse particle coordinates x and y to complete the description

- Consistent change in axial velocity ($\beta_b c$) due to E_z^a must be evaluated
  - Typically due to RF cavities and/or induction cells
- Restrict analysis to fields from applied focusing structures

Intense beam accelerators and transport lattices are designed to optimize *linear* applied focusing forces with terms:

$$\begin{aligned}
\text{Electric:} \quad & E_x^a \simeq (\text{function of } s) \times (x \text{ or } y) \\
& E_y^a \simeq (\text{function of } s) \times (x \text{ or } y) \\
\text{Magnetic:} \quad & B_x^a \simeq (\text{function of } s) \times (x \text{ or } y) \\
& B_y^a \simeq (\text{function of } s) \times (x \text{ or } y) \\
& B_z^a \simeq (\text{function of } s)
\end{aligned}$$

Common situations that realize these linear applied focusing forms will be overviewed:

- Continuous Focusing (see: S2B)
- Quadrupole Focusing
  - Electric (see: S2C)
  - Magnetic (see: S2D)
- Solenoidal Focusing (see: S2E)

Other situations that will not be covered (typically more nonlinear optics):

- Einzel Lens (see: J.J. Barnard, Intro Lectures)
- Plasma Lens
- Wire guiding

Why design around linear applied fields ?

- Linear oscillators have well understood physics allowing formalism to be developed that can guide design
- Linear fields are "lower order" so it should be possible for a given source amplitude to generate field terms with greater strength than for "higher order" nonlinear fields

S2B: Continuous Focusing

Assume constant electric field applied focusing force:

$$\mathbf{B}^a = 0$$

$$\mathbf{E}^a_\perp = E^a_x \hat{\mathbf{x}} + E^a_y \hat{\mathbf{y}} = -\frac{m\gamma_b\beta_b^2 c^2 k_{\beta 0}^2}{q}\mathbf{x}_\perp$$

$$k_{\beta 0}^2 \equiv \text{const} > 0$$

$$[k_{\beta 0}] = \frac{\text{rad}}{\text{m}}$$

Continuous focusing equations of motion:

Insert field components into linear applied field equations and collect terms

$$\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp + k_{\beta 0}^2 \mathbf{x}_\perp = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial\mathbf{x}_\perp}$$

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + k_{\beta 0}^2 x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + k_{\beta 0}^2 y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

Equivalent Component Form

Even this simple model can become complicated

- Space charge: ϕ must be calculated consistent with beam evolution
- Acceleration: acts to damp orbits (see: S10)

Simple model in limit of no acceleration ($\gamma_b\beta_b \simeq \text{const}$) and negligible space-charge ($\phi \simeq \text{const}$):

$$\mathbf{x}''_\perp + k_{\beta 0}^2 \mathbf{x}_\perp = 0 \quad \Longrightarrow \text{orbits simple harmonic oscillatons}$$

General solution is elementary:

$$\mathbf{x}_\perp = \mathbf{x}_\perp(s_i)\cos[k_{\beta 0}(s-s_i)] + [\mathbf{x}'_\perp(s_i)/k_{\beta 0}]\sin[k_{\beta 0}(s-s_i)]$$

$$\mathbf{x}'_\perp = -k_{\beta 0}\mathbf{x}_\perp(s_i)\sin[k_{\beta 0}(s-s_i)] + \mathbf{x}'_\perp(s_i)\cos[k_{\beta 0}(s-s_i)]$$

$$\mathbf{x}_\perp(s_i) = \text{Initial coordinate}$$

$$\mathbf{x}'_\perp(s_i) = \text{Initial angle}$$

In terms of a transfer map in the *x*-plane (*y*-plane analogous):

$$\begin{bmatrix} x \\ x' \end{bmatrix}_s = \mathbf{M}_x(s|s_i) \cdot \begin{bmatrix} x \\ x' \end{bmatrix}_{s_i}$$

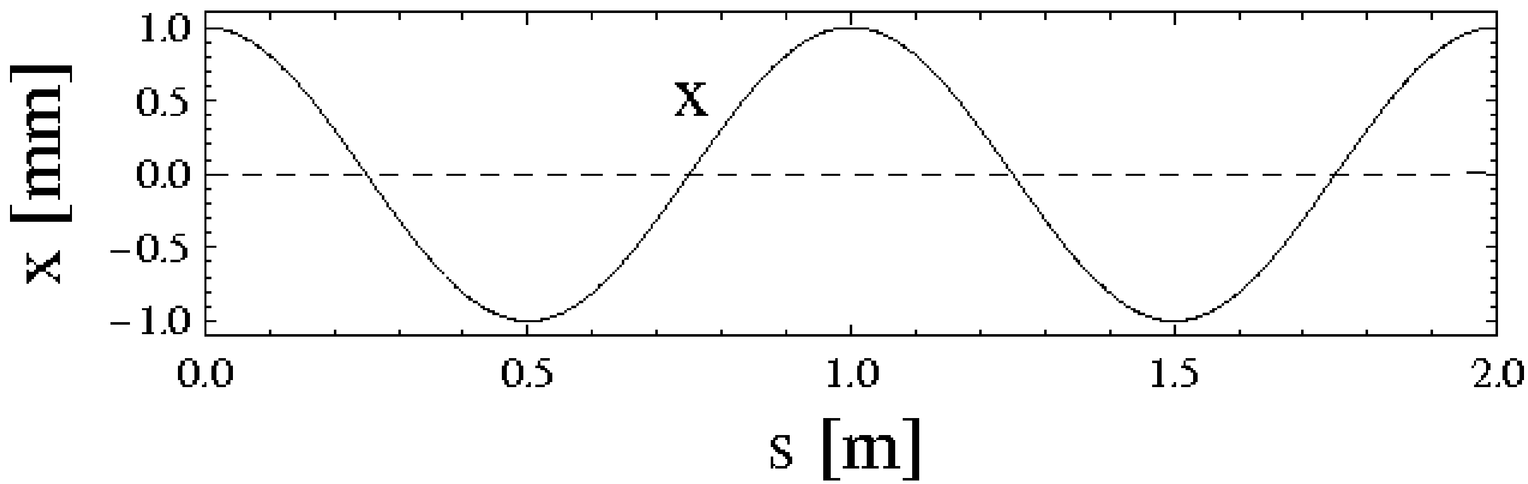
$$\mathbf{M}_x(s|s_i) = \begin{bmatrix} \cos[k_{\beta 0}(s-s_i)] & \frac{1}{k_{\beta_0}}\sin[k_{\beta 0}(s-s_i)] \\ -k_{\beta 0}\sin[k_{\beta 0}(s-s_i)] & \cos[k_{\beta 0}(s-s_i)] \end{bmatrix}$$

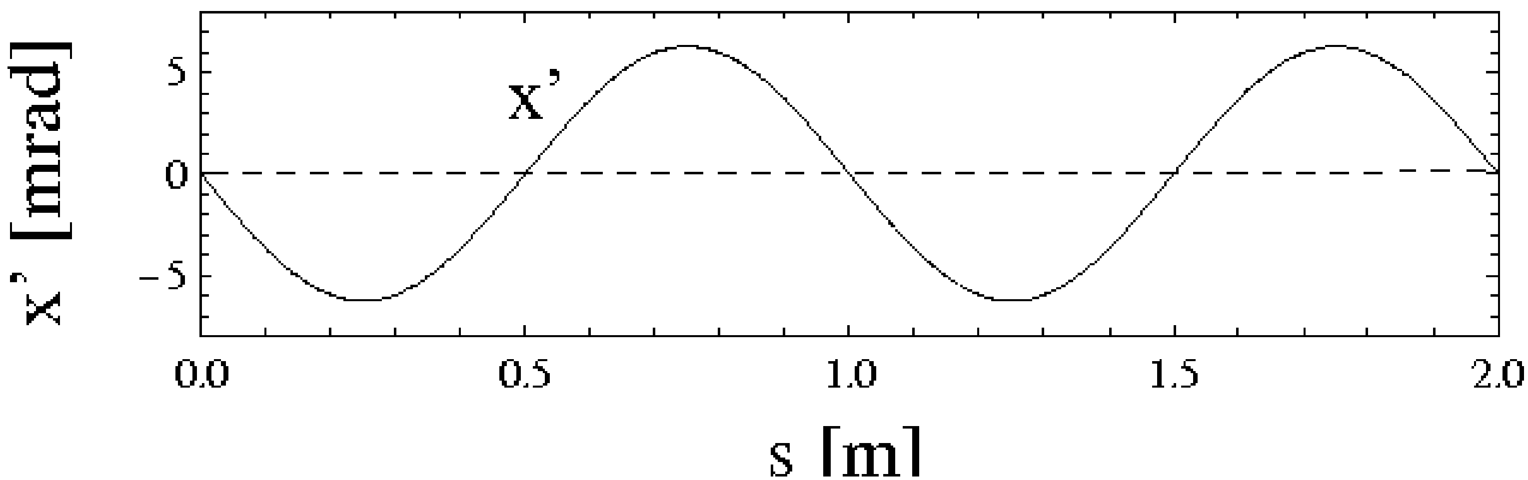
/// Example: Particle Orbits in Continuous Focusing

Particle phase-space in x-x' with only applied field

$$k_{\beta 0} = 2\pi \text{ rad/m} \qquad x(0) = 1 \text{ mm} \qquad y(0) = 0$$

$$\phi \simeq 0 \qquad \gamma_b \beta_b = \text{const} \qquad x'(0) = 0 \qquad y'(0) = 0$$





- Orbits in the applied field are just simple harmonic oscillators ///

Problem with continuous focusing model:

The continuous focusing model is realized by a stationary ($m \rightarrow \infty$) partially neutralizing uniform background of charges filling the beam pipe. To see this apply Maxwell's equations to the applied field to calculate an applied charge density:

$$\rho^a = \epsilon_0 \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{E}^a = -\frac{2m\epsilon_0\gamma_b\beta_b^2 c^2 k_{\beta 0}^2}{q} = \text{const}$$

- Unphysical model, but commonly employed since it represents the average action of more physical focusing fields in a simpler to analyze model
  - Demonstrate later in simple examples and problems given
- Continuous focusing can provide reasonably good estimates for more realistic periodic focusing models if $k_{\beta 0}^2$ is appropriately identified in terms of "equivalent" parameters *and* the periodic system is stable.
  - See lectures that follow and homework problems for examples

In more realistic models, one requires that *quasi-static* focusing fields in the machine aperture satisfy the vacuum Maxwell equations

$$\nabla \cdot \mathbf{E}^a = 0 \qquad \nabla \cdot \mathbf{B}^a = 0$$
$$\nabla \times \mathbf{E}^a = 0 \qquad \nabla \times \mathbf{B}^a = 0$$

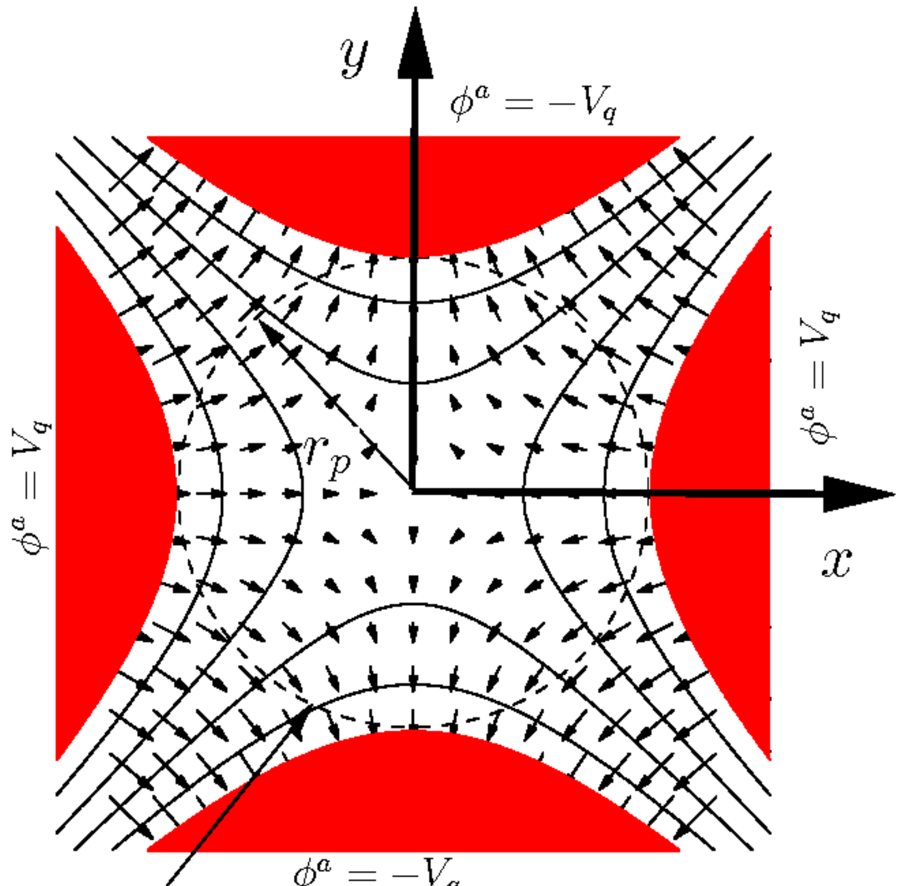
- Require in the region of the beam
- Applied field sources outside of the beam region

The vacuum Maxwell equations constrain the 3D form of applied fields resulting from spatially localized lenses. The following cases are commonly exploited to optimize linear focusing strength in physically realizable systems while keeping the model relatively simple:

1) Alternating Gradient Quadrupoles with transverse orientation
   - Electric Quadrupoles (see: S2C)
   - Magnetic Quadrupoles (see: S2D)
2) Solenoidal Magnetic Fields with longitudinal orientation (see: S2E)
3) Einzel Lenses (see J.J. Barnard, Introductory Lectures)

S2C: Alternating Gradient Quadrupole Focusing
Electric Quadrupoles

In the axial center of a long electric quadrupole, model the fields as 2D transverse



Electrodes Outside of Circle $r = r_p$
Electrodes: $x^2 - y^2 = \mp r_p^2$

- Electrodes hyperbolic
- Structure infinitely extruded along z

2D Transverse Fields

$$\mathbf{B}^a = 0$$

$$E_x^a = -Gx$$
$$E_y^a = \quad Gy$$

$$G \equiv \frac{2V_q}{r_p^2} = -\frac{\partial E_x^a}{\partial x} = \frac{\partial E_y^a}{\partial y}$$
$= \text{Electric Gradient}$

$V_q = \text{Pole Voltage}$

$r_p = \text{Pipe Radius}$
(clear aperture)

//Aside: How can you calculate these fields?

Fields satisfy within vacuum aperture:

$$\nabla \cdot \mathbf{E}^a = 0 \qquad \nabla \times \mathbf{E}^a = 0 \qquad \Longrightarrow \qquad \mathbf{E}^{\mathbf{a}} = -\nabla\phi^a$$

Choose a long axial structure with 2D hyperbolic potential surfaces:

$$\phi^a = \text{const}(x^2 - y^2)$$

Require: $\phi^a = V_q$ at $x = r_p, y = 0 \quad \Longrightarrow \quad \text{const} = V_q/r_p^2$

$$\phi^a = \frac{V_q}{r_p^2}(x^2 - y^2)$$

$$\Longrightarrow \quad E_x^a = -\frac{\partial \phi^a}{\partial x} = \frac{-2V_q}{r_p^2}x \equiv -Gx$$

$$E_y^a = -\frac{\partial \phi^a}{\partial y} = \frac{2V_q}{r_p^2}y \equiv Gy \qquad\qquad G \equiv \frac{2V_q}{r_p^2}$$

Realistic geometries can be considerably more complicated

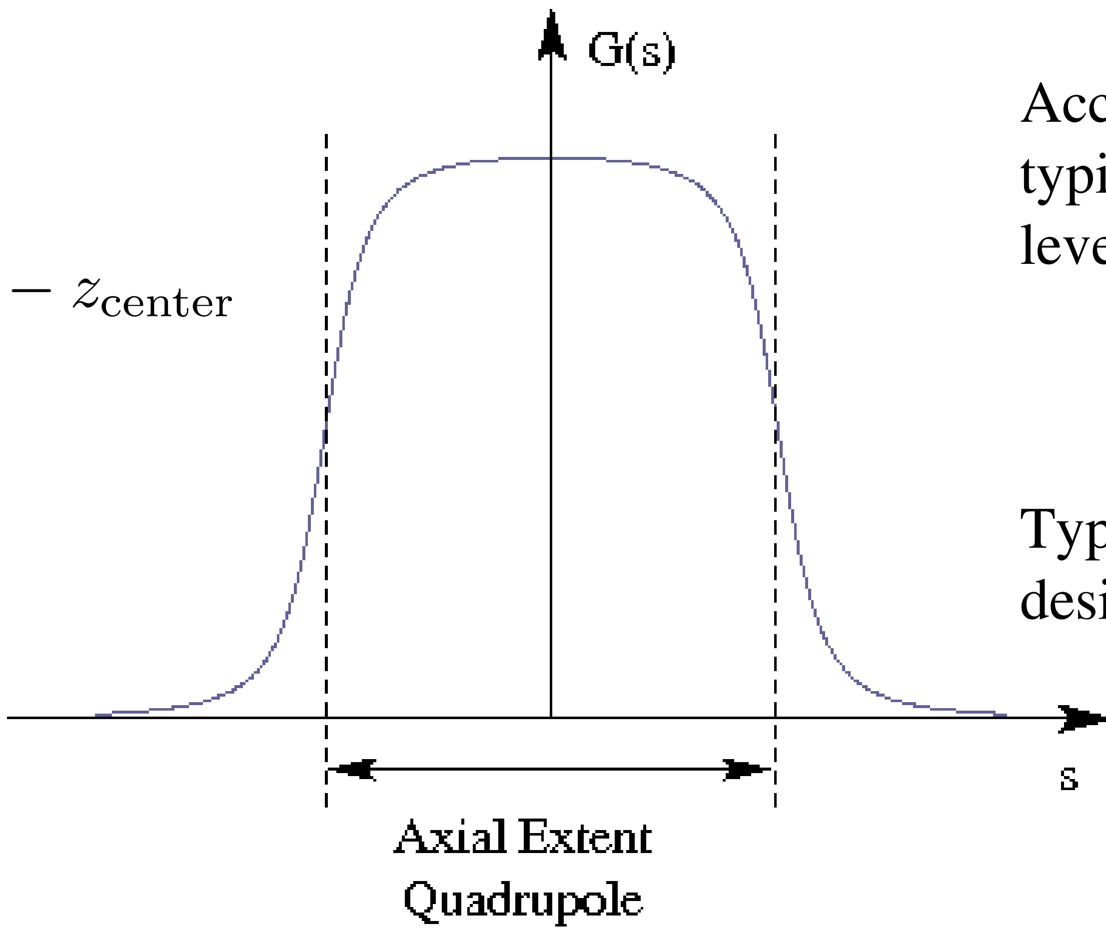
- Truncated hyperbolic electrodes transversely, truncated structure in *z*

//

Quadrupoles actually have finite axial length in z. Model this by taking the gradient G to vary in s, i.e., $G = G(s)$ with $s = z - z_{\text{center}}$ (straight section)

- Variation is called the fringe-field of the focusing element
- Variation will violate the Maxwell Equations in 3D
  - Provides a reasonable first approximation in many applications
- Usually quadrupole is long, and $G(s)$ will have a flat central region and rapid variation near the ends

$s = z - z_{\text{center}}$



Accurate fringe calculation typically requires higher level modeling:

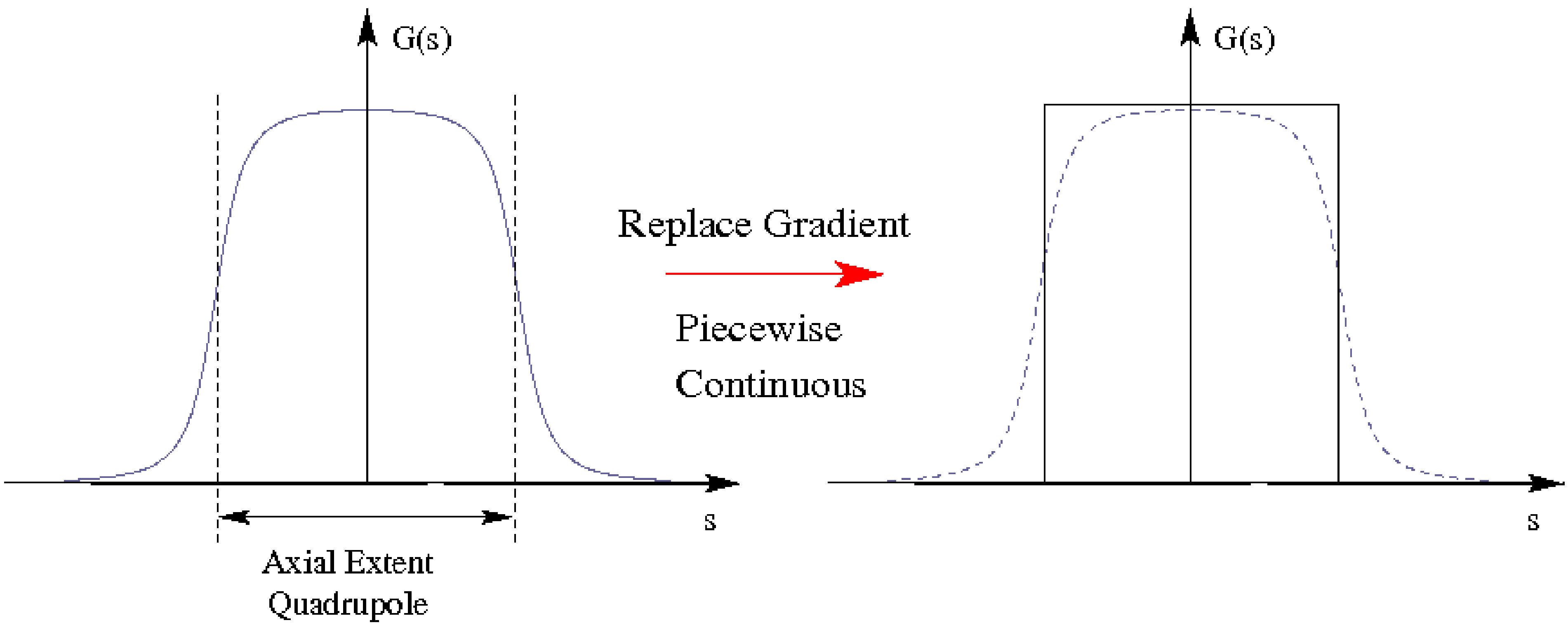
3D analysis

Detailed geometry

Typically employ magnetic design codes

For many applications the actual quadrupole fringe function $G(s)$ is replaced by a simpler function to allow more idealized modeling

- Replacements should be made in an "equivalent" parameter sense to be detailed later (see: lectures on Transverse Centroid and Envelope Modeling)
- Fringe functions often replaced in design studies by piecewise constant $G(s)$
  - Commonly called "hard-edge" approximation
- See S3 and Lund and Bukh, PRSTAB 7 924801 (2004), Appendix C for more details on equivalent models



Electric quadrupole equations of motion:

- Insert applied field components into linear applied field equations and collect terms

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa(s)x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' - \kappa(s)y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$\kappa(s) = \frac{qG}{m\gamma_b\beta_b^2c^2} = \frac{G}{\beta_b c[B\rho]}$$

$$G = -\frac{\partial E_x^a}{\partial x} = \frac{\partial E_y^a}{\partial y} = \frac{2V_q}{r_p^2} \qquad [B\rho] \equiv \frac{\gamma_b\beta_b mc}{q} = \text{Rigidity}$$

$$\beta_b c[B\rho] \equiv \text{Electric Rigidity}$$

- For positive/negative κ , the applied forces are Focusing/deFocusing in the *x*- and *y*-planes
- The *x*- and *y*-equations are decoupled
- Valid whether the the focusing function κ is piecewise constant or incorporates a fringe model

Simple model in limit of no acceleration ($\gamma_b \beta_b \simeq \text{const}$) and negligible space-charge ($\phi \simeq \text{const}$) and $\kappa = \text{const}$:

$$x'' + \kappa x = 0$$
$$y'' - \kappa y = 0$$

$\Longrightarrow$ orbits harmonic or hyperbolic depending on sign of κ

General solution:

$\kappa > 0$:

$$x = x_i \cos[\sqrt{\kappa}(s - s_i)] + (x'_i/\sqrt{\kappa}) \sin[\sqrt{\kappa}(s - s_i)]$$
$$x' = -\sqrt{\kappa} x_i \sin[\sqrt{\kappa}(s - s_i)] + x'_i \cos[\sqrt{\kappa}(s - s_i)]$$
$$x(s_i) = x_i = \text{ Initial coordinate}$$
$$x'(s_i) = x'_i = \text{ Initial angle}$$
$$y = y_i \cosh[\sqrt{\kappa}(s - s_i)] + (y'_i/\sqrt{\kappa}) \sinh[\sqrt{\kappa}(s - s_i)]$$
$$y' = \sqrt{\kappa} y_i \sinh[\sqrt{\kappa}(s - s_i)] + y'_i \cosh[\sqrt{\kappa}(s - s_i)]$$
$$y(s_i) = y_i = \text{ Initial coordinate}$$
$$y'(s_i) = y'_i = \text{ Initial angle}$$

$\kappa < 0$:

$$Exchange x and y in \kappa > 0 \text{ case.}$$

In terms of a transfer maps:

$\kappa > 0$:

$$\begin{bmatrix} x \\ x' \end{bmatrix}_s = \mathbf{M}_x(s|s_i) \cdot \begin{bmatrix} x \\ x' \end{bmatrix}_{s_i}$$

$$\begin{bmatrix} y \\ y' \end{bmatrix}_s = \mathbf{M}_y(s|s_i) \cdot \begin{bmatrix} y \\ y' \end{bmatrix}_{s_i}$$

$$\mathbf{M_x}(s|s_i) = \begin{bmatrix} \cos[\sqrt{\kappa}(s-s_i)] & \frac{1}{\sqrt{\kappa}}\sin[\sqrt{\kappa}(s-s_i)] \\ -\sqrt{\kappa}\sin[\sqrt{\kappa}(s-s_i)] & \cos[\sqrt{\kappa}(s-s_i)] \end{bmatrix}$$

$$\mathbf{M_y}(s|s_i) = \begin{bmatrix} \cosh[\sqrt{\kappa}(s-s_i)] & \frac{1}{\sqrt{\kappa}}\sinh[\sqrt{\kappa}(s-s_i)] \\ \sqrt{\kappa}\sinh[\sqrt{\kappa}(s-s_i)] & \cosh[\sqrt{\kappa}(s-s_i)] \end{bmatrix}$$

$\kappa < 0$:

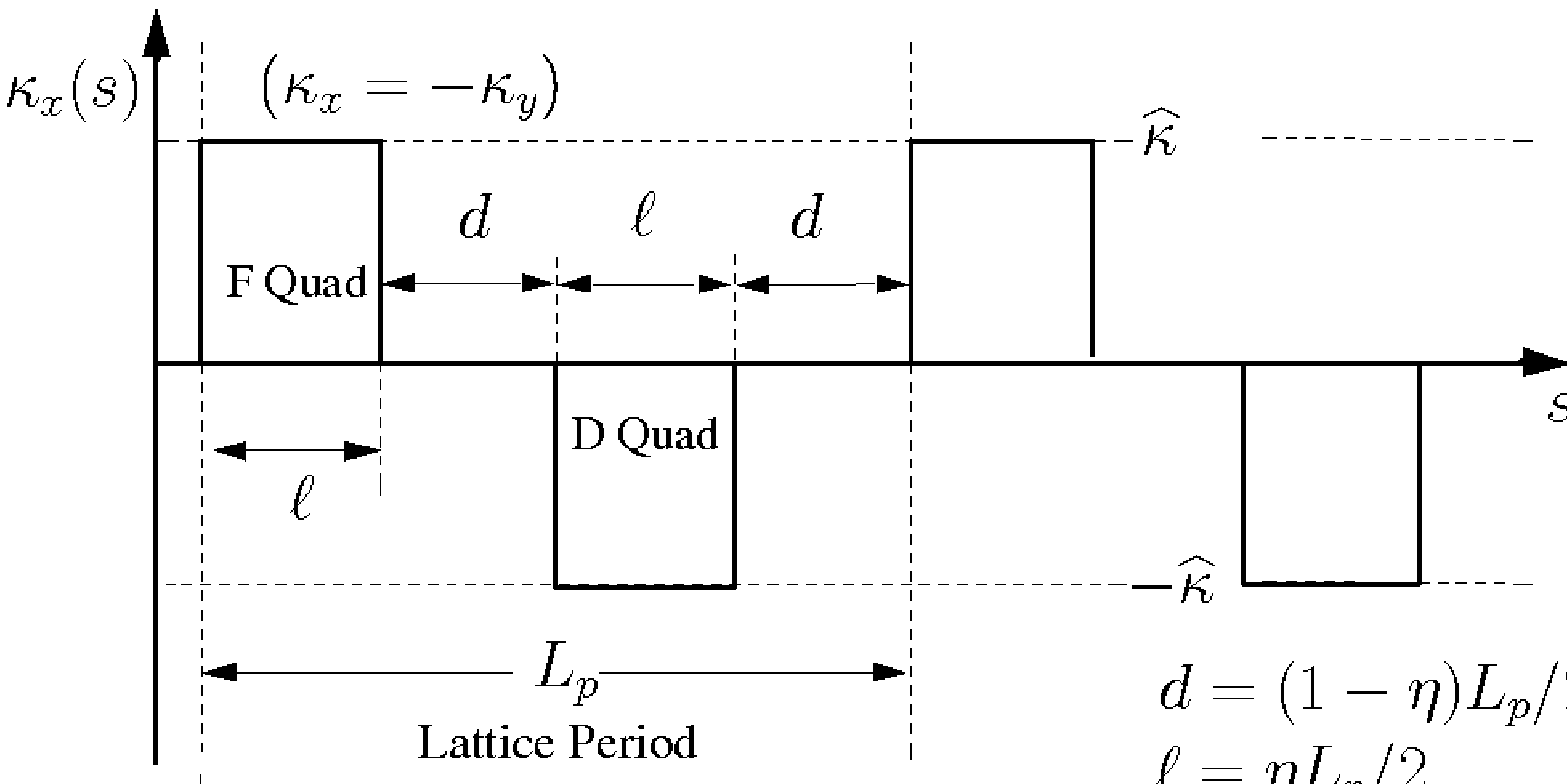
Exchange x and y in $\kappa > 0$ case.

Quadrupoles must be arranged in a lattice where the particles traverse a sequence of optics with alternating gradient to focus strongly in both transverse directions

- Alternating gradient necessary to provide focusing in both *x*- and *y*-planes
- Alternating Gradient Focusing often abbreviated "AG" and is sometimes called "Strong Focusing"
- FODO is acronym:
  - F (Focus) in plane placed where excursions (on average) are small
  - D (deFocus) placed where excursions (on average) are large
  - O (drift) allows axial separation between elements
- Focusing lattices often (but not necessarily) periodic
  - Periodic expected to give optimal efficiency in focusing with quadrupoles
- Drifts between F and D quadrupoles allow space for: acceleration cells, beam diagnostics, vacuum pumping, ...
- Focusing strength must be limited for stability (see S5)

Example Quadrupole FODO periodic lattices with piecewise constant κ

- FODO: [Focus drift(O) DeFocus Drift(O)] has equal length drifts and same length F and D quadrupoles
- FODO is simplest possible realization of "alternating gradient" focusing
  - Can also have thin lens limit of finite axial length magnets in FODO lattice



$$d = (1-\eta)L_p/2$$
$$\ell = \eta L_p/2$$

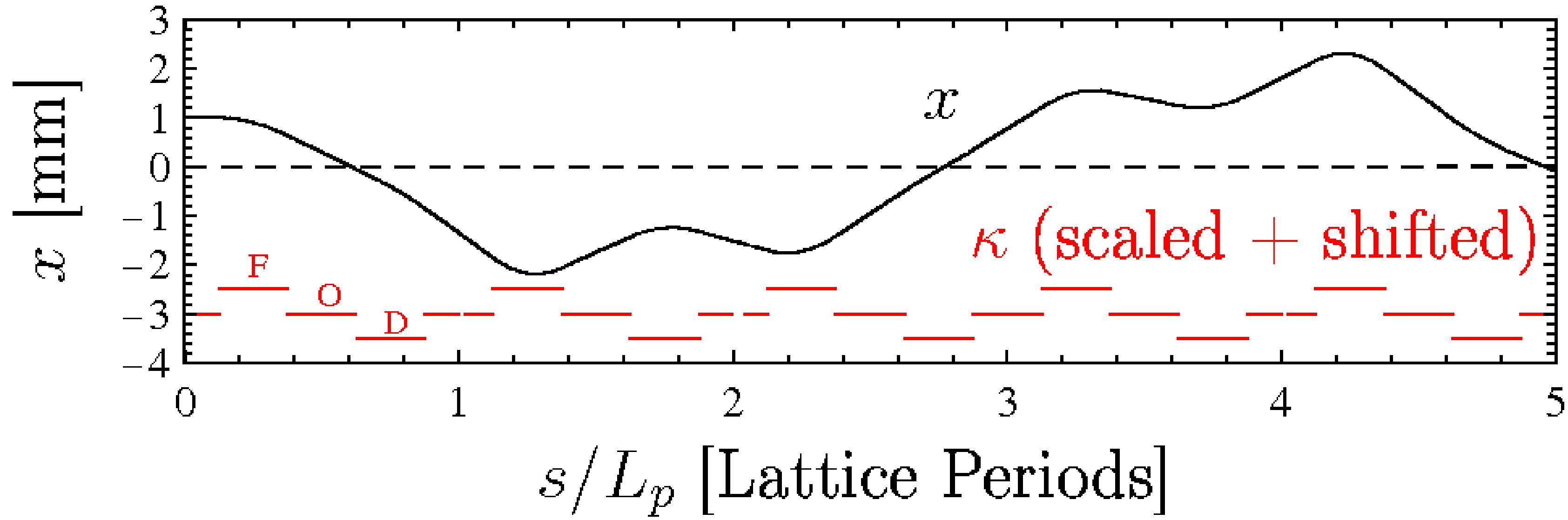
$$\eta = \text{Occupancy} \in (0, 1]$$

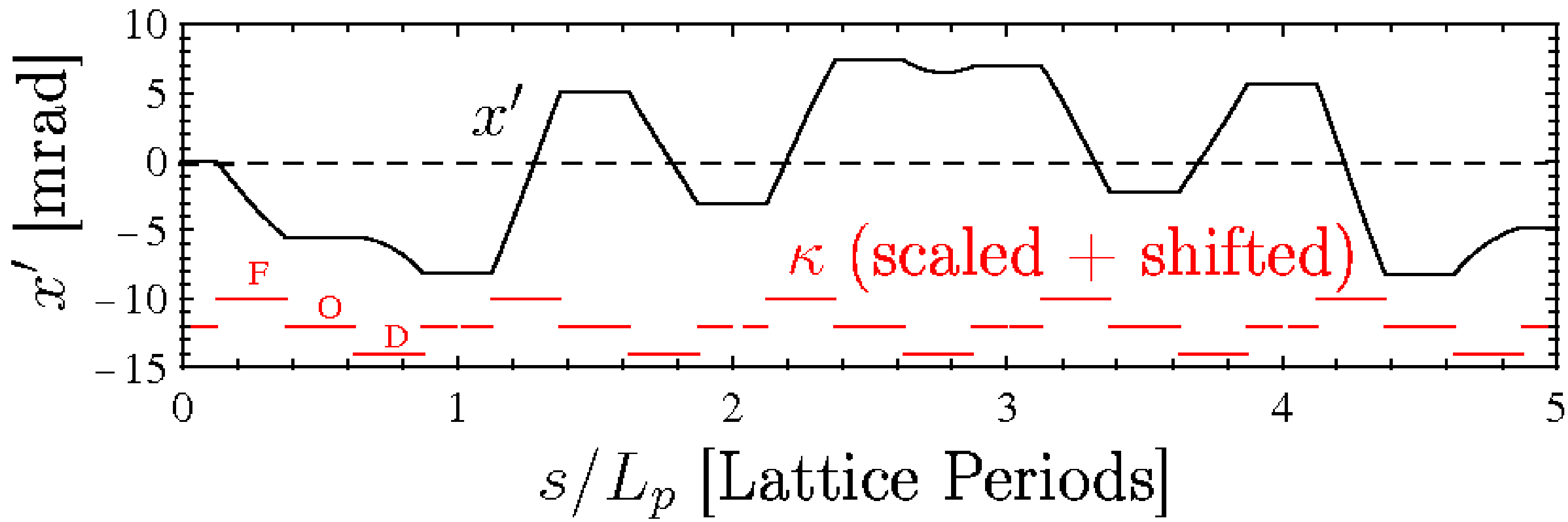
/// Example: Particle Orbits in a FODO Periodic Quadrupole Focusing Lattice:

Particle phase-space in x-x' with only hard-edge applied field

$L_p = 0.5$ m $\quad \kappa = \pm 50$ rad/m^2 in Quads $\quad x(0) = 1$ mm $\quad y(0) = 0$

$\eta = 0.5 \quad \phi \simeq 0 \quad \gamma_b\beta_b = \text{const} \quad x'(0) = 0 \quad y'(0) = 0$





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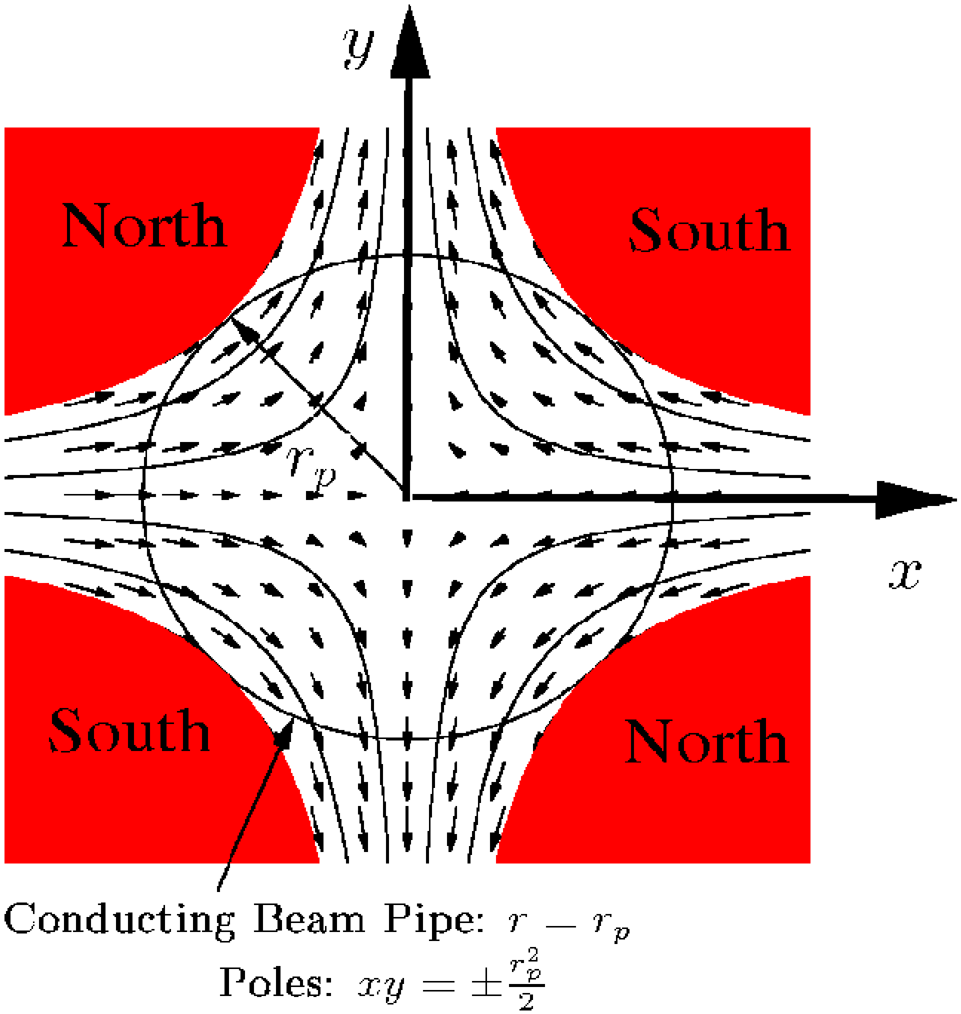
Comments on Orbits:

- Orbits strongly deviate from simple harmonic form due to AG focusing
  - Multiple harmonics present
- Orbit tends to be farther from axis in focusing quadrupoles and closer to axis in defocusing quadrupoles to provide net focusing
- Will find later that if the focusing is sufficiently strong, the orbit can become unstable (see: S5)
- *y*-orbit has the same properties as *x*-orbit due to the periodic structure and AG focusing
- If quadrupoles are rotated about their z-axis of symmetry, then the *x*- and *y*-equations become cross-coupled. This is called quadrupole skew coupling (see: Appendix A) and complicates the dynamics.

Some properties of particle orbits in quadrupoles with $\kappa = \text{const}$ will be analyzed in the problem sets

S2D: Alternating Gradient Quadrupole Focusing
Magnetic Quadrupoles

In the axial center of a long magnetic quadrupole, model fields as 2D transverse



2D Transverse Fields

$$\mathbf{E}^a_\perp = 0$$

$$B^a_x = Gy$$
$$B^a_y = Gx$$
$$B^a_z = 0$$

$$G \equiv \frac{B_q}{r_p} = \frac{\partial B^a_x}{\partial y} = \frac{\partial B^a_y}{\partial x}$$
$$= \text{Magnetic Gradient}$$

$$B_q = |\mathbf{B}^a|_{r=r_p} = \text{Pole Field}$$
$$r_p = \text{Pipe Radius}$$

- Magnetic (ideal iron) poles hyperbolic
- Structure infinitely extruded along z

//Aside: How can you calculate these fields?

Fields satisfy within vacuum aperture:

$$\nabla \cdot \mathbf{B}^a = 0 \qquad \nabla \times \mathbf{B}^a = 0 \quad \Longrightarrow \quad \mathbf{B}^{\mathbf{a}} = -\nabla \phi^a$$

Analogous to electric case, BUT magnetic force is different so rotate potential surfaces by 45 degrees:

Electric

$$\mathbf{F}_\perp = -q \frac{\partial \phi^a}{\partial \mathbf{x}_\perp}$$

$$\phi^a = \text{const}(x^2 - y^2)$$

Magnetic

$$\mathbf{F}_\perp = -q\beta_b c \hat{\mathbf{z}} \times \frac{\partial \phi^a}{\partial \mathbf{x}_\perp}$$

expect electric potential form rotated by 45 degrees ...

$$x \rightarrow \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} y$$

$$y \rightarrow \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y$$

$$\phi^a \rightarrow \phi^a = -\text{const} \cdot xy$$

$$\Longrightarrow \begin{aligned} B_x^a &= -\frac{\partial \phi^a}{\partial x} = \text{const} \cdot y \\ B_y^a &= -\frac{\partial \phi^a}{\partial y} = \text{const} \cdot x \end{aligned}$$

Require: $|\mathbf{B}^a| = B_p$ at $r = \sqrt{x^2 + y^2} = r_p$ $\Longrightarrow$ $\text{const} = B_p/r_p$

$$\Longrightarrow \phi^a = -\frac{B_p}{r_p} xy \qquad G = \frac{B_p}{r_p}$$

Realistic geometries can be considerably more complicated

- Truncated hyperbolic poles, truncated structure in *z*
- Both effects give nonlinear focusing terms

//

Analogously to the electric quadrupole case, take $G = G(s)$

- Same comments made on electric quadrupole fringe in S2C are directly applicable to magnetic quadrupoles

Magnetic quadrupole equations of motion:

- Insert field components into linear applied field equations and collect terms

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa(s)x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' - \kappa(s)y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$\kappa(s) = \frac{qG}{m\gamma_b\beta_b c} = \frac{G}{[B\rho]}$$

$$G = \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x} = \frac{B_q}{r_p} \qquad [B\rho] \equiv \frac{\gamma_b\beta_b mc}{q} = \text{Rigidity}$$

- Equations identical to the electric quadrupole case in terms of $\kappa(s)$
- All comments made on electric quadrupole focusing lattice are immediately applicable to magnetic quadruples: just apply different κ definitions in design
- Scaling of κ with energy different than electric case impacts applicability

$$\kappa = \begin{cases} \frac{G}{\beta_b c[B\rho]} & \text{Electric Focusing;} \quad G = \frac{\partial E_y^a}{\partial y} = \frac{2V_q}{r_p^2} \\ \frac{G}{[B\rho]} & \text{Magnetic Focusing;} \quad G = \frac{\partial B_x^a}{\partial y} = \frac{B_q}{r_p} \end{cases}$$

- Electric focusing weaker for higher particle energy (larger β_b)
- Technical limit values of gradients
  - Voltage holding for electric
  - Material properties (iron saturation, superconductor limits, ...) for magnetic
- See JJB Intro lectures for discussion on focusing technology choices

Different energy dependence also gives different dispersive properties when beam has axial momentum spread:

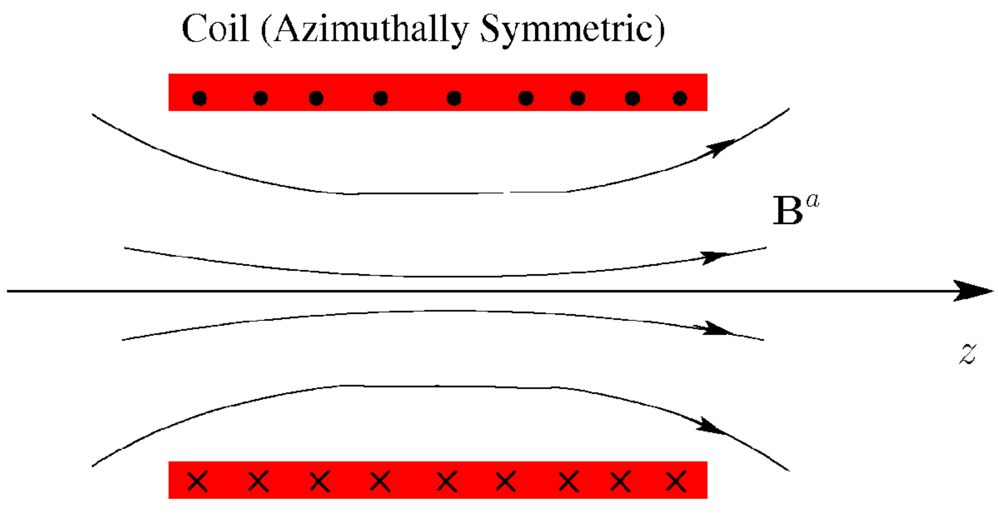
$$\delta \equiv \frac{\delta p}{p_0} = \text{Fractional Momentum Error}$$

$$\kappa \rightarrow \begin{cases} \frac{\kappa}{(1+\delta)^2} & \text{Electric Focusing} \\ \frac{\kappa}{1+\delta} & \text{Magnetic Focusing} \end{cases}$$

- Electric case further complicated because δ couples to the transverse motion since particles crossing higher electrostatic potentials are accelerated/deaccelerated

S2E: Solenoidal Focusing

The field of an ideal magnetic solenoid is invariant under transverse rotations about it's axis of symmetry (z) can be expanded in terms of the on-axis field as as:



Vacuum Maxwell equations:

$$\nabla \cdot \mathbf{B}^a = 0$$
$$\nabla \times \mathbf{B}^a = 0$$

Imply $\mathbf{B}^a$ can be expressed in terms of on-axis field $\mathbf{B}^a_z(r = 0, z)$

$$\mathbf{E}^a = 0$$

$$\mathbf{B}^a_\perp = \frac{1}{2} \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} \frac{\partial^{2\nu-1} B_{z0}(z)}{\partial z^{2\nu-1}} \left(\frac{|\mathbf{x}_\perp|}{2} \right)^{2\nu-2} \mathbf{x}_\perp$$

$$B^a_z = B_{z0}(z) + \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} B_{z0}(z)}{\partial z^{2\nu}} \left(\frac{|\mathbf{x}_\perp|}{2} \right)^{2\nu}$$

$$B_{z0}(z) \equiv B^a_z(\mathbf{x}_\perp = 0, z) = \text{On-Axis Field}$$

See Appendix D or Reiser, *Theory and Design of Charged Particle Beams*, Sec. 3.3.1

Writing out explicitly the terms of this expansion:

$$\mathbf{B}^a(r,z) = \hat{\mathbf{r}} B_r^a(r,z) + \hat{\mathbf{z}} B_z^a(r,z)$$
$$= (-\hat{\mathbf{x}} \sin\theta + \hat{\mathbf{y}} \cos\theta) B_r^a(r,z) + \hat{\mathbf{z}} B_z^a(r,z)$$

$$r = \sqrt{x^2+y^2}$$

where

$$B_r^a(r,z) = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} B_{z0}^{(2\nu-1)}(z) \left(\frac{r}{2}\right)^{2\nu-1}$$
$$= \boxed{-\frac{B'_{z0}(z)}{2} r} + \frac{B_{z0}^{(3)}(z)}{16} r^3 - \frac{B_{z0}^{(5)}(z)}{384} r^5 + \frac{B_{z0}^{(7)}(z)}{18432} r^7 - \frac{B_{z0}^{(9)}(z)}{1474560} r^9 + \ldots$$

$$B_z^a(r,z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} B_{z0}^{(2\nu)}(z) \left(\frac{r}{2}\right)^{2\nu}$$
$$= \boxed{B_{z0}(z)} - \frac{B''_{z0}(z)}{4} r^2 + \frac{B_{z0}^{(4)}(z)}{64} r^4 - \frac{B_{z0}^{(6)}(z)}{2304} r^6 + \frac{B_{z0}^{(8)}(z)}{147456} r^8 + \ldots$$

$$B_{z0}(z) \equiv B_z^a(r=0,z) = \text{On-axis Field}$$

$\boxed{\cdots}$ Linear Terms

$$B_{z0}^{(n)}(z) \equiv \frac{\partial^n B_{z0}(z)}{\partial z^n} \qquad B'_{z0}(z) \equiv \frac{\partial B_{z0}(z)}{\partial z} \qquad B''_{z0}(z) \equiv \frac{\partial^2 B_{z0}(z)}{\partial z^2}$$

For modeling, we truncate the expansion using only leading-order terms to obtain:

- Corresponds to linear dynamics in the equations of motion

$$B^a_x = -\frac{1}{2}\frac{\partial B_{z0}(z)}{\partial z}x$$

$$B^a_y = -\frac{1}{2}\frac{\partial B_{z0}(z)}{\partial z}y$$

$$B^a_z = B_{z0}(z)$$

$$B_{z0}(z) \equiv B^a_z(\mathbf{x}_\perp = 0, z) = \text{On-Axis Field}$$

Note that this truncated expansion is divergence free:

$$\nabla \cdot \mathbf{B}^a = -\frac{1}{2}\frac{\partial B_{z0}}{\partial z}\frac{\partial}{\partial \mathbf{x}_\perp}\cdot \mathbf{x}_\perp + \frac{\partial}{\partial z}B_{z0} = 0$$

but not curl free within the vacuum aperture:

$$\nabla \times \mathbf{B}^a = \frac{1}{2}\frac{\partial^2 B_{z0}(z)}{\partial z^2}(-\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)$$

$$= \frac{1}{2}\frac{\partial^2 B_{z0}(z)}{\partial z^2}r(-\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta) = \frac{1}{2}\frac{\partial^2 B_{z0}(z)}{\partial z^2}r\hat{\theta}$$

- Nonlinear terms needed to satisfy 3D Maxwell equations

Solenoid equations of motion:

- Insert field components into equations of motion and collect terms

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' - \frac{B'_{z0}(s)}{2[B\rho]}y - \frac{B_{z0}(s)}{[B\rho]}y' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

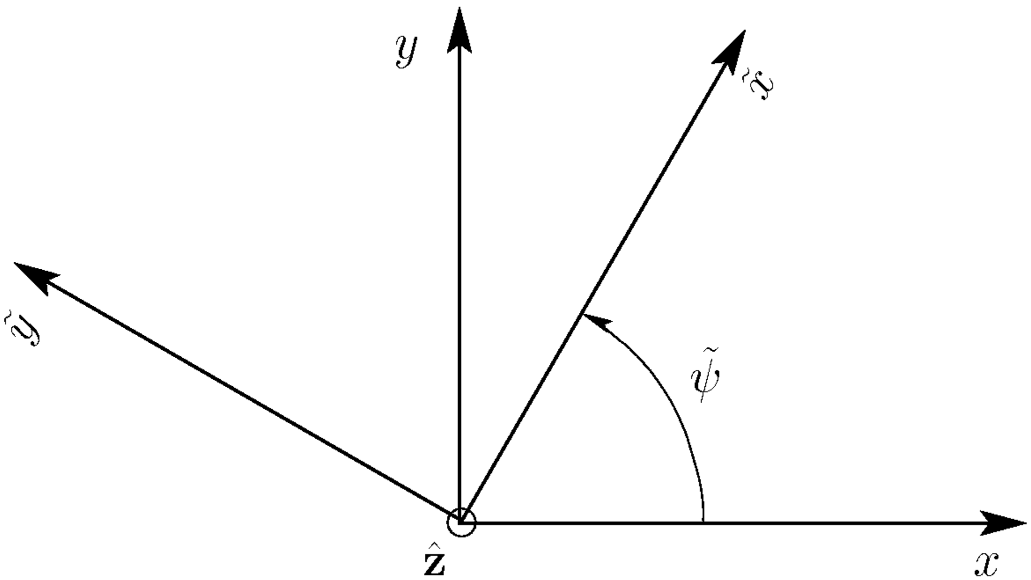
$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \frac{B'_{z0}(s)}{2[B\rho]}x + \frac{B_{z0}(s)}{[B\rho]}x' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$[B\rho] \equiv \frac{\gamma_b\beta_b mc}{q} = \text{Rigidity} \qquad \frac{B_{z0}(s)}{[B\rho]} = \frac{\omega_c(s)}{\gamma_b\beta_b c}$$

$$\omega_c(s) = \frac{qB_{z0}(s)}{m} = \text{Cyclotron Frequency (in applied axial magnetic field)}$$

- Equations are linearly cross-coupled in the applied field terms
  - x equation depends on y, y'
  - y equation depends on x, x'

It can be shown (see: Appendix B) that the linear cross-coupling in the applied field can be removed by an s-varying transformation to a rotating "Larmor" frame:



$$\tilde{x} = \quad x \cos \tilde{\psi}(s) + y \sin \tilde{\psi}(s)$$
$$\tilde{y} = -x \sin \tilde{\psi}(s) + y \cos \tilde{\psi}(s)$$

$$\tilde{\psi}(s) = - \int_{s_i}^{s} d\bar{s}\; k_L(\bar{s})$$

$$k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]} = \frac{\omega_c(s)}{2\gamma_b \beta_b c}$$
= Larmor wave number

$s = s_i$ defines initial condition

$\tilde{\ldots}$ used to denote rotating frame variables

If the beam space-charge is *axisymmetric*:

$$\frac{\partial \phi}{\partial \mathbf{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial \mathbf{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\mathbf{x}_\perp}{r}$$

then the space-charge term also decouples under the Larmor transformation and the equations of motion can be expressed in fully uncoupled form:

$$\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa(s)\tilde{x} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial r}\frac{\tilde{x}}{r}$$

$$\tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa(s)\tilde{y} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial r}\frac{\tilde{y}}{r}$$

$$\kappa(s) = k_L^2(s) \equiv \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2$$

Will demonstrate this in problems for the simple case of:

$$B_{z0}(s) = \text{const}$$

- Because Larmor frame equations are in the same form as continuous and quadrupole focusing with a different κ, for solenoidal focusing we implicitly work in the Larmor frame and simplify notation by dropping the tildes:

$$\tilde{\mathbf{x}}_\perp \to \mathbf{x}_\perp$$

/// Aside: Notation:

A common theme of this class will be to introduce new effects and generalizations while keeping formulations looking as similar as possible to the the most simple representations given. When doing so, we will often use “tildes” to denote transformed variables to stress that the new coordinates have, in fact, a more complicated form that must be interpreted in the context of the analysis being carried out. Some examples:

- Larmor frame transformations for Solenoidal focusing

  See: Appendix B
- Normalized variables for analysis of accelerating systems

  See: S10
- Coordinates expressed relative to the beam centroid

  See: S.M. Lund, lectures on Transverse Centroid and Envelope Model
- Variables used to analyze Einzel lenses

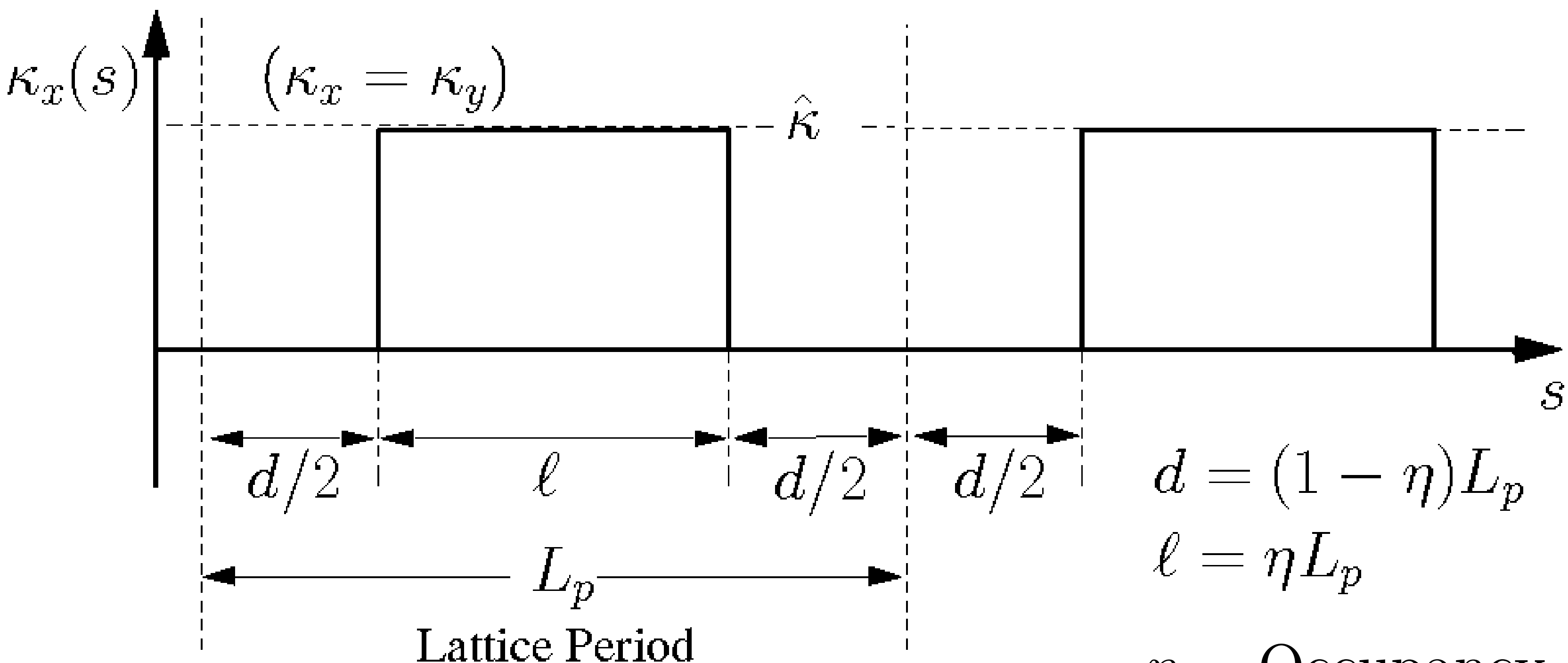
  See: J.J. Barnard, Introductory Lectures

///

Solenoid periodic lattices can be formed similarly to the quadrupole case

- Drifts placed between solenoids of finite axial length
  - Allows space for diagnostics, pumping, acceleration cells, etc.
- Analogous equivalence cases to quadrupole
  - Piecewise constant κ often used
- Fringe can be more important for solenoids

Simple hard-edge solenoid lattice with piecewise constant κ



$$d = (1 - \eta)L_p$$

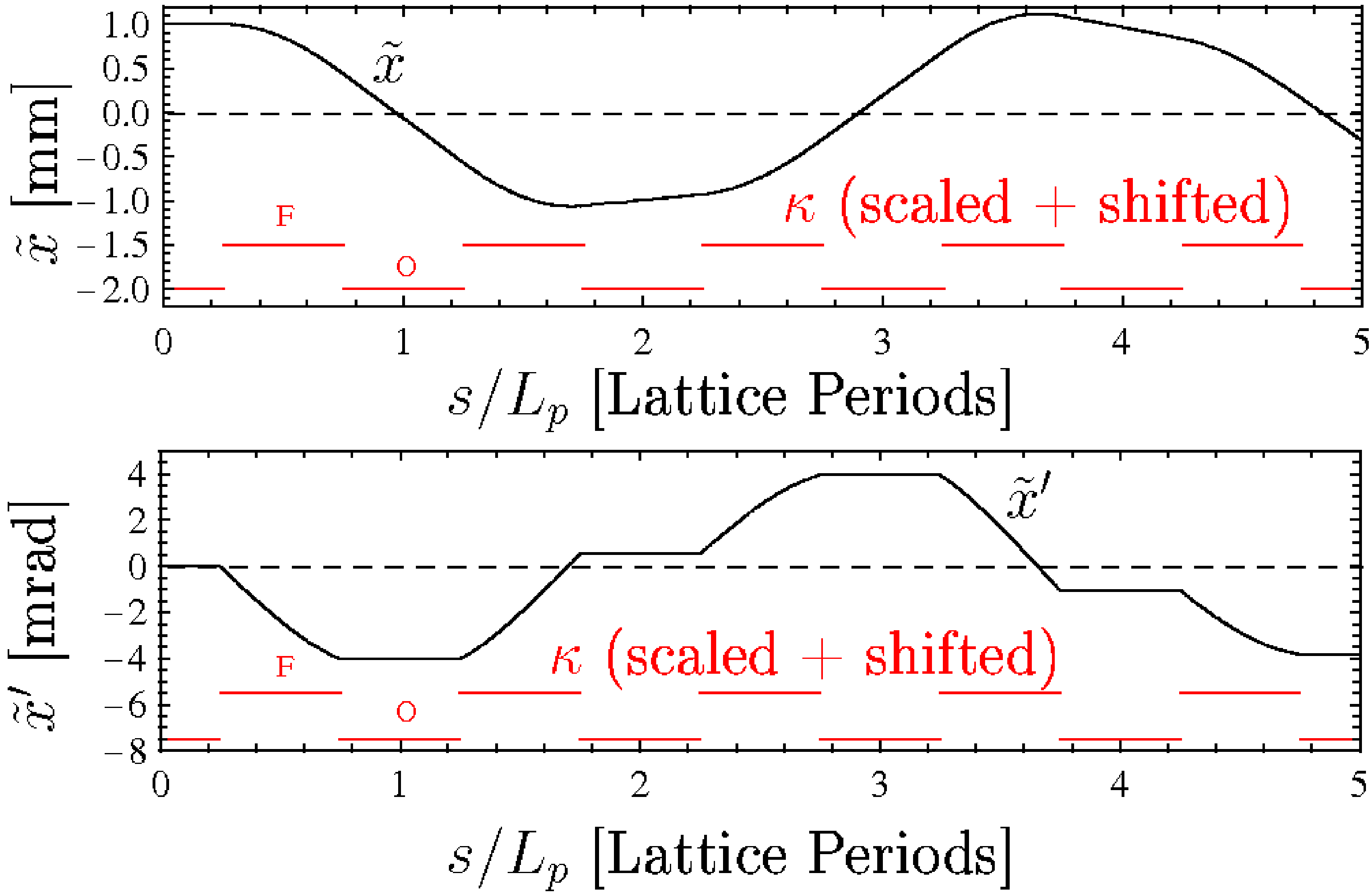
$$\ell = \eta L_p$$

$$\eta = \text{Occupancy} \in (0, 1]$$

/// Example: Larmor Frame Particle Orbits in a Periodic Solenoidal Focusing Lattice: $\tilde{x} - \tilde{x}'$ phase-space for hard edge elements and applied fields

$L_p = 0.5$ m $\quad \kappa = 20\ \text{rad/m}^2$ in Solenoids $\quad \tilde{x}(0) = 1$ mm $\quad \tilde{y}(0) = 0$

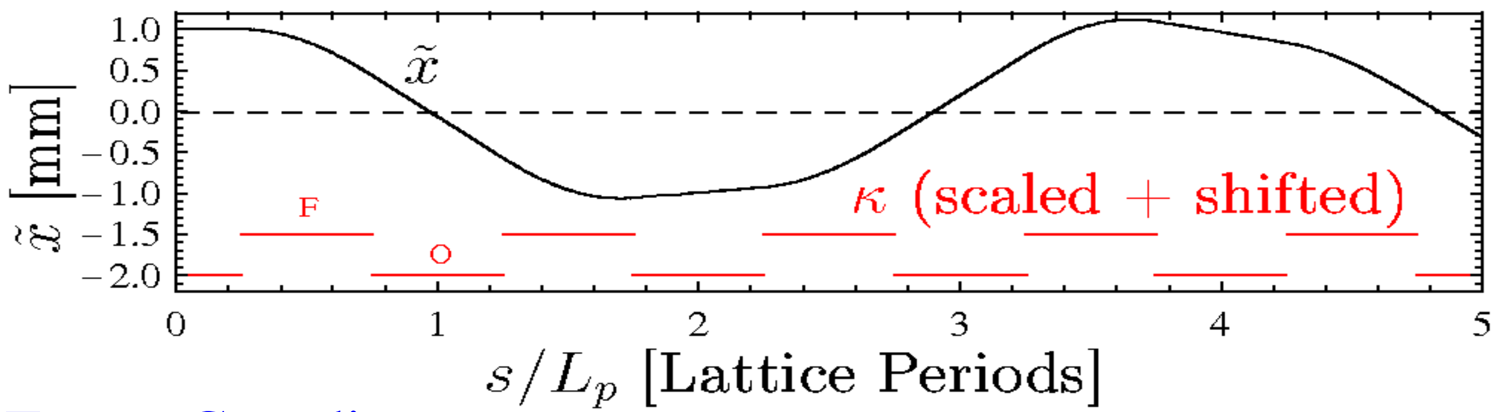
$\eta = 0.5 \quad \phi \simeq 0 \quad \gamma_b \beta_b = \text{const} \quad \tilde{x}'(0) = 0 \quad \tilde{y}'(0) = 0$



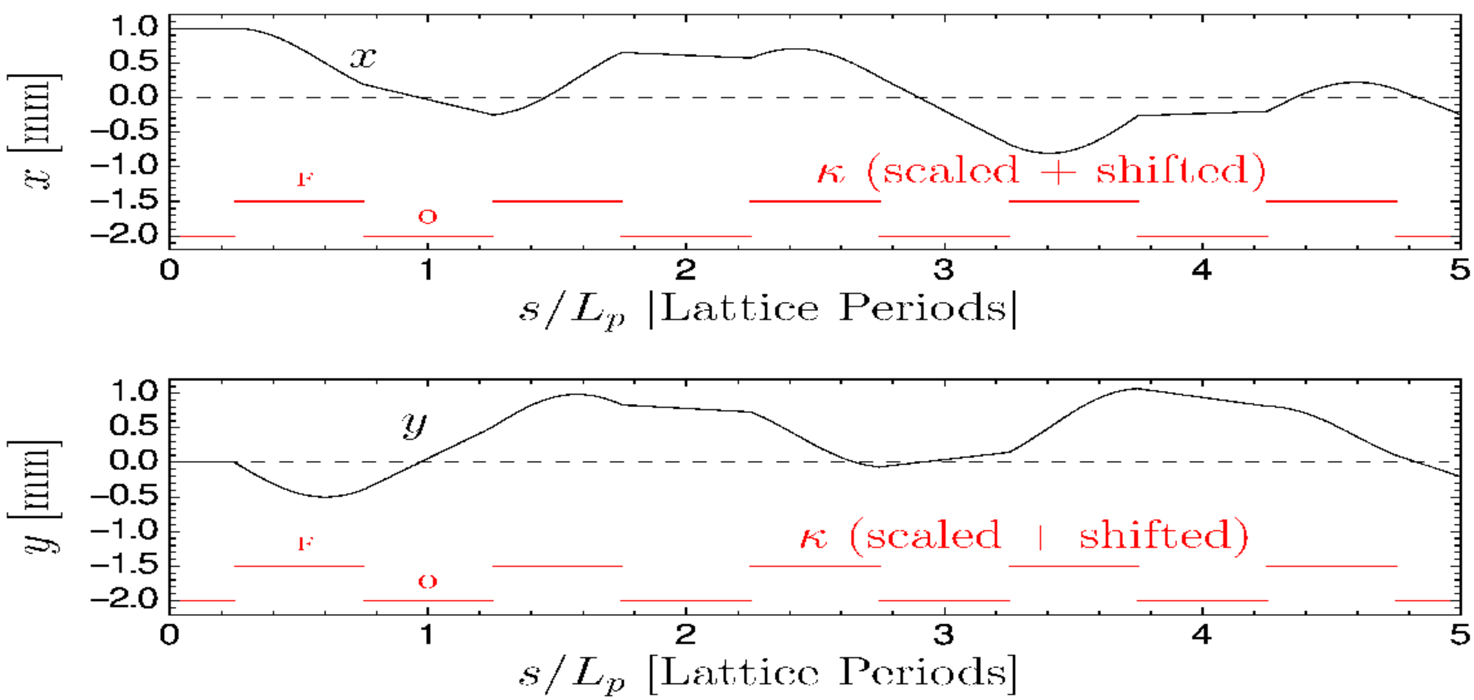
Contrast of Larmor-Frame and Lab-Frame Orbits

- Same initial condition

Larmor-Frame Coordinate   Orbit in transformed *x*-plane only



Lab-Frame Coordinate   Orbit in both *x*- and *y*-planes

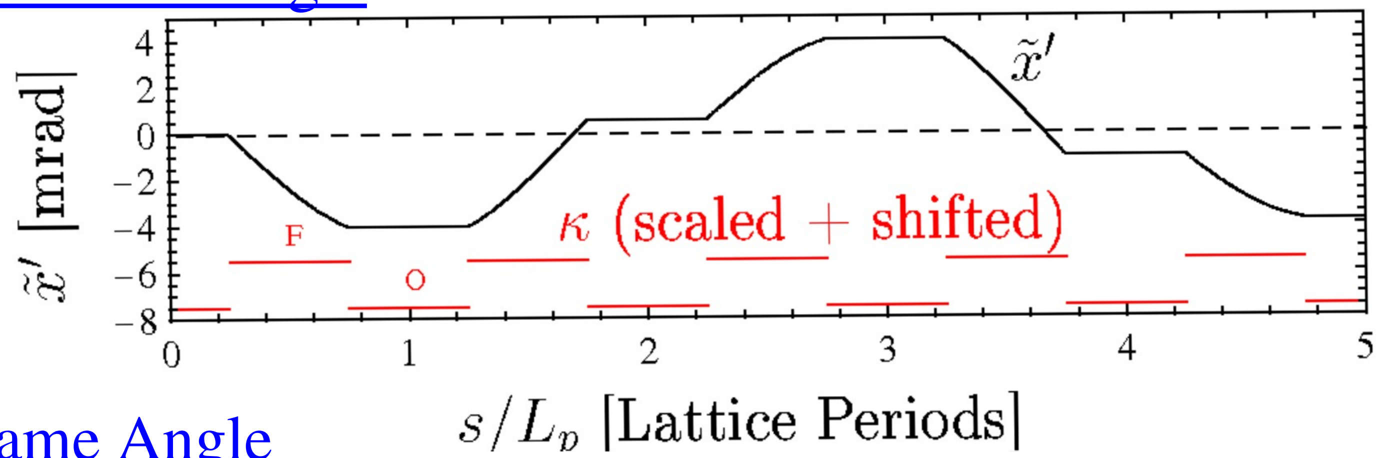


Calculate using transfer matrices in Appendix C

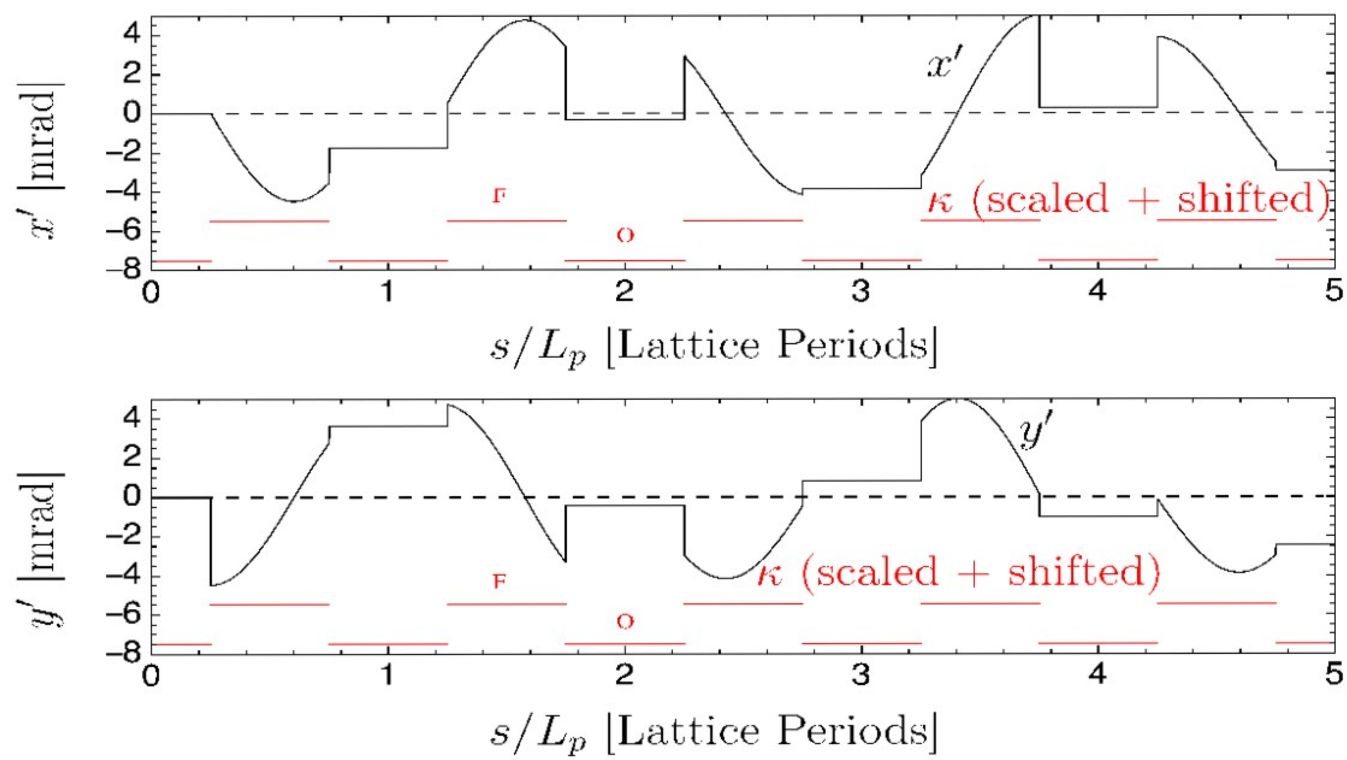
Contrast of Larmor-Frame and Lab-Frame Orbits

- Same initial condition

Larmor-Frame Angle



Lab-Frame Angle

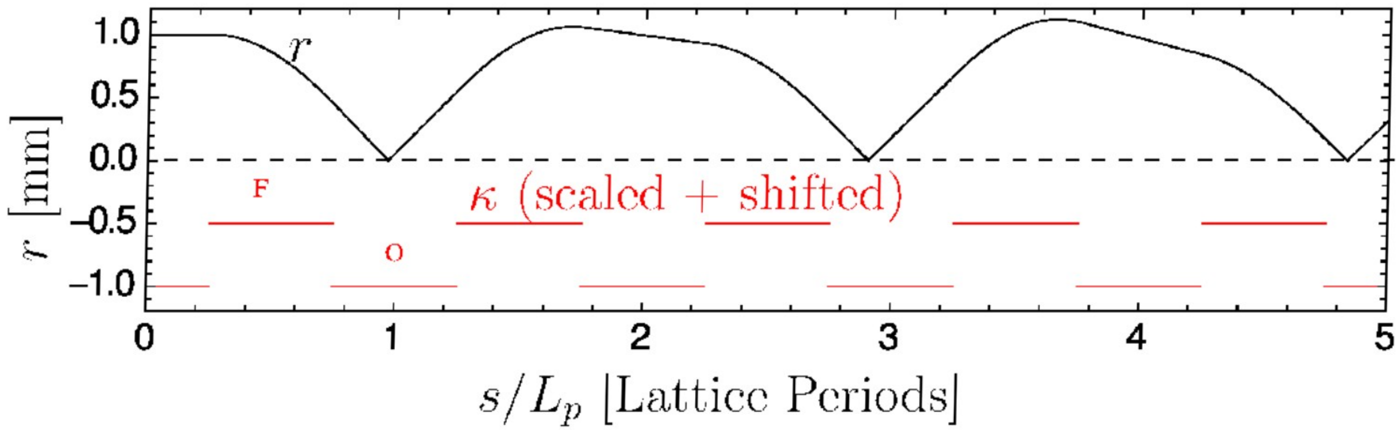


Calculate using transfer matrices in Appendix C

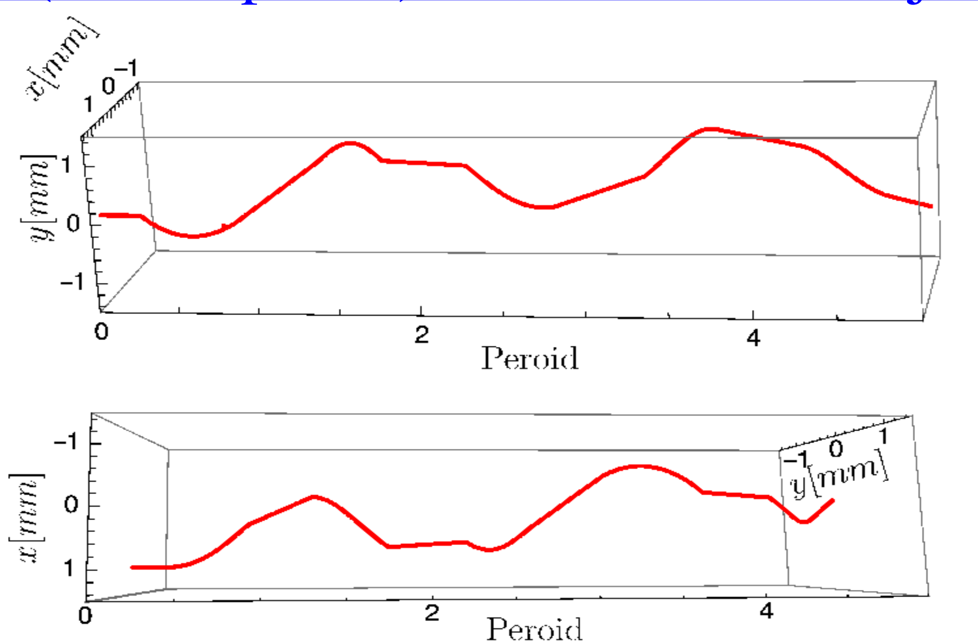
Additional perspectives of particle orbit in solenoid transport channel

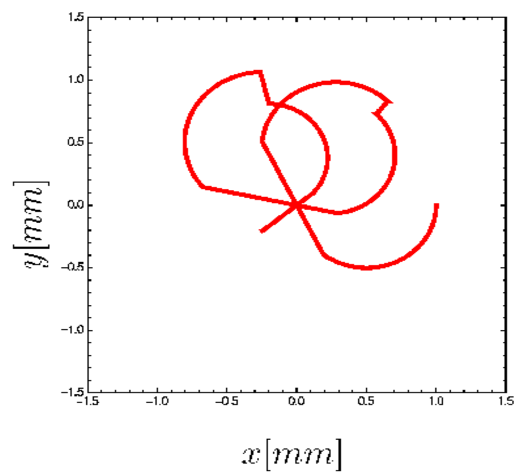
- Same initial condition

Radius evolution (Lab or Larmor Frame: radius same)



Side- (2 view points) and End-View Projections of 3D Lab-Frame Orbit





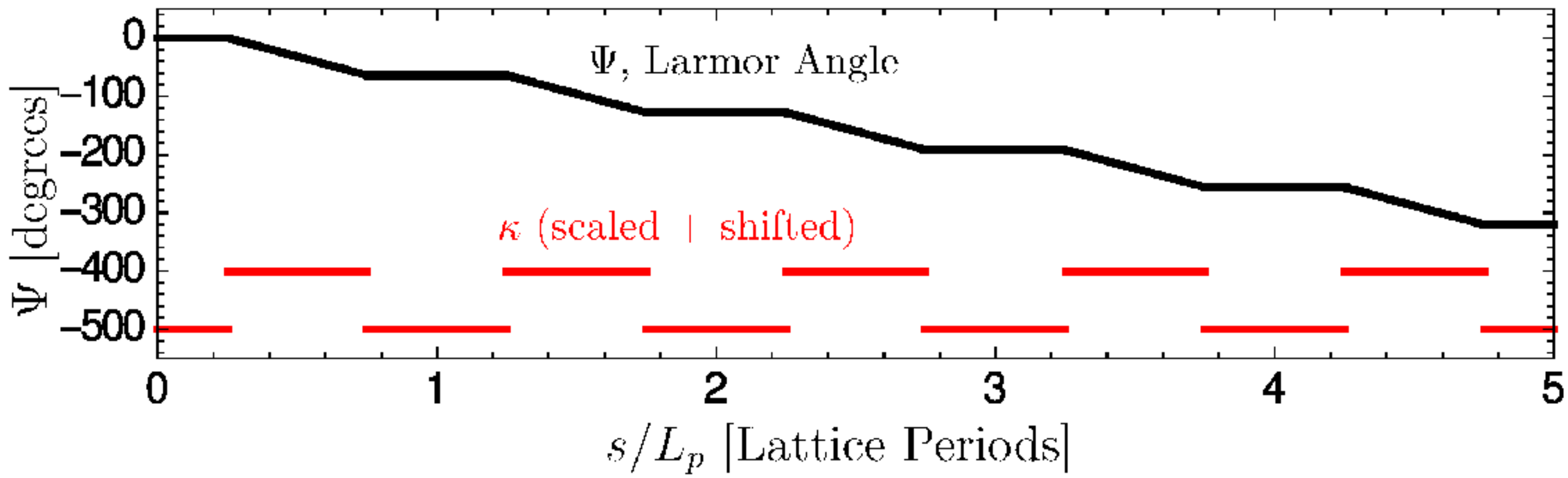
Calculate using transfer matrices in Appendix C

Larmor angle and angular momentum of particle orbit in solenoid transport channel

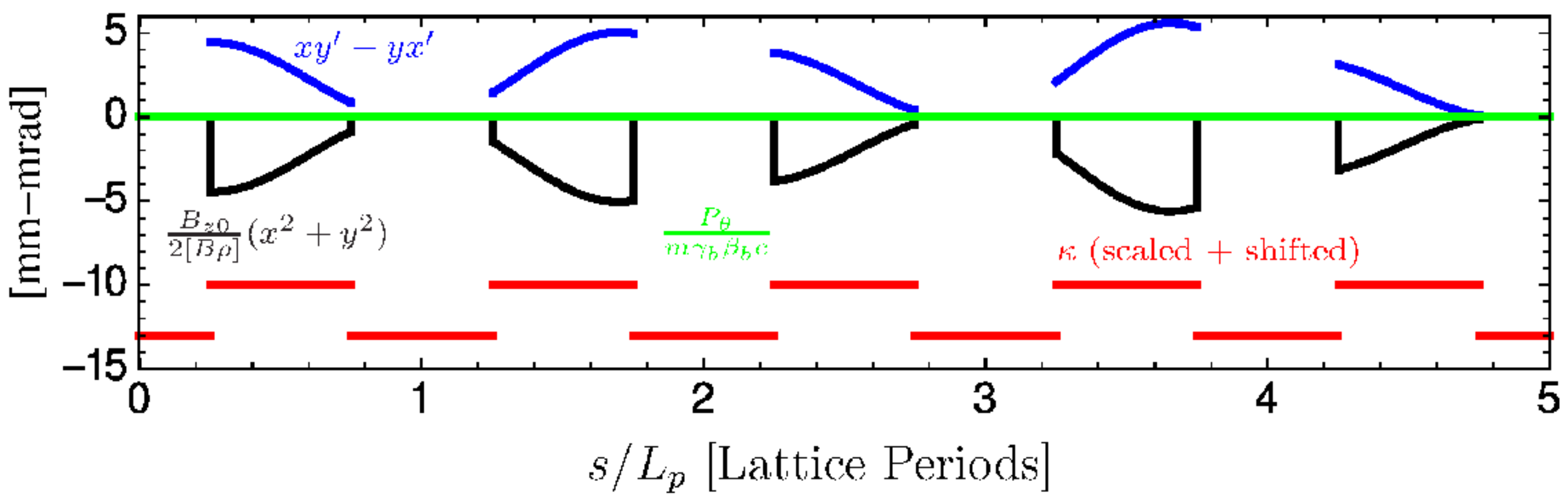
- Same initial condition

Larmor Angle

$$\tilde{\psi}(s) = -\int_{s_i}^{s} d\bar{s}\ k_L(\bar{s}) \qquad k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]}$$



Angular Momentum and Canonical Angular Momentum (see Sec. S2G)



///

Comments on Orbits:

- See Appendix C for details on calculation
  - Discontinuous fringe of hard-edge model must be treated carefully if integrating in the laboratory-frame.
- Larmor-frame orbits strongly deviate from simple harmonic form due to periodic focusing
  - Multiple harmonics present
  - Less complicated than quadrupole AG focusing case when interpreted in the Larmor frame due to the optic being focusing in both planes
- Orbits transformed back into the Laboratory frame using Larmor transform (see: Appendix B and Appendix C)
  - Laboratory frame orbit exhibits more complicated x-y plane coupled oscillatory structure
- Will find later that if the focusing is sufficiently strong, the orbit can become unstable (see: S5)
- Larmor frame y-orbits have same properties as the x-orbits due to the equations being decoupled and identical in form in each plane
  - In example, Larmor y-orbit is zero due to simple initial condition in x-plane
  - Lab y-orbit is nozero due to x-y coupling

Comments on Orbits (continued):

- Larmor angle advances continuously even for hard-edge focusing
- Mechanical angular momentum jumps discontinuously going into and out of the solenoid
  - Particle spins up and down going into and out of the solenoid
  - No mechanical angular momentum outside of solenoid due to the choice of intial condition in this example (initial x-plane motion)
- Canonical angular momentum P_θ is conserved in the 3D orbit evolution
  - As expected from analysis in S2G
  - Invariance provides a good check on dynamics
  - P_θ in example has zero value due to the specific (x-plane) choice of initial condition. Other choices can give nonzero values and finite mechanical angular momentum in drifts.

Some properties of particle orbits in solenoids with piecewise $\kappa = \text{const}$ will be analyzed in the problem sets

///

S2F: Summary of Transverse Particle Equations of Motion

In linear applied focusing channels, without momentum spread or radiation, the particle equations of motion in both the *x*- and *y*-planes expressed as:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial x}\phi$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial y}\phi$$

$$\kappa_x(s) = x\text{-focusing function of lattice}$$

$$\kappa_y(s) = y\text{-focusing function of lattice}$$

Common focusing functions:

Continuous:

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

Quadrupole (Electric or Magnetic):

$$\kappa_x(s) = -\kappa_y(s) = \kappa(s)$$

Solenoidal (equations must be interpreted in Larmor Frame: see Appendix B):

$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

Although the equations have the same form, the couplings to the fields are different which leads to different regimes of applicability for the various focusing technologies with their associated technology limits:

Focusing:

Continuous:

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

Good qualitative guide (see later material/lecture)

BUT not physically realizable (see S2B)

Quadrupole:

$$\kappa_x(s) = -\kappa_y(s) = \begin{cases} \dfrac{G(s)}{\beta_b c[B\rho]}, & \text{Electric} \\ \dfrac{G(s)}{c[B\rho]}, & \text{Magnetic} \end{cases} \qquad [B\rho] = \frac{m\gamma_b\beta_b c}{q}$$

G is the field gradient which for linear applied fields is:

$$G(s) = \begin{cases} -\dfrac{\partial E_x^a}{\partial x} = \dfrac{\partial E_y^a}{\partial y} = \dfrac{2V_q}{r_p^2}, & \text{Electric} \\ \dfrac{\partial B_x^a}{\partial y} = \dfrac{\partial B_y^a}{\partial x} = \dfrac{B_p}{r_p}, & \text{Magnetic} \end{cases}$$

Solenoid:

$$\kappa_x(s) = \kappa_y(s) = k_L^2(s) = \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2 \qquad \omega_c(s) = \frac{qB_{z0}(s)}{m}$$

It is instructive to review the structure of solutions of the transverse particle equations of motion in the absence of:

Space-charge: $\frac{\partial \phi}{\partial x} \sim \frac{\partial \phi}{\partial y} \sim 0$

Acceleration: $\gamma_b \beta_b \simeq \text{const} \qquad \Longrightarrow \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \simeq 0$

In this simple limit, the *x* and *y*-equations are of the same Hill's Equation form:

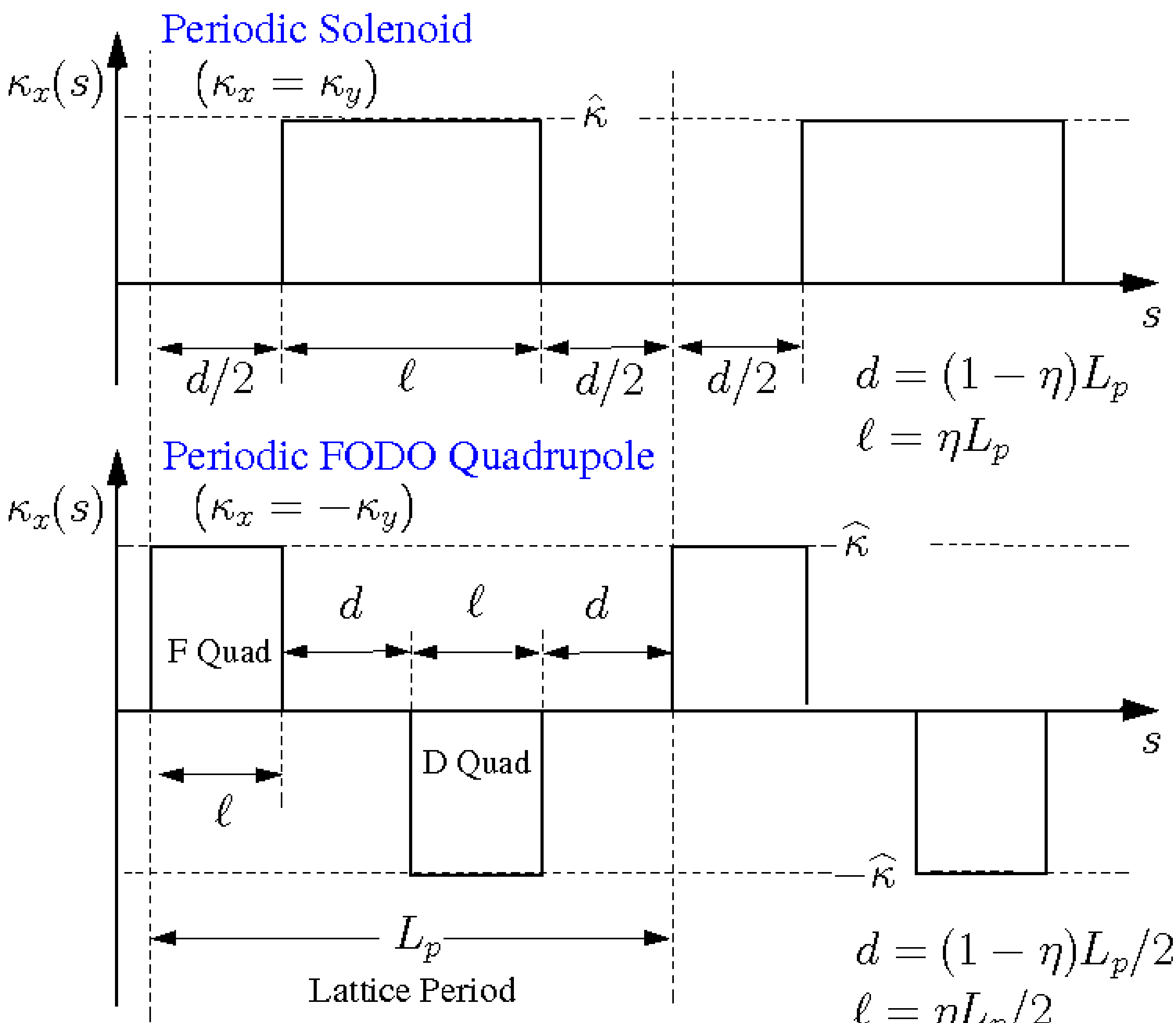
$$x'' + \kappa_x(s) x = 0$$
$$y'' + \kappa_y(s) y = 0$$

- These equations are central to transverse dynamics in conventional accelerator physics (weak space-charge and acceleration)
  - Will study how solutions change with space-charge in later lectures

In many cases beam transport lattices are designed where the applied focusing functions are periodic:

$$\kappa_x(s + L_p) = \kappa_x(s) \qquad L_p = \text{Lattice Period}$$
$$\kappa_y(s + L_p) = \kappa_y(s)$$

Common, simple examples of periodic lattices:



However, the focusing functions need not be periodic:

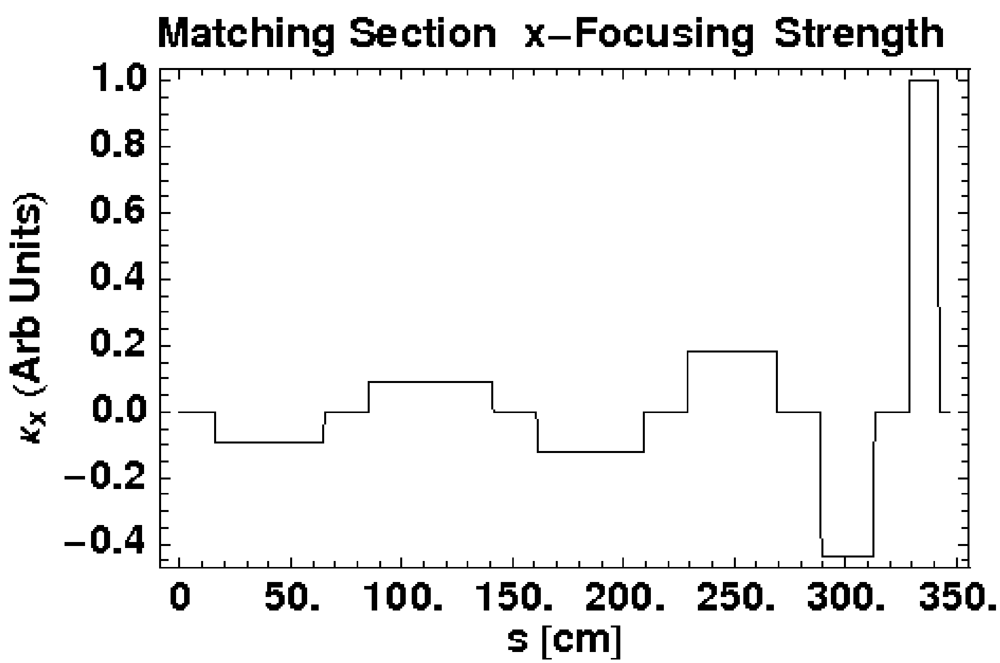
- Often take periodic or continuous in this class for simplicity of interpretation

Focusing functions can vary strongly in many common situations:

- Matching and transition sections
- Strong acceleration
- Significantly different elements can occur within periods of lattices in rings
  - "Panofsky" type (wide aperture along one plane) quadrupoles for beam insertion and extraction in a ring

Example of Non-Periodic Focusing Functions: Beam Matching Section

Maintains alternating-gradient structure but not quasi-periodic



Example corresponds to High Current Experiment Matching Section (hard edge equivalent) at LBNL (2002)

Equations presented in this section apply to a single particle moving in a beam under the action of linear applied focusing forces. In the remaining sections, we will (mostly) neglect space-charge ($\phi \rightarrow 0$) as is conventional in the standard theory of low-intensity accelerators.

- What we learn from treatment will later aid analysis of space-charge effects
  - Appropriate variable substitutions will be made to apply results
- Important to understand basic applied field dynamics since space-charge complicates
  - Results in plasma-like collective response

/// Example: We will see in Transverse Centroid and Envelope Descriptions of Beam Evolution that the linear particle equations of motion can be applied to analyze the evolution of a beam when image charges are neglected

$$x \rightarrow x_c \equiv \langle x \rangle_\perp \quad x - \text{centroid}$$

$$y \rightarrow y_c \equiv \langle y \rangle_\perp \quad y - \text{centroid}$$

///

S2G: Conservation of Angular Momentum in Axisymmetric Focusing Systems

Background:

Goal: find an invariant for axisymmetric focusing systems which can help us further interpret/understand the dynamics.

In Hamiltonian descriptions of beam dynamics one must employ proper canonical conjugate variables such as (x-plane):

$$\begin{aligned} x &= && \text{Canonical Coordinate} \\ P_x = p_x + qA_x &= && \text{Canonical Momentum} \end{aligned}$$

+ analogous y-plane

Here, $\boldsymbol{A}$ denotes the vector potential of the (static for cases of field models considered here) applied magnetic field with:

$$\mathbf{B}^a = \nabla \times \mathbf{A}$$

For the cases of linear applied magnetic fields in this section, we have:

$$\mathbf{A} = \begin{cases} \hat{\mathbf{z}}\frac{G}{2}(y^2 - x^2), & \text{Magnetic Quadrupole Focusing} \\ -\hat{\mathbf{x}}\frac{1}{2}B_{z0}y + \hat{\mathbf{y}}\frac{1}{2}B_{z0}x, & \text{Solenoidal Focusing} \\ 0, & \text{Otherwise} \end{cases}$$

For continuous, electric or magnetic quadrupole focusing without acceleration $(\gamma_b \beta_b = \text{const})$, it is straightforward to verify that *x,x'* and *y,y'* are canonical coordinates and that the correct equations of motion are generated by the Hamiltonian:

$$H_\perp = \frac{1}{2}x'^2 + \frac{1}{2}y'^2 + \frac{1}{2}\kappa_x x^2 + \frac{1}{2}\kappa_y y^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2 c^3}$$

$$\frac{d}{ds}x = \frac{\partial H_\perp}{\partial x'} \qquad \frac{d}{ds}x = \frac{\partial H_\perp}{\partial y'}$$

$$\frac{d}{ds}x' = -\frac{\partial H_\perp}{\partial x} \qquad \frac{d}{ds}y' = -\frac{\partial H_\perp}{\partial y}$$

Giving the familiar equations of motion:

$$x'' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \kappa_y y = -\frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial\phi}{\partial y}$$

For solenoidal magnetic focusing without acceleration, it can be verified that we can take (tilde) canonical variables:

- Tildes *do not* denote Larmor transform variables here !

$$\tilde{x} = x \qquad \tilde{y} = y$$

$$\tilde{x}' = x' - \frac{B_{z0}}{2[B\rho]} y \qquad \tilde{y}' = y' + \frac{B_{z0}}{2[B\rho]} x \qquad [B\rho] \equiv \frac{m\gamma_b\beta_b c}{q}$$

With Hamiltonian:

$$\tilde{H}_\perp = \frac{1}{2}\left[\left(\tilde{x}' + \frac{B_{z0}}{2[B\rho]}\tilde{y}\right)^2 + \left(\tilde{y}' - \frac{B_{z0}}{2[B\rho]}\tilde{x}\right)^2\right] + \frac{q\phi}{m\gamma_b^3\beta_b^2c^3}$$

$$\frac{d}{ds}\tilde{x} = \frac{\partial \tilde{H}_\perp}{\partial \tilde{x}'} \qquad \frac{d}{ds}\tilde{y} = \frac{\partial \tilde{H}_\perp}{\partial \tilde{y}'}$$

$$\frac{d}{ds}\tilde{x}' = -\frac{\partial H_\perp}{\partial \tilde{x}} \qquad \frac{d}{ds}\tilde{y}' = -\frac{\partial H_\perp}{\partial \tilde{y}}$$

Caution:
Primes do not mean *d/ds* in tilde variables here: just notation to distinguish “momentum” variable!

Giving (after some algebra) the familiar equations of motion:

$$x'' - \frac{B'_{z0}(s)}{2[B\rho]} y - \frac{B_{z0}(s)}{[B\rho]} y' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial x}$$

$$y'' + \frac{B'_{z0}(s)}{2[B\rho]} x + \frac{B_{z0}(s)}{[B\rho]} x' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial y}$$

Canonical angular momentum

One expects from general considerations (Noether's Theorem in dynamics) that systems with a symmetry have a conservation constraint associated with the generator of the symmetry. So for systems with azimuthal symmetry $(\partial/\partial\theta = 0)$, one expects there to be a conserved canonical angular momentum (generator of rotations). Based on the Hamiltonian dynamics structure, examine:

$$P_\theta \equiv [\mathbf{x} \times \mathbf{P}] \cdot \hat{\mathbf{z}} = [\mathbf{x} \times (\mathbf{p} + q\mathbf{A})] \cdot \hat{\mathbf{z}}$$

This is exactly equivalent to

- Here γ factor is exact (*not* paraxial)

$$\begin{aligned} P_\theta &= (xp_y - yp_x) + q(xA_y - yA_x) \\ &= r(p_\theta + qA_\theta) = m\gamma r^2\dot{\theta} + qrA_\theta \end{aligned}$$

Or employing the usual paraxial approximation steps:

$$\begin{aligned} P_\theta &\simeq m\gamma_b\beta_b c(xy' - yx') + q(xA_y - yA_x) \\ &= m\gamma_b\beta_b c r^2\theta' + qrA_\theta \end{aligned}$$

Inserting the vector potential components consistent with linear approximation solenoid focusing in the paraxial expression gives:

- Applies to (superimposed or separately) to continuous, magnetic or electric quadrupole, or solenoidal focusing since $A_\theta \neq 0$ only for solenoidal focusing

$$\begin{aligned} P_\theta &\simeq m\gamma_b\beta_b c(xy' - yx') + \frac{qB_{z0}}{2}(x^2 + y^2) \\ &= m\gamma_b\beta_b c r^2\theta' + \frac{qB_{z0}}{2}r^2 \end{aligned}$$

For a coasting beam $(\gamma_b\beta_b = \text{const})$, it is often convenient to analyze:

- Later we will find this is analogous to use of "unnormalized" variables used in calculation of ordinary emittance rather than normalized emittance

$$\begin{aligned} \frac{P_\theta}{m\gamma_b\beta_b c} &= xy' - yx' + \frac{B_{z0}}{2[B\rho]}(x^2 + y^2) \\ &= r^2\theta' + \frac{B_{z0}}{2[B\rho]}r^2 \end{aligned}$$

$$[B\rho] \equiv \frac{m\gamma_b\beta_b c}{q}$$

Conservation of canonical angular momentum

To investigate situations where the canonical angular momentum is a constant of the motion for a beam evolving in linear applied fields, we differentiate P_θ with respect to *s* and apply equations of motion

Equations of Motion:

Including acceleration effects again, we summarize the equations of motion as:

- Applies to continuous, quadrupole (electric + magnetic), and solenoid focusing as expressed
- Several types of focusing can also be superimposed
  - Show for superimposed solenoid

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x - \frac{B'_{z0}(s)}{2[B\rho]}y - \frac{B_{z0}(s)}{[B\rho]}y' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y y + \frac{B'_{z0}(s)}{2[B\rho]}x + \frac{B_{z0}(s)}{[B\rho]}x' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$[B\rho] = \frac{m\gamma_b\beta_b c}{q} \qquad \kappa_x(s) = \begin{cases} k_{\beta 0}^2 = \text{const}, & \text{Continuous Focus } (\kappa_y = \kappa_x) \\ \frac{G(s)}{\beta_b c[B\rho]}, & \text{Electric Quadrupole Focus } (\kappa_y = -\kappa_x) \\ \frac{G(s)}{c[B\rho]}, & \text{Magnetic Quadrupole Focus } (\kappa_y = -\kappa_x) \end{cases}$$

Employ the paraxial form of P_θ consistent with the possible existence of a solenoid magnetic field:

- Formula also applies as expressed to continuous and quadrupole focusing

$$\mathrm{P}_\theta = m\gamma_b\beta_b c(xy' - yx') + \frac{qB_{z0}}{2}(x^2 + y^2)$$

Differentiate and apply equations of motion:

- Intermediate algebraic steps not shown

$$\frac{d}{ds}P_\theta = mc(\gamma_b\beta_b)'(xy' - yx') + mc(\gamma_b\beta_b)(xy'' - yx'')$$

$$+ \frac{qB'_{z0}}{2}(x^2 + y^2) + qB_{z0}(xx' + yy')$$

$$= mc(\gamma_b\beta_b)[\kappa_x - \kappa_y]xy - \frac{q}{\gamma_b^2\beta_b c}\left(x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x}\right)$$

So *IF*:

1) $\kappa_x = \kappa_y$

- Valid continuous or solenoid focusing
- Invalid for quadrupole focusing

2) $x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\theta} = 0$

- Axisymmetric beam

$$\frac{d}{ds}P_\theta = 0 \qquad \Longrightarrow \qquad P_\theta = \text{const}$$

For:

- Continuous focusing
- Linear optics solenoid magnetic focusing
- Other axisymmetric electric optics not covered such as Einzel lenses …

$$\mathrm{P}_\theta = m\gamma_b\beta_b c(xy' - yx') + \frac{qB_{z0}}{2}(x^2 + y^2) = \text{const}$$

$$m\gamma_b\beta_b c(xy' - yx') = \text{Mechanical Angular Momentum Term}$$

$$\frac{qB_{z0}}{2}(x^2 + y^2) = \text{Vector Potential Angular Momentum Term}$$

In S2E we plot for solenoidal focusing :

- Mechanical angular momentum $\propto xy' - yx'$
- Larmor rotation angle $\tilde{\psi}$
- Canonical angular momentum (constant) P_θ

Comments:

- Where valid, $P_\theta = \text{const}$ provides a powerful constraint to check dynamics
- If $P_\theta = \text{const}$ for all particles, then $\langle P_\theta \rangle = \text{const}$ for the beam as a whole and it is found in envelope models that canonical angular momentum can act effectively act phase-space area (emittance-like term) defocusing the beam
- Valid for acceleration: similar to a “normalized emittance”: see S10

Example: solenoidal focusing channel

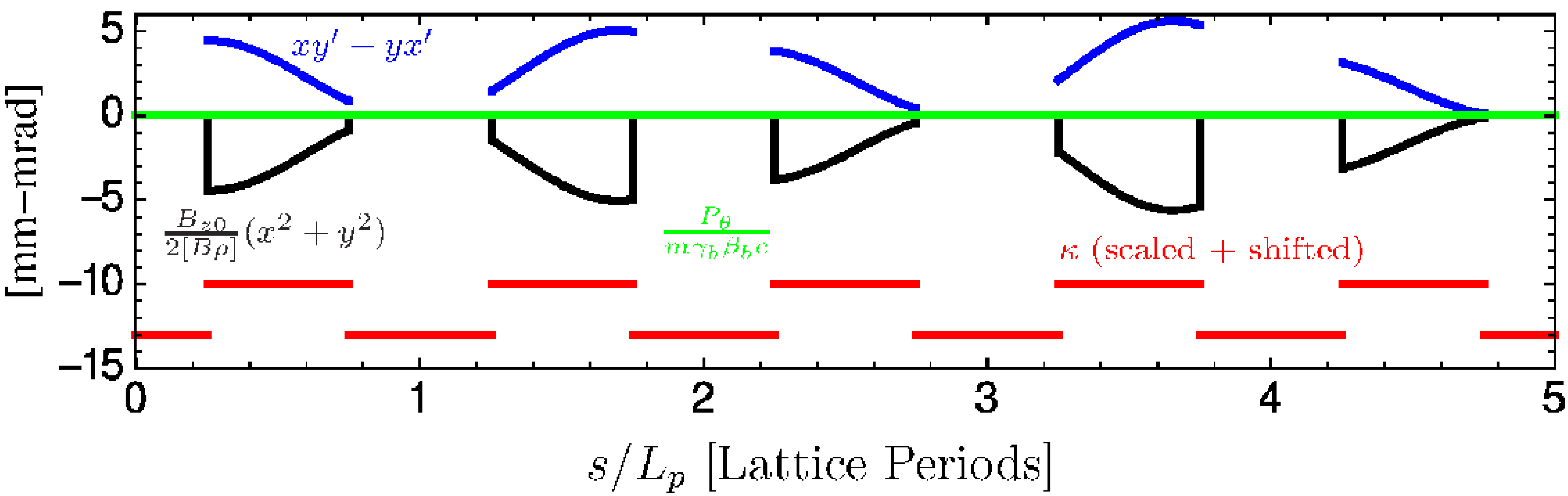
Employ the solenoid focusing channel example in S2E and plot:

- Mechanical angular momentum $\propto xy' - yx'$
- Vector potential contribution to canonical angular momentum $\propto B_{z0}(x^2 + y^2)$
- Canonical angular momentum (constant) P_θ

(green) $$\frac{P_\theta}{m\gamma_b\beta_b c} = xy' - yx' + \frac{B_{z0}}{2[B\rho]}(x^2 + y^2) = \text{const} = \text{Canonical Angular Momentum}$$

(black) $$xy' - yx' = r^2\theta' = \text{Mechanical Angular Momentum}$$

(blue) $$\frac{B_{z0}}{2[B\rho]}(x^2 + y^2) = \sqrt{\kappa}(x^2 + y^2) = \text{Vector Potential Component Canonical Angular Momentum}$$



Comments on Orbits (see also info in S2E on 3D orbit):

- Mechanical angular momentum jumps discontinuously going into and out of the solenoid
  - Particle spins up (θ' jumps) and down going into and out of the solenoid
  - No mechanical angular momentum outside of solenoid due to the choice of intial condition in this example (initial x-plane motion)
- Canonical angular momentum P_θ is conserved in the 3D orbit evolution
  - Invariance provides a strong check on dynamics
  - P_θ in example has zero value due to the specific (x-plane) choice of initial condition of the particle. Other choices can give nonzero values and finite mechanical angular momentum in drifts.
- Solenoid provides focusing due to radial kicks associated with the "fringe" field entering the solenoid
  - Kick is abrupt for hard-edge solenoids
  - Details on radial kick/rotation structure can be found in Appendix C

Alternative expressions of canonical angular momentum

It is insightful to express the canonical angular momentum in (denoted tilde here) in the solenoid focusing canonical variables used earlier in this section and rotating Larmor frame variables:

- See Appendix B for Larmor frame transform
- Might expect simpler form of expressions given the relative simplicity of the formulation in canonical and Larmor frame variables

Canonical Variables:

$$\tilde{x} = x \qquad\qquad \tilde{y} = y$$

$$\tilde{x}' = x' - \frac{B_{z0}}{2[B\rho]} y \qquad\qquad \tilde{y}' = y' + \frac{B_{z0}}{2[B\rho]} x$$

$$\Longrightarrow \quad \begin{aligned} \frac{P_\theta}{m\gamma_b\beta_b c} &\equiv xy' - yx' + \frac{B_{z0}}{2[B\rho]}(x^2 + y^2) \\ &= \tilde{x}\tilde{y}' - \tilde{x}\tilde{y}' \end{aligned}$$

- Applies to acceleration also since just employing transform as a definition here

Larmor (Rotating) Frame Variables:

Larmor transform following formulation in Appendix B:

- Here tildes denote Larmor frame variables

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix} = \begin{bmatrix} \cos\tilde{\psi} & 0 & -\sin\tilde{\psi} & 0 \\ k_L \sin\tilde{\psi} & \cos\tilde{\psi} & k_L \cos\tilde{\psi} & -\sin\tilde{\psi} \\ \sin\tilde{\psi} & 0 & \cos\tilde{\psi} & 0 \\ -k_L \cos\tilde{\psi} & \sin\tilde{\psi} & k_L \sin\tilde{\psi} & \cos\tilde{\psi} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{bmatrix}$$

$$\tilde{\psi}(s) = -\int_{s_i}^{s} d\bar{s}\; k_L(\bar{s})$$

$$k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]}$$

gives after some algebra:

$$\Longrightarrow \qquad x^2 + y^2 = \tilde{x}^2 + \tilde{y}^2$$

$$xy' - yx' = \tilde{x}\tilde{y}' - \tilde{y}\tilde{x}' - \frac{B_{z0}}{2[B\rho]}(\tilde{x}^2 + \tilde{y}^2)$$

Showing that:

$$\frac{P_\theta}{m\gamma_b\beta_b c} \equiv xy' - yx' + \frac{B_{z0}}{2[B\rho]}(x^2 + y^2)$$
$$= \tilde{x}\tilde{y}' - \tilde{x}\tilde{y}'$$

- Same form as previous canonical variable case due to notation choices. However, steps/variables and implications different in this case !

Bush's Theorem expression of canonical angular momentum conservation

Take:

$$\mathbf{B}^a = \nabla \times \mathbf{A}$$

and apply Stokes Theorem to calculate the magnetic flux Ψ through a circle of radius *r*:

$$\Psi = \int_r d^2x \, \mathbf{B}^a \cdot \hat{\mathbf{z}} = \int_r d^2x \, (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{z}} = \oint_r \mathbf{A} \cdot d\vec{\ell}$$

For a nonlinear, but axisymmetric solenoid, one can always take:

- Also applies to linear field component case

$$\mathbf{A} = \hat{\theta} A_\theta(r, z)$$

$$\Longrightarrow \quad \mathbf{B}^a = -\hat{\mathbf{r}} \frac{\partial A_\theta}{\partial z} + \hat{\mathbf{z}} \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta)$$

Thus:

$$\Psi = 2\pi r A_\theta$$

// Aside: Nonlinear Application of Vector Potential

Given the magnetic field components

$$B_r^a(r,z) \qquad B_z^a(r,z)$$

the equations

$$B_r^a(r,z) = -\frac{\partial}{\partial z}A_\theta(r,z)$$
$$B_z^a(r,z) = \frac{1}{r}\frac{\partial}{\partial r}[rA_\theta(r,z)]$$

can be integrated for a single isolated magnet to obtain *equivalent* expressions for A_θ

$$A_\theta(r,z) = -\int_{-\infty}^{z} d\tilde{z}\ B_r^a(r,\tilde{z})$$
$$A_\theta(r,z) = \frac{1}{r}\int_0^r d\tilde{r}\tilde{r}\ B_z^a(\tilde{r},z)$$

- Resulting A_θ contains consistent nonlinear terms with magnetic field

//

Then the exact form of the canonical angular momentum for for solenoid focusing can be expressed as:

- Here γ factor is exact (not paraxial)

$$P_\theta = m\gamma r^2\dot{\theta} + qrA_\theta$$
$$= m\gamma r^2\dot{\theta} + \frac{q\Psi}{2\pi}$$

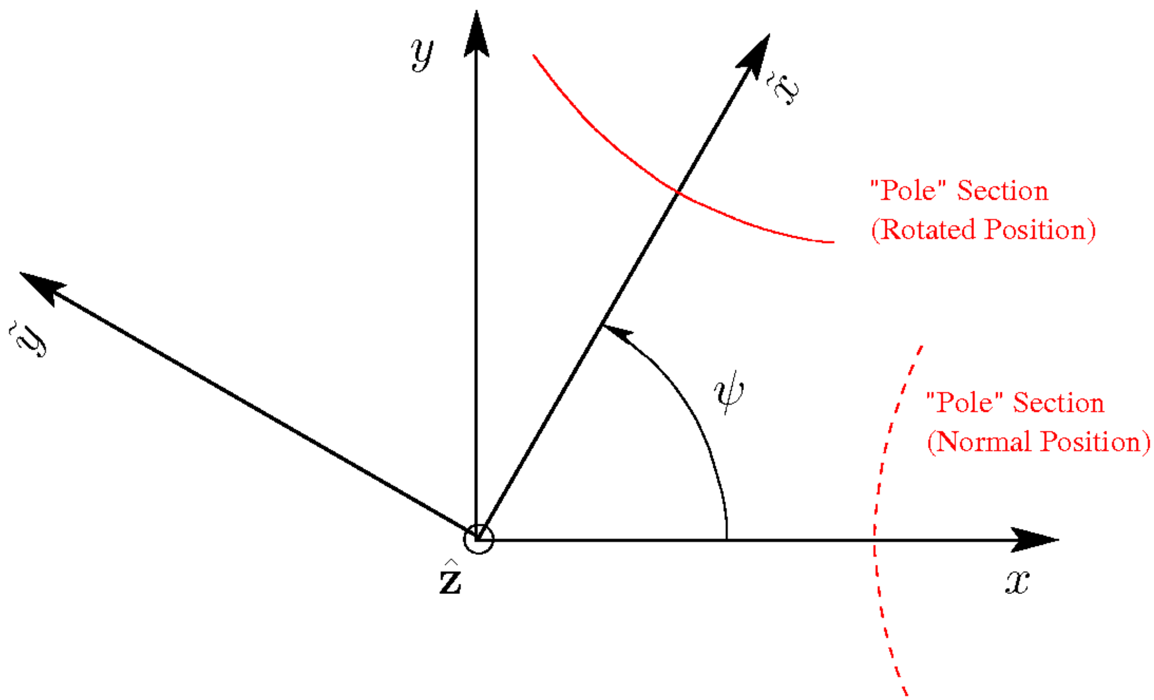
This form is often applied in solenoidal focusing and is known as "Bush's Theorem" with

$$\mathrm{P}_\theta = m\gamma r^2\dot{\theta} + \frac{q\Psi}{2\pi} = \mathrm{const}$$

- In a static applied magnetic field, $\gamma = \mathrm{const}$ further simplifying use of eqn
- Exact as expressed, but easily modified using familiar steps for paraxial form and/or linear field components
- Expresses how a particle "spins up" when entering a solenoidal magnetic field

Appendix A: Quadrupole Skew Coupling

Consider a quadrupole actively rotated through an angle ψ about the z-axis:



Transforms

$$\begin{aligned} \tilde{x} &= \quad x\cos\psi + y\sin\psi \\ \tilde{y} &= -x\sin\psi + y\cos\psi \end{aligned}$$

$$\begin{aligned} x &= \quad \tilde{x}\cos\psi - \tilde{y}\sin\psi \\ y &= \quad \tilde{x}\sin\psi + \tilde{y}\cos\psi \end{aligned}$$

Normal Orientation Fields

Electric

$$\begin{aligned} E_x^a &= -Gx \\ E_y^a &= \quad Gy \end{aligned}$$

Magnetic

$$\begin{aligned} B_x^a &= Gy \\ B_y^a &= Gx \end{aligned}$$

$$\begin{aligned} G &= G(s) \\ &= \text{Field Gradient (Electric or Magnetic)} \end{aligned}$$

Note: units of G different in electric and magnetic cases

A1

Rotated Fields

Electric

$$E_x^a = E_{\tilde{x}}^a \cos\psi - E_{\tilde{y}}^a \sin\psi \qquad E_{\tilde{x}}^a = -G\tilde{x} = -G(\ \ x\cos\psi + y\sin\psi)$$

$$E_y^a = E_{\tilde{x}}^a \sin\psi + E_{\tilde{y}}^a \cos\psi \qquad E_{\tilde{y}}^a = \ \ G\tilde{y} = \ \ G(-x\sin\psi + y\cos\psi)$$

Combine equations, collect terms, and apply trigonometric identities to obtain:

$$E_x^a = -G\cos(2\psi)x - G\sin(2\psi)y$$
$$E_y^a = -G\sin(2\psi)x + G\cos(2\psi)y$$

$$2\sin\psi\cos\psi = \sin(2\psi)$$
$$\cos^2\psi - \sin^2\psi = \cos(2\psi)$$

Magnetic

$$B_x^a = B_{\tilde{x}}^a \cos\psi - B_{\tilde{y}}^a \sin\psi \qquad B_{\tilde{x}}^a = G\tilde{y} = G(-x\sin\psi + y\cos\psi)$$

$$B_y^a = B_{\tilde{x}}^a \sin\psi + B_{\tilde{y}}^a \cos\psi \qquad B_{\tilde{y}}^a = G\tilde{x} = G(\ \ x\cos\psi + y\sin\psi)$$

Combine equations, collect terms, and apply trigonometric identities to obtain:

$$B_x^a = -G\sin(2\psi)x + G\cos(2\psi)y$$
$$B_y^a = \ \ G\cos(2\psi)x + G\sin(2\psi)y$$

A2

For *both* electric and magnetic focusing quadrupoles, these field component projections can be inserted in the linear field Eqns of motion to obtain:

Skew Coupled Quadrupole Equations of Motion

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa\cos(2\psi)x + \kappa\sin(2\psi)y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' - \kappa\cos(2\psi)y + \kappa\sin(2\psi)x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$\kappa = \begin{cases} \frac{G}{\beta_b c[B\rho]}, & \text{Electric Focusing} \\ \frac{G}{[B\rho]}, & \text{Magnetic Focusing} \end{cases}$$

System is skew coupled:

- *x*-equation depends on *y*, *y′* and *y*-equation on *x*, *x′* for $\psi \neq n\pi/2$ (n integer)

Skew-coupling considerably complicates dynamics

- Unless otherwise specified, we consider only quadrupoles with “normal” orientation with $\psi = n\pi/2$
- Skew coupling errors or intentional skew couplings can be important
  - Leads to transfer of oscillations energy between *x* and *y*-planes
  - Invariants much more complicated to construct/interpret

A3

The skew coupled equations of motion can be alternatively derived by actively rotating the quadrupole equation of motion in the form:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa(s)x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' - \kappa(s)y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

- Steps are then identical whether quadrupoles are electric *or* magnetic

A4