09.lec Momentum Spread Effects in Bending and Focusing*

Prof. Steven M. Lund
Physics and Astronomy Department
Facility for Rare Isotope Beams (FRIB)
Michigan State University (MSU)

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Steven M. Lund and Yue Hao

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Outline

Review see: 09.rev.momentum_spread

Dispersive and Chromatic Effects: Particle Equations of Motion

Particles do not necessarily have the design momentum p_0

$$p = p_0 + \delta p$$

p_0 = Design momentum
δp = "off Momentum" (momentum deviation)

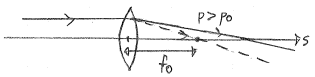
This results in different focal lengths and bend radii in optical elements since the particle rigidity (Bp) changes:

$$(Bp) = \frac{p}{q} = \left(\frac{p_0}{q}\right)\left(\frac{p}{p_0}\right) = (Bp)_0\left(\frac{p}{p_0}\right)$$

$$(Bp)_0 = \frac{p_0}{q} = \text{Design Rigidity}$$

Focusing (Thin Lens)

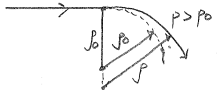
$$\frac{1}{f} \simeq \kappa\ell = \frac{B'\ell}{(Bp)}$$



design $\frac{1}{f_0} = \frac{B'\ell}{(Bp)_0}$ $\quad f > f_0 \;\; p > p_0$ $\quad f < f_0 \;\; p < p_0$

Bending

$$\frac{1}{\rho} = \frac{By(0)}{(Bp)}$$



$\frac{1}{\rho_0} = \frac{By(0)}{(Bp)_0}$ $\quad \rho_0 < \rho \;\; p > p_0$ $\quad \rho_0 > \rho \;\; p < p_0$

Need to account for this effect in the particle equations of motion. This is relatively straightforward to do: Simply retrace steps in derivation of design momentum equations presented in lecture note set 04.lecture.pdf and allow momentum deviations while measuring s along the reference (design) trajectory.

We previously obtained

$$\hat{x}: \quad x'' - \frac{(p_0 + x)}{\rho_0^2} = -\frac{By}{(Bp)}\left(1 + \frac{x}{\rho_0}\right)^2$$

$$\hat{y}: \quad y'' = \frac{Bx}{(Bp)}\left(1 + \frac{x}{\rho_0}\right)^2$$

Here we take

Fields:

$$B_x = B'y$$
$$B_y = By(0) + B'x$$

Dipole Bend; Quadrupole

Design Bend:

$$\frac{1}{\rho_0} = \frac{By(0)}{(Bp)_0}$$

★ Comment: Denoting design bend with subscript "0" to avoid confusion.

Rigidity:

$$(Bp) = (Bp)_0\frac{p}{p_0}$$

Design

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Insert these expressions

$$x'' - \frac{(p_0+x)}{\beta_0^2} = -\frac{(B_y(0)+B'x)}{(B\rho)_0}\left(\frac{p_0}{p}\right)\left(1+\frac{x}{\beta_0}\right)^2$$

$$y'' = \frac{B'y}{(B\rho)_0}\left(\frac{p_0}{p}\right)\left(1+\frac{x}{\beta_0}\right)^2$$

Expand keeping only linear order terms:

$$x'' + \left[-\frac{1}{\beta_0^2} + \frac{2B_y(0)}{\beta_0(B\rho)_0}\left(\frac{p_0}{p}\right) + \frac{B'}{(B\rho)_0}\left(\frac{p_0}{p}\right)\right]x = \frac{1}{\beta_0} - \frac{B_y(0)}{(B\rho)_0}\left(\frac{p_0}{p}\right)$$

$$y'' - \frac{B'}{(B\rho)_0}\left(\frac{p_0}{p}\right)y = 0$$

Denote: $\kappa \equiv \frac{B'}{(B\rho)_0}$ = Quadrupole coupling defined wrt design momentum

and apply $\frac{1}{\beta_0} = \frac{B_y(0)}{(B\rho)_0}$

to obtain:

$$x'' + \left[\frac{1}{\beta_0^2}\left(-1+2\left(\frac{p_0}{p}\right)\right) + \frac{\kappa}{(p/p_0)}\right]x = \frac{1}{\beta_0}\left[1-\frac{p_0}{p}\right]$$

$$y'' - \frac{\kappa}{(p/p_0)}y = 0$$

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Use:

$$\left(\frac{p}{p_0}\right) = \frac{p_0+\delta p}{p_0} = 1 + \frac{\delta p}{p_0} \equiv 1+\delta$$

$\delta \equiv \frac{\delta p}{p_0}$ = Fractional Momentum Error

Then

$$-1 + 2\left(\frac{p_0}{p}\right) = \frac{-p+2p_0}{p} = \frac{-(p_0+\delta p)+2p_0}{p_0+\delta p} = \frac{p_0-\delta p}{p_0+\delta p} = \frac{1-\delta}{1+\delta}$$

$$1 - \frac{p_0}{p} = \frac{p-p_0}{p} = \frac{p_0+\delta p-p_0}{p_0+\delta p} = \frac{\delta p}{p_0+\delta p} = \frac{\delta}{1+\delta}$$

and the equations of motion including off design momentum become:

$$x'' + \left[\frac{1}{\beta_0^2}\frac{1-\delta}{1+\delta} + \frac{\kappa}{1+\delta}\right]x = \frac{\delta}{1+\delta}\frac{1}{\beta_0}$$

$$y'' - \frac{\kappa}{1+\delta}y = 0$$

$\kappa \equiv \frac{B'}{(B\rho)_0}$ $\quad \frac{1}{\beta_0} = \frac{B_y(0)}{(B\rho)_0}$ $\quad (B\rho)_0 = \frac{p_0}{q}$

* often the "0" subscript is dropped on p_0 and $(B\rho)_0$ and the quantities are taken implicitly to be defined wrt the design momentum.

S9: Momentum Spread Effects in Bending and Focusing

S9A: Formulation

Except for brief digressions in S1 and S4, we have concentrated on particle dynamics where all particles have the design longitudinal momentum at a value of s in the lattice:

$$p_s = m\gamma_b\beta_b c = \text{same for every particle}$$

Realistically, there will always be a finite spread of particle momentum within a beam slice, so we take:

$$p_s = p_0 + \delta p$$
$$p_0 \equiv m\gamma_b\beta_b c = \text{Design Momentum}$$
$$\delta p \equiv \text{Off Momentum}$$

Typical values of momentum spread in a beam with a single species of particles with conventional sources and accelerating structures:

$$\frac{|\delta p|}{p_0} \sim 10^{-2} \to 10^{-6}$$

The spread of particle momentum can modify particle orbits, particularly when dipole bends are present since the bend radius depends strongly on the particle momentum

To better understand this effect, we analyze the particle equations of motion with leading-order momentum spread (see: S1H) effects retained:

$$x''(s) + \left[\frac{1}{R^2(s)}\frac{1-\delta}{1+\delta} + \frac{\kappa_x(s)}{(1+\delta)^n}\right]x(s) = \frac{\delta}{1+\delta}\frac{1}{R(s)}$$

$$y''(s) + \frac{\kappa_y(s)}{(1+\delta)^n}y(s) = 0$$

$R(s) =$ Local Bend Radius for design momentum p_0 ($R \to \infty$ in straight sections)

$\delta \equiv \frac{\delta p}{p_0}$ $\qquad$ $\kappa_{x,y} =$ Focusing Functions (using design momentum)

$$n = \begin{cases} 1, & \text{Magnetic Quadrupoles} \\ 2, & \text{Solenoids, Electric Quadrupoles} \end{cases}$$

Magnetic Dipole Bend

$$\frac{1}{R(s)} = \frac{B_y^a|_{\text{dipole}}}{[B\rho]}$$

$$[B\rho] = \frac{p_0}{q}$$

Neglects:

- Space-charge: $\phi \to 0$
- Nonlinear applied focusing: $\mathbf{E}^a$, $\mathbf{B}^a$ contain only linear focus terms
- Acceleration: $p_0 = mc\gamma_b\beta_b = \text{const}$

In the equations of motion, it is important to understand that B_y^a of the magnetic bends are set from the radius *R* required by the design particle orbit (see: S1 for details)

- Equation relating *R* to fields must be modified for electric bends (see S1)
- *y*-plane bends also require modification of eqns (analogous to *x*-plane case)

The focusing strengths are defined with respect to the design momentum:

$$\kappa_x = \begin{cases} \frac{qG}{m\gamma_b\beta_b^2c^2}, & G = -\partial E_x^a/\partial x = \partial E_y^a/\partial y = \text{Electric Quad. Grad.} \\ \frac{qG}{m\gamma_b\beta_b c}, & G = \partial B_x^a/\partial y = \partial B_y^a/\partial x = \text{Magnetic Quad. Grad.} \\ \frac{qB_{z0}}{4m\gamma_b^2\beta_b^2c^2}, & B_{z0} = \text{Solenoidal Magnetic Field} \end{cases}$$

$$\gamma_b,\ \beta_b \text{ calculated from } p_0$$

Terms in the equations of motion associated with momentum spread (δ) can be lumped into two classes:

S.9B: Dispersive -- Associated with Dipole Bends

S.9C: Chromatic -- Associated with Applied Focusing (κ)

S9B: Dispersive Effects

Present only in the *x*-equation of motion and result from bending. Neglecting chromatic terms:

$$x''(s) + \left[\frac{1}{R^2(s)}\frac{1-\delta}{1+\delta} + \kappa_x(s)\right] x(s) = \frac{\delta}{1+\delta}\frac{1}{R(s)}$$

Term 1 Term 2

Particles are bent at different radii when the momentum deviates from the design value ($\delta \neq 0$) leading to changes in the particle orbit

- Dispersive terms contain the bend radius *R*

Generally, the bend radii *R* are large and δ is small, and we can take to leading order:

Term 1: $\left[\frac{1}{R^2}\frac{1-\delta}{1+\delta} + \kappa_x\right] x \simeq \kappa_x x + \Theta(1/R^2) + \Theta(\delta/R^2)$

Term 2: $\frac{\delta}{1+\delta}\frac{1}{R} \simeq \frac{\delta}{R} + \Theta(\delta^2/R)$

Careful if *R* not large as might be the case in low-energy beam lines

The equations of motion then become:

$$x''(s) + \kappa_x(s)x(s) = \frac{\delta}{R(s)}$$
$$y''(s) + \kappa_y(s)y(s) = 0$$

- The y-equation is not changed from the usual Hill's Equation

The *x*-equation is typically solved for *periodic* ring lattices by exploiting the linear structure of the equation and linearly resolving:

$$x(s) = x_h(s) + x_p(s)$$

$x_h \equiv$ Homogeneous Solution

$x_p \equiv$ Particular Solution

where x_h is the *general* solution to the Hill's Equation:

$$x_h''(s) + \kappa_x(s)x_h(s) = 0$$

and x_p is the *periodic* solution to:

$$x_p = \delta \cdot D \qquad D''(s) + \kappa_x(s)D(s) = \frac{1}{R(s)}$$

$D \equiv$ Dispersion Function $\qquad D(s + L_p) = D(s)$

This convenient resolution of the orbit *x*(*s*) can *always* be made because the homogeneous solution will be adjusted to match any initial condition

Note that x_p provides a measure of the offset of the particle orbit relative to the design orbit resulting from a small deviation of momentum (δ)

x(*s*) = 0 defines the design orbit

$[[D]] = \text{meters}$

$\delta \cdot D = \text{Dispersion induced orbit offset in meters}$

Comments:

- It can be shown (see Appendix B) that *D* is unique given a focusing function κ_x for a periodic lattice provided that $\frac{\sigma_{0x}}{2\pi} \neq \text{integer}$
  - In this context *D* is interpreted as a Lattice Function similarly to the betatron function
  - Consequently, δD gives the closed orbit of an off-momentum particle in a ring due to dispersive effects
- The case of how to interpret and solve for *D* in a non-periodic lattice (transfer line) will be covered
  - In this case initial conditions of *D* will matter

Extended 3x3 Transfer Matrix Form for Dispersion Function

Can solve D in

$$D'' + \kappa_x D = \frac{1}{R}$$

by taking

$$D = D_h + D_p$$

D_h = Homogeneous Solution

D_p = Particular Solution

Homogeneous solution is the general solution to

- Usual Hill's equation with solution expressed in terms of principle functions in 2x2 matrix form

$$D_h'' + \kappa_x D_h = 0$$

$$\begin{bmatrix} D_h \\ D_h' \end{bmatrix}_s = \mathbf{M}(s|s_i) \cdot \begin{bmatrix} D_h \\ D_h' \end{bmatrix}_{s_i} = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \cdot \begin{bmatrix} D_h \\ D_h' \end{bmatrix}_{s_i}$$

Particular solution take to be the zero initial condition solution to

- Homogeneous part used to adjust for general initial conditions: always integrate from zero initial value and angle

$$D_p'' + \kappa_x D_p = \frac{1}{R}$$

$$D_p(s_i) = 0 = D_p'(s_i)$$

Denote solution as from zero initial value and angle at $s = s_i$ as $D_p(s) = D_p(s|s_i)$

Can superimpose the homogeneous and particular solutions to form a generalized 3x3 transfer matrix for the Dispersion function D as:

- Initial condition absorbed on homogeneous solution

$$\begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_s = \begin{bmatrix} C(s|s_i) & S(s|s_i) & D_p(s|s_i) \\ C'(s|s_i) & S'(s|s_i) & D_p'(s|s_i) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s_i} = \begin{bmatrix} [\mathbf{M}(s|is_i)] & D_p(s|s_i) \\ & D_p'(s|s_i) \\ 0 \quad 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s_i} \equiv \mathbf{M}_3(s|s_i) \cdot \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s_i}$$

For a periodic solution:

$$D(s_i + L_p) = D(s_i)$$
$$D'(s_i + L_p) = D'(s_i)$$

This gives two constraints to determine the needed initial condition for periodicity

- Third row trivial

$$D(s_i) - C(s_i + L_p|s_i)D(s_i) - S(s_i + L_p|s_i)D'(s_i) = D_p(s_i + L_p|s_i)$$
$$D'(s_i) - C'(s_i + L_p|s_i)D(s_i) - S'(s_i + L_p|s_i)D'(s_i) = D_p'(s_i + L_p|s_i)$$

Solving this using matrix methods (inverse by minor) and simplifying the result with the Wronskian invariant (S5C)

$$W = C(s|s_i)S'(s|s_i) - S(s|s_i)C'(s|s_i) = 1$$

and the definition of phase advance in the periodic lattice (S6G)

$$\cos\sigma_{0x} = \frac{1}{2}\mathrm{Tr}\ \mathbf{M}(s_i + L_p|s_i) = \frac{1}{2}[C(s_i + L_p|s_i) + S'(s_i + L_p|s_i)]$$

Yields:

$$\begin{bmatrix} D \\ D' \end{bmatrix}_{s_i} = \frac{1}{2(1-\cos\sigma_{0x})} \begin{bmatrix} 1 - S'(s_i + L_p|s_i) & S(s_i + L_p|s_i) \\ C'(s_i + L_p|s_i) & 1 - C(s_i + L_p|s_i) \end{bmatrix} \cdot \begin{bmatrix} D_p(s_i + L_p|s_i) \\ D_p'(s_i + L_p|s_i) \end{bmatrix}$$

- Resulting solution for D from this initial condition will have the periodicity of the lattice. These values always exist for real σ_{0x} $(\sigma_{0x} < 180^\circ)$
- Values of $D(s_i)$, $D'(s_i)$ depend on location of choice of s_i in lattice period
- Can use 3x3 transfer matrix to find D anywhere in the lattice
- Formulation assumes that the underlying lattice is stable with $\sigma_{0x} < 180^\circ$

Alternatively, take $s_i = s$ to show that

$$D(s) = \frac{[1 - S'(s + L_p|s)]\, D_p(s + L_p|s) + S(s + L_p|s)D_p'(s + L_p|s)}{2(1 - \cos\sigma_{0x})}$$
$$D'(s) = \frac{C'(s + L_p|s)D_p(s + L_p|s) + [1 - C(s + L_p|s)]\, D_p'(s + L_p|s)}{2(1 - \cos\sigma_{0x})}$$

Particular Solution for the Dispersion Function in a Periodic Lattice

To solve the particular function of the dispersion from a zero initial condition,

$$D_p'' + \kappa_x D_p = \frac{1}{R} \qquad D_p(s_i) = 0 = D_p'(s_i)$$

A Green's function method can be applied (see Appendix A) to express the solution in terms of projection on the principal orbits of Hill's equation as:

$$D_p(s) = \int_{s_i}^{s} d\tilde{s}\ \frac{1}{R(\tilde{s})} G(s,\tilde{s})$$
$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$
$$\mathcal{C}(s|s_i) = \text{Cosine-like Principal Trajectory}$$
$$\mathcal{S}(s|s_i) = \text{Sine-like Principal Trajectory}$$

Cosine-Like Solution

$$\mathcal{C}''(s|s_i) + \kappa(s)\mathcal{C}(s|s_i) = 0$$
$$\mathcal{C}(s_i|s_i) = 1$$
$$\mathcal{C}'(s_i|s_i) = 0$$

Sine-Like Solution

$$\mathcal{S}''(s|s_i) + \kappa(s)\mathcal{S}(s|s_i) = 0$$
$$\mathcal{S}(s_i|s_i) = 0$$
$$\mathcal{S}'(s_i|s_i) = 1$$

Discussion:

- The Green's function solution for D_p, together with the 3x3 transfer matrix can be used to solve explicitly for D from an initial value
- The initial values $D(s_i)$, $D'(s_i)$ found will yield the *unique* solution for D with the periodicity of the lattice

The periodic lattice solution for the dispersion function can be expressed in terms of the betatron function of the periodic lattice as follows:

From S7C:

$$\mathbf{M}(s|s_i) = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{\frac{\beta(s)}{\beta_i}}[\cos\Delta\psi(s) + \alpha_i \sin\Delta\psi(s)] & \sqrt{\beta_i\beta}\sin\Delta\psi(s) \\ -\frac{\alpha(s)-\alpha_i}{\sqrt{\beta_i\beta(s)}}\cos\Delta\psi(s) - \frac{1+\alpha_i\alpha(s)}{\sqrt{\beta_i\beta(s)}}\sin\Delta\psi(s) & \sqrt{\frac{\beta_i}{\beta(s)}}[\cos\Delta\psi(s) - \alpha\sin\Delta\psi(s)] \end{bmatrix}$$

and using

$$D_p(s) = \int_{s_i}^{s} d\tilde{s}\ \frac{1}{R(\tilde{s})} G(s,\tilde{s}) \qquad G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

along with periodicity of the lattice functions β, α

along with considerable algebraic manipulations show that the dispersion function D for the periodic lattice can be expressed as:

$$D(s) = \frac{\sqrt{\beta(s)}}{2\sin(\sigma_{0x}/2)} \int_{s}^{s+L_p} d\tilde{s}\ \frac{\sqrt{\beta(\tilde{s})}}{R(\tilde{s})} \cos[\Delta\psi(\tilde{s}) - \Delta\psi(s) - \sigma_{0x}/2]$$

$$D'(s) - \frac{\alpha(s)}{\beta(s)} D(s)$$
$$= \frac{1}{2\sqrt{\beta(s)}\sin(\sigma_{0x}/2)} \int_{s}^{s+L_p} d\tilde{s}\ \frac{\sqrt{\beta(\tilde{s})}}{R(\tilde{s})} \sin[\Delta\psi(\tilde{s}) - \Delta\psi(s) - \sigma_{0x}/2]$$

- Formulas and related information can be found in SY Lee, *Accelerator Physics* and Conte and MacKay, *Introduction to the Physics of Particle Accelerators*
- Provides periodic dispersion function D as an integral of betatron function describing the linear optics of the lattice

Full Orbit Resolution in a Periodic Dispersive Lattice

Taking a particle initial condition,

$$x(s = s_i) \equiv x_i \qquad \delta = \frac{\Delta p}{p_0}$$
$$x'(s = s_i) \equiv x_i'$$

and using the homogeneous (Hill's Equation Solution) and particular solutions (Dispersion function) of the periodic lattice, the orbit can be resolved as

$$x(s) = x_h + x_p = x_i C(s|s_i) + x_i' S(s|s_i) + \delta[D(s) - D_i C(s|s_i) - D_i' S(s|s_i)]$$
$$x'(s) = x_h' + x_p' = x_i C'(s|s_i) + x_i' S'(s|s_i) + \delta[D'(s) - D_i C'(s|s_i) - D_i' S'(s|s_i)]$$

here,

$$D(s = s_i) \equiv D_i$$
$$D'(s = s_i) \equiv D_i'$$

Transfer Matrices for Dispersion Function

In problems, will derive 3x3 transfer matrices:

- Summarize results here for completeness
- Can apply to any initial conditions $D_i, \quad D'_i$
  - Only specific initial conditions will yield D periodic with lattice
  - Useful in general form for applications to transfer lines, achromatic bends, etc.

$$\begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_s \equiv \mathbf{M}_3(s|s_i) \cdot \begin{bmatrix} D \\ D' \\ 1 \end{bmatrix}_{s_i}$$

Drift: $\kappa_x(s) = 0$

$$\mathbf{M}_3(s|s_i) = \begin{bmatrix} 1 & (s-s_i) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thin Lens: located at $s = s_i$ with focal strength f (no superimposed bend)

$$\kappa_x(s) = -\frac{1}{f}\delta(s-s_i)$$

$$\mathbf{M}_3(s_i^+|s_i^-) = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thick Focus Lens: with $\kappa_x = \hat{\kappa} = \text{const} > 0$ (no superimposed bend)

$$\mathbf{M}_3(s|s_i) = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] & 0 \\ -\sqrt{\hat{\kappa}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] & \cos[\sqrt{\hat{\kappa}}(s-s_i)] & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thick deFocus Lens: with $\kappa_x = -\hat{\kappa} = \text{const} < 0$ (no superimposed bend)

$$\mathbf{M}_3(s|s_i) = \begin{bmatrix} \cosh[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] & 0 \\ \sqrt{\hat{\kappa}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] & \cosh[\sqrt{\hat{\kappa}}(s-s_i)] & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Bend with Focusing: $R = \text{const}, \quad \kappa_x = \hat{\kappa} = \text{const} > 0$

$$\mathbf{M}_3(s|s_i) = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\hat{\kappa}}\left\{1-\cos[\sqrt{\hat{\kappa}}(s-s_i)]\right\} \\ -\sqrt{\hat{\kappa}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] & \cos[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sin[\sqrt{\hat{\kappa}}(s-s_i)] \\ 0 & 0 & 1 \end{bmatrix}$$

Bend with deFocusing: $R = \text{const}, \quad \kappa_x = -\hat{\kappa} = \text{const} < 0$

$$\mathbf{M}_3(s|s_i) = \begin{bmatrix} \cosh[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{R\hat{\kappa}}\left\{-1+\cosh[\sqrt{\hat{\kappa}}(s-s_i)]\right\} \\ \sqrt{\hat{\kappa}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] & \cosh[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{R\sqrt{\hat{\kappa}}}\sinh[\sqrt{\hat{\kappa}}(s-s_i)] \\ 0 & 0 & 1 \end{bmatrix}$$

For the special case of a sector bend of axial length ℓ the bend with focsusing result reduces for transport through the full bend to:

$$\ell = R\theta, \quad \theta = \text{Bend Angle}$$

$$\mathbf{M}_3 = \begin{bmatrix} \cos\theta & R\sin\theta & R(1-\cos\theta) \\ -\frac{\sin\theta}{R} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

For a small angle bend with $|\theta| \ll 1$ this further reduces to:

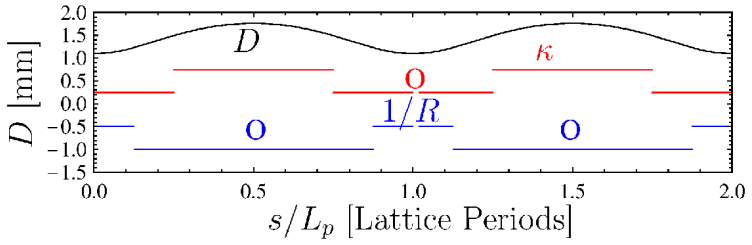
$$\mathbf{M}_3 \simeq \begin{bmatrix} 1 & \ell & \ell\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{bmatrix}$$

// Example: Dispersion function for a simple periodic lattice

For purposes of a simple illustration we here use an imaginary FO (Focus-Drift) piecewise-constant lattice where the *x*-plane focusing is like the focus-plane of a quadrupole with one thick lens focus optic per lattice period and a single drift with the bend in the middle of the drift

- Focus element implemented by $\kappa > 0$ *x*-plane quadrupole transfer matrix in S5B.

$$L_p = 0.5 \text{ m} \quad \kappa = 20/\text{m}^2 \text{ in Focusing}$$
$$\eta = 0.5 \qquad R = 15 \text{ m, in bend, } 25\% \text{ Occupancy}$$



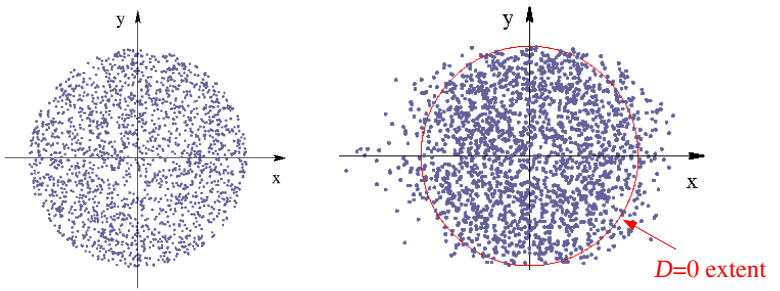
//

// Example: Dispersion broadens the distribution in x

Uniform Bundle of particles D = 0

Same Bundle of particles D nonzero

- Gaussian distribution of momentum spread distorts the *x*-*y* distribution extents in *x* but not in *y*



//

// Example: Continuous Focusing in a Continuous Bend

$$\kappa_x(s) = k_{\beta 0}^2 = \text{const}$$

$$R(s) = R = \text{const}$$

Dispersion equation becomes:

$$D'' + k_{\beta 0}^2 D = \frac{1}{R}$$

With constant solution:

$$D = \frac{1}{k_{\beta 0}^2 R} = \text{const}$$

From this result we can crudely estimate the average value of the dispersion function in a ring with periodic focusing by taking:

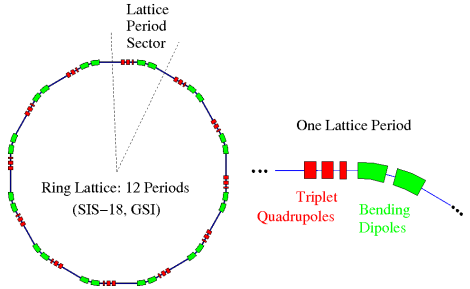
$$R = \text{Avg Radius Ring}$$
$$L_p = \text{Lattice Period (Focusing)}$$
$$\sigma_{0x} = x\text{-Plane Phase Advance}$$

$$\Longrightarrow \quad k_{\beta 0} \sim \frac{\sigma_0}{L_p} \qquad \Longrightarrow \quad D \sim \frac{L_p^2}{\sigma_0^2 R}$$

//

Many rings are designed to focus the dispersion function *D*(s) to small values in straight sections even though the lattice has strong bends

- Desirable since it allows smaller beam extents at locations near where *D* = 0 and these locations can be used to insert and extract (kick) the beam into and out of the ring with minimal losses and/or accelerate the beam
  - Since average value of *D* is dictated by ring size and focusing strength (see example next page) this variation in values can lead to *D* being larger in other parts of the ring
- Quadrupole triplet focusing lattices are often employed in rings since the use of 3 optics per period (vs 2 in doublet) allows more flexibility to tune *D* while simultaneously allowing particle phase advances to also be adjusted



Dispersive Effects in Transfer Lines with Bends

It is common that a beam is transported through a single or series of bends in applications rather than a periodic ring lattice. In such situations, dispersive corrections to the particle orbit are analyzed differently. In this case, the same particular + homogeneous solution decomposition is used as in the ring case with the Dispersion function satisfying:

$$\boxed{D''(s) + \kappa_x(s) D(s) = \frac{1}{R(s)}}$$

However, in this case D is solved from an initial condition. Usually (but not always) from a dispersion-free initial condition $s = s_i$ upstream of the bends with:

$$\boxed{D(s_i) = 0 = D'(s_i)}$$

If the bends and focusing elements can be configured such that on transport through the bend $(s = s_d)$ that

$$\boxed{D(s_d) = 0 = D'(s_d)}$$

Then the bend system is first order achromatic meaning there will be no final orbit deviation to 1st order in δ on traversing the system.

This equation has the form of a *Driven Hill's Equation*:

$$x'' + \kappa(s)x = p(s) \qquad \begin{array}{l} x \to D \\ p \to 1/R \end{array}$$

The general solution to this equation can be solved analytically using a Green function method (see Appendix A) based on principle orbits of the homogeneous Hill's equation as:

$$x(s) = x(s_i)\mathcal{C}(s|s_i) \; + \; x'(s_i)\mathcal{S}(s|s_i) \; + \; \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})p(\tilde{s})$$
$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

Cosine-Like Solution	Sine-Like Solution
$\mathcal{C}''(s\|s_i) + \kappa(s)\mathcal{C}(s\|s_i) = 0$	$\mathcal{S}''(s\|s_i) + \kappa(s)\mathcal{S}(s\|s_i) = 0$
$\mathcal{C}(s_i\|s_i) = 1$	$\mathcal{S}(s_i\|s_i) = 0$
$\mathcal{C}'(s_i\|s_i) = 0$	$\mathcal{S}'(s_i\|s_i) = 1$

$$x(s_i) = \text{Initial value } x$$
$$x'(s_i) = \text{Initial value } x'$$

Green function effectively casts driven equation in terms of homogeneous solution projections of Hill's equation.

Using this Green function solution from the dispersion-free initial condition gives

$$D(s) = \mathcal{S}(s|s_i)\int_{s_i}^{s} d\tilde{s}\;\frac{1}{R(\tilde{s})}\mathcal{C}(\tilde{s}|s_i) \; - \; \mathcal{C}(s|s_i)\int_{s_i}^{s} d\tilde{s}\;\frac{1}{R(\tilde{s})}\mathcal{S}(\tilde{s}|s_i)$$

$$\mathcal{C}(s|s_i) = \text{Cosine-like Principal Trajectory}$$
$$\mathcal{S}(s|s_i) = \text{Sine-like Principal Trajectory}$$

- Alternatively, the 3x3 transfer matrices previously derived can also be applied to advance D from a dispersion free point in the the linear lattice

The full particle orbit consistent with dispersive effects is given by

$$x(s) = x(s_i)\mathcal{C}(s|s_i) \; + \; x'(s_i)\mathcal{S}(s|s_i) \; + \; \delta D(s)$$
$$x'(s) = x(s_i)\mathcal{C}'(s|s_i) \; + \; x'(s_i)\mathcal{S}'(s|s_i) \; + \; \delta D'(s)$$

For a 1st order achromatic system we requite for no leading-order dispersive corrections to the orbit on transiting the lattice ($s_i \to s_d$). This requires:

$$0 = \int_{s_i}^{s_d} d\tilde{s}\;\frac{1}{R(\tilde{s})}\mathcal{C}(\tilde{s}|s_i)$$
$$0 = \int_{s_i}^{s_d} d\tilde{s}\;\frac{1}{R(\tilde{s})}\mathcal{S}(\tilde{s}|s_i)$$

Various lattices consisting of regular combinations of bends and focusing optics can be made achromatic to 1st order by meeting these criteria.

- Higher-order achromats also possible under more detailed analysis. See, for examples: Rusthoi and Wadlinger, 1991 PAC, 607

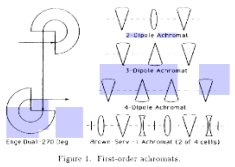


Figure 1. First-order achromats.

Examples are provided in the following slides for achromatic bends as well as bend systems to maximize/manipulate dispersive properties for species separation. Further examples can be found in the literature

Symmetries in Achromatic Lattice Design

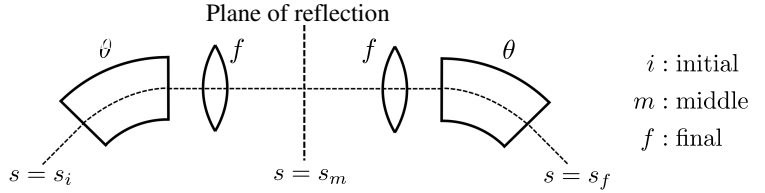
Input from C.Y. Wong, MSU

Symmetries are commonly exploited in the design of achromatic lattices to:

- Simplify the lattice design
- Reproduce (symmetrically) initial beam conditions downstream

Example lattices will be given after discussing general strategies:

Approach 1: beam line with reflection symmetry about its mid-plane

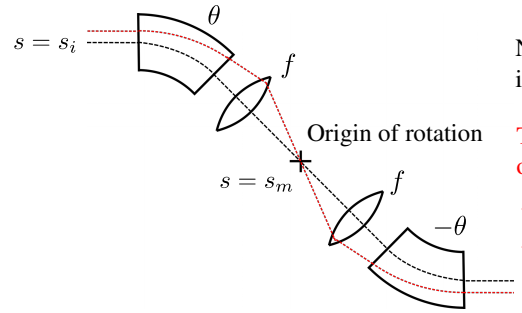


If $g'(s_m) = 0, \quad$ then $g(s_i) = g(s_f),\; g'(s_i) = -g'(s_f)$

where g can be $\beta_x,\; \beta_y$ or D

After the mid-plane, the beam traverses the same lattice elements in reverse order. So if the lattice function angle (d/ds) vanishes at mid-plane, the lattice function undergoes "time reversal" in the 2nd half of the beam line exiting downstream at the symmetric axial location with the same initial value and opposite initial angle.

Approach 2: beam line with rotational symmetry about the mid-point:



Note that the dipoles bend in *different* directions

Trajectory in red: ideal off-momentum particle

$x(s_i) = D(s_i)\delta \neq 0$

$x'(s_i) = D'(s_i)\delta = 0$

Focusing properties of dipoles are independent of bend direction (sign θ). Same reasoning as Approach 1 gives:

If $\beta'_{x,y}(s_m) = 0$, then $\beta_{x,y}(s_i) = \beta_{x,y}(s_f)$, $\beta'_{x,y}(s_i) = -\beta'_{x,y}(s_f)$

Dispersive properties of dipoles change with bend direction. See Appendix C.

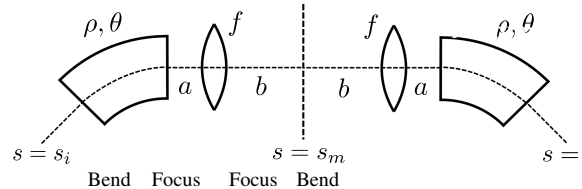
If $D(s_m) = 0$ (instead of D'), then $D(s_i) = -D(s_f)$, $D'(s_i) = D'(s_f)$

If *D* vanishes at mid-plane, the dispersive shift of an off-momentum particle also exhibits rotational symmetry about the mid-point

Example: Achromatic Bend with Thin Lens Focusing

Input from C.Y. Wong, MSU

Apply Approach 1 with simple round numbers:

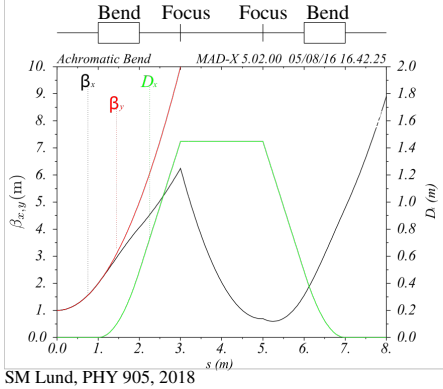


$\theta = \pi/6$

$\rho = 6/\pi$ m

$a = 1$ m

$b = 1$ m



For $D(s_i) = 0 = D'(s_i)$,

$D'(s_m) = 0$ if $f = \rho \tan\frac{\theta}{2} + a$

(see next slide)

$\Rightarrow f = 1.51$ m

The bending system is achromatic, but the betatron functions are asymmetric due to insufficient lattice parameters to tune.

- Add more elements to address

For incident beam with $D(s_i) = 0 = D'(s_i)$, the dispersion function only evolves once the beam enters the dipole

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{s_m} = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{s_m - b} = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{M} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

Note that the drift *b* after the thin lens focus does not affect D'

$$D'(s_m) = D'(s_m - b) = 0 \text{ if } M_{23} = 0$$

Solution gives:

$$\Longrightarrow f = \rho \tan\frac{\theta}{2} + a$$

Discussion:

- Only have to design half the beam-line by exploiting symmetries:
  - One constraint at mid-point satisfies two constraints at the end of the beam line if an asymmetric design approach was taken
  - Symmetric lattic easier to set/tune: strengths in 1st half of the beam line identical to mirror pair in the 2nd half
- It is *possible* to achieve the same final conditions with an asymmetric beam line, but this is generally not preferred
- There should be more lattice strength parameters that can be turned than constraints – needs more optics elements than this simple example
  - Except in simplest of cases, parameters often found using numerical procedures and optimization criteria

Discussion Continued:

- Usually Approach 1 and Approach 2 are applied for transfer line bends with

$$D(s_i) = 0 = D(s_f),\ D'(s_i) = 0 = D'(s_f)$$

  However, this is not necessary
- Common applications with $D(s_i) = 0 = D'(s_i)$ for linacs and transfer lines:
  - Approach 1: fold a linac, or create dispersion at mid-plane to collimate / select species from a multi-species beam
  - Approach 2: translate the beam
- Common applications for rings:
  - Approach 1: Minimize dispersion in straight sections to reduce aberrations in RF cavities, wigglers/undulators, injection/extraction, etc.
- Not only is it desirable to minimize the dispersion at cavities for acceleration purposes, an accelerating section has no effect on the dispersion function up to 1st order only if $D = D' = 0$
  - Consider an off-momentum particle with $x'_D = \delta D' = 0\ ,\ x_D = \delta D \neq 0$ undergoing purely longitudinal acceleration
  - δ changes while x_D does not, while entails that D changes

Example: Simplified Fragment Separator

Input from C.Y. Wong, MSU

Heavy ion beams impinge on a production target to produce isotopes for nuclear physics research. Since many isotopes are produced, a fragment separator is needed downstream to serve two purposes:

- Eliminate unwanted isotopes
- Select and focus isotope of interest onto a transport line to detectors

Different isotopes have different rigidities, which are exploited to achieve isotope selection

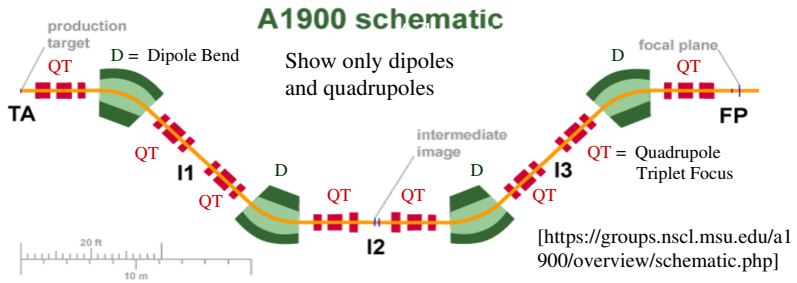
Rigidity $[B\rho] = \dfrac{p}{q} = \dfrac{\gamma m v}{q}$ ref particle (isotope) sets parameters in lattice transfer matrices

$\delta = \left(\dfrac{\Delta p}{p}\right)_{\text{eff}} = \dfrac{\Delta[B\rho]}{[B\rho]_0}$ Deviation from the reference rigidity treated as an effective momentum difference

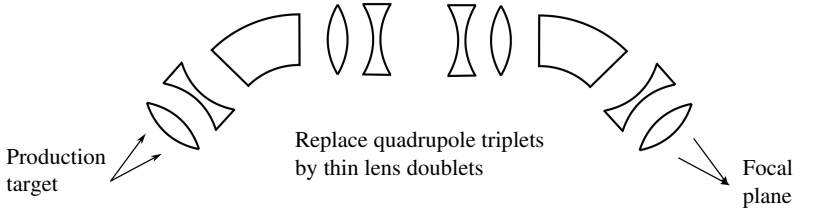
- Applied fields fixed for all species

Dispersion exploited to collimate off-rigidity fragments

NSCL A1900 Fragment Separator: Simplified Illustration



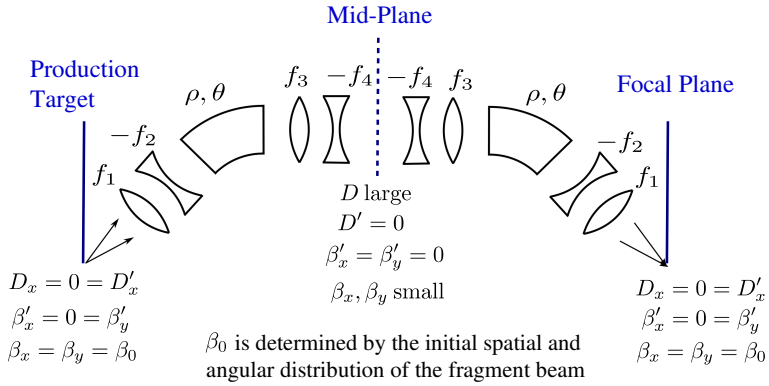
Further Simplified Example: 2 segment version



Design Goals:

- Dipoles set so desired isotope traverses center of all elements
- Dispersion function D is: *large at collimation* for rigidity resolution
  small elsewhere to minimize losses
- β_x β_y should be small at collimation point and focal plane

Apply Approach 1 by requiring $D' = \beta'_x = \beta'_y = 0$ at mid-plane



β_0 is determined by the initial spatial and angular distribution of the fragment beam

Supplementary: Parameters for Simplified Fragment Separator

f_1 $-f_2$ ρ,θ f_3 $-f_4$ $-f_4$ f_3 ρ,θ $-f_2$ f_1

0.6m 1m 1m 1.4m 2m 1m 2m 1m 2m 1.4m 1m 1m 0.6m

Desired isotope: $^{31}S^{16+}$ from ^{40}Ar(140 MeV/u) on Be target

Energy: 120 MeV/u

Rigidity: 3.15 Tesla-m

Dipole ρ,θ are fixed

$\rho = 1.78$ m $\quad \theta = \pi/4$

Thus $B_y(0)$ is uniquely determined by $[B\rho]$

$B_y(0) = 1.7$ Tesla

Initial conditions at production target:

$\sqrt{\langle x^2\rangle} = 1$ mm $\quad \sqrt{\langle x'^2\rangle} = 10$ mrad

$\epsilon_x = 10$ mm-mrad

Impose constraints and solve f's numerically:

$f_1 = 1.12$ m

$f_2 = f_1$

$f_3 = 1.79$ m

$f_4 = 4.17$ m

Quadrupole gradients $\Longleftrightarrow$ for lengths $\ell = 20$ cm

$G_1 = 13.9$ T/m

$G_2 = 13.9$ T/m

$G_3 = 8.7$ T/m

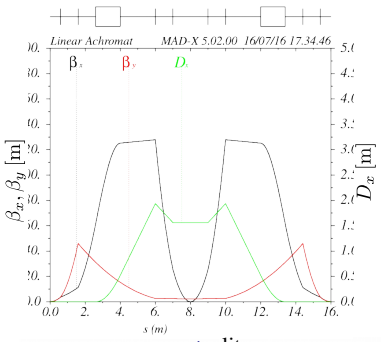
$G_4 = 3.7$ T/m

For other isotopes:

If initial $\langle x^2\rangle, \langle x'^2\rangle$ are same, scale all fields to match rigidity $[B\rho]$

If not, the f's also have to be re-tuned to meet the constraints

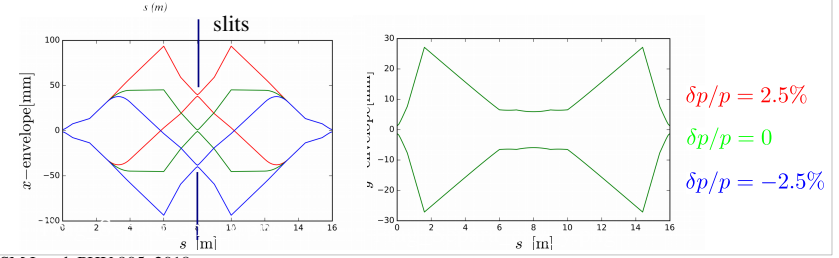
Lattice functions and beam envelope



- Slits at mid-plane where dispersion large to collimate unwanted isotopes and discriminate momentum
- x-envelope plotted for 3 momentum values:

$$x_{\rm env} = \pm\sqrt{\beta_x \epsilon_x} + \delta D$$

- Aperture sizes and D (properties of lattice), determine the angular and momentum acceptance of the fragment separator



Example: Charge Selection System of the FRIB Front End

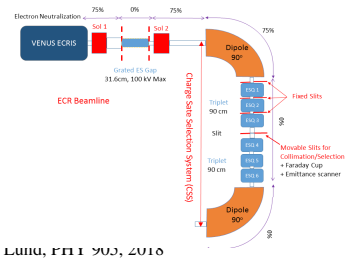
Input from C.Y. Wong, MSU

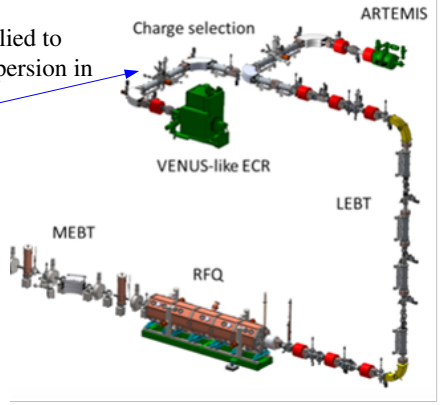
An ECR ion source produces a many-species beam

A charge selection system (CSS) is placed shortly downstream of each source to select the desired species for further transport and collimate the rest

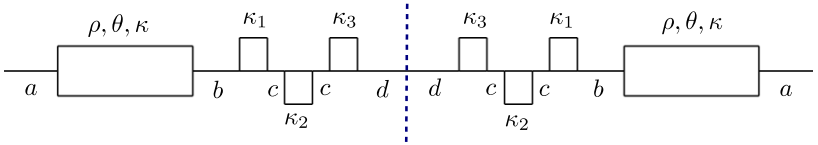
- The CSS consists of two quadrupole triplets and two 90-degree sector dipoles
- The dipoles have slanted poles applied to increase x-focusing to enhance dispersion in the CSS

FRIB CSS





Parameters for the CSS



Dipole:

$\theta = \pi/2 \quad \rho = 2/\pi$ m

$\kappa_x = 0.1/\rho^2 \quad \kappa_y = 0.9/\rho^2$

Mid-plane conditions:

$$\alpha_x(s_m) = \alpha_y(s_m) = D' = 0$$

where field index $n = 0.9$ from :

$$x'' + \kappa_x x = x'' + \frac{1-n}{\rho^2} x = 0$$

$$y'' + \kappa_y y = y'' + \frac{n}{\rho^2} y = 0$$

Quadrupoles:

$l_{\rm quad} = 0.2$ m

$\kappa_{1x} = -\kappa_{1y} = 8.30$ m^{-2}

$\kappa_{2x} = -\kappa_{2y} = -15.60$ m^{-2}

$\kappa_{3x} = -\kappa_{3y} = 7.51$ m^{-2}

Drifts:

$a = 0.4$ m

$b = 0.35$ m

$c = 0.13$ m

$d = 0.19$ m

Initial Conditions:

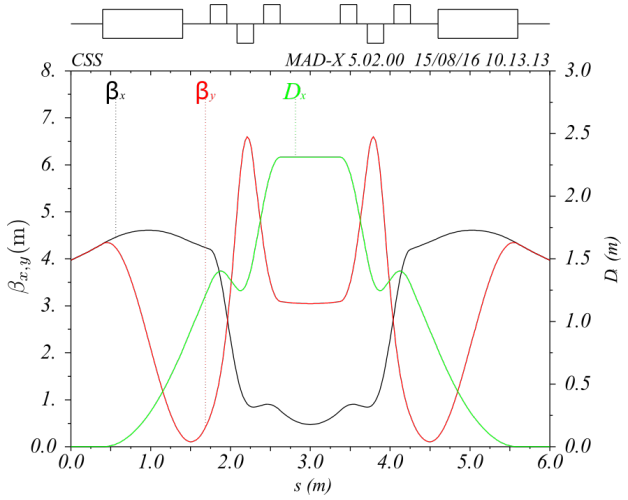
$\beta_x(s_i) = \beta_y(s_i) = 3.971$ m

$\alpha_x(s_i) = \alpha_y(s_i) = -0.380$

$D(s_i) = D'(s_i) = 0$

Lattice Functions of CSS

Large dispersion and small beam size in x at mid-plane facilitates the collimation of unwanted species



S9C: Chromatic Effects

Present in both *x*- and *y*-equations of motion and result from applied focusing strength changing with deviations in momentum:

$$x''(s) + \frac{\kappa_x(s)}{(1+\delta)^n} x(s) = 0$$

$$y''(s) + \frac{\kappa_y(s)}{(1+\delta)^n} y(s) = 0$$

$R \to \infty$ to neglect bending terms

$\kappa_{x,y}$ = Focusing Functions with γ_b, β_b calculated from p_0

- Generally of lesser importance (smaller corrections) relative to dispersive terms (S9C) in linacs *except* where the beam is focused onto a target (small spot) or when momentum spreads are large
- Can be important in rings where precise control of tunes (betratron oscillations per ring lap) are needed to avoid resonances: see Transverse Particle Resonances
- J.J. Barnard in Application Lectures: Heavy Ion Fusion and Final Focusing will overview consequences of chromatic effects on the achievable beam spot in his analysis on final focus optics

Can analyze by redefining kappa function to incorporate off-momentum:

$$\frac{\kappa_x(s)}{(1+\delta)^n} \to \kappa_{x,\mathrm{new}}(s)$$

However, this would require calculating new amplitude/betatron functions for each particle off-momentum value δ in the distribution to describe the evolution of the orbits. That would not be efficient.

Rather, need a *perturbative formula* to calculate the small amplitude correction to the nominal particle orbit with design momentum due to the small amplitude correction due to the off-momentum δ .

Either the *x*- and *y*-equations of motion can be put in the form:

$$x''(s) + \frac{\kappa(s)}{(1+\delta)^n} x(s) = 0$$

Expand to leading order in δ :

$$x''(s) + \kappa(s)(1 - n\delta)x(s) = 0$$

Set:

$$x(s) = x_0(s) + \eta(s)$$

$x_0(s)$ = Orbit Solution for $\delta = 0$

$\eta(s)$ = Orbit Correction to x_0 for $\delta \neq 0$

Giving:

$$x_0'' + \kappa x_0 = 0$$

$$(x_0 + \eta)'' + \kappa(1 - n\delta)(x_0 + \eta) = 0$$

Insert 1st equation in 2nd equation and neglect 2nd order term in $\delta \cdot \eta$ to obtain a linear equation for η :

$$\eta'' + \kappa\eta = n\delta\kappa x_0$$

This equation has the form of a *Driven Hill's Equation*:

$$x'' + \kappa(s)x = p(s)$$

$$x \to \eta$$
$$p \to n\delta\kappa x_0$$

The general solution to this equation can be solved analytically using a Green function method (see Appendix A) as:

$$x(s) = x(s_i)\mathcal{C}(s|s_i) \; + \; x'(s_i)\mathcal{S}(s|s_i) \; + \; \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})p(\tilde{s})$$
$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

Cosine-Like Solution

$$\mathcal{C}''(s|s_i) + \kappa(s)\mathcal{C}(s|s_i) = 0$$
$$\mathcal{C}(s_i|s_i) = 1$$
$$\mathcal{C}'(s_i|s_i) = 0$$

Sine-Like Solution

$$\mathcal{S}''(s|s_i) + \kappa(s)\mathcal{S}(s|s_i) = 0$$
$$\mathcal{S}(s_i|s_i) = 0$$
$$\mathcal{S}'(s_i|s_i) = 1$$

$$x(s_i) = \text{Initial value } x$$
$$x'(s_i) = \text{Initial value } x'$$

Using this result, the general solution for the chromatic correction to the particle orbit can be expressed as:

$$\eta(s) = \eta(s_i)\mathcal{C}(s|s_i) + \eta'(s_i)\mathcal{S}(s|s_i) + n\delta\int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})\kappa(\tilde{s})x_0(\tilde{s})$$
$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$
$$\eta(s_i) = \text{Initial value } \eta$$
$$\eta'(s_i) = \text{Initial value } \eta'$$

Chromatic orbit perturbations are typically measured from a point in the lattice where they are initially zero like a drift where the orbit was correct before focusing quadrupoles. In this context, can take:

$$\eta(s_i) = 0 = \eta'(s_i)$$

$$\eta(s) = n\delta\int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})\kappa(\tilde{s})x_0(\tilde{s})$$

The Green function can be simplified using results from S6F:

$$C(s|s_i) = \frac{w(s)}{w_i}\cos\Delta\psi(s) - w_i' w(s)\sin\Delta\psi(s)$$
$$S(s|s_i) = w_i w(s)\sin\Delta\psi(s)$$

$$\Delta\psi(s) \equiv \int_{s_i}^{s}\frac{d\tilde{s}}{w^2(\tilde{s})}$$
$$w_i \equiv w(s = s_i)$$
$$w_i' \equiv w'(s = s_i)$$

Giving after some algebra:

$$\begin{aligned} G(s,\tilde{s}) &= \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i) \\ &= w(s)w(\tilde{s})[\sin\Delta\psi(s)\cos\Delta\psi(\tilde{s}) - \cos\Delta\psi(s)\sin\Delta\psi(\tilde{s})] \\ &= w(s)w(\tilde{s})\sin[\Delta\psi(s) - \Delta\psi(\tilde{s})] \end{aligned}$$

Using this and the phase amplitude form of the orbit:

$$\begin{aligned} x_0(s) &= A_i w(s)\cos[\psi(s)] \\ &= \sqrt{\epsilon}w(s)\cos[\Delta\psi(s) + \psi_i] \end{aligned}$$

- Initial phase ψ_i implicitly chosen (can always do) for initial amplitude $A_i \geq 0$

the orbit deviation from chromatic effects can be calculated as:

$$\begin{aligned} \eta(s) &= n\delta\int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})\kappa(\tilde{s})x_0(\tilde{s}) \\ &= n\delta\sqrt{\epsilon}w(s)\int_{s_i}^{s} d\tilde{s}\; w^2(\tilde{s})\sin[\Delta\psi(s) - \Delta\psi(\tilde{s})]\cos[\Delta\psi(\tilde{s}) + \psi_i] \end{aligned}$$

Formula applicable to all types of focusing lattices:

- Quadrupole: electric and magnetic
- Solenoid (Larmor frame)
- Linac and rings

Add examples in future editions of notes …

Comments:

- Perturbative formulas can be derived to calculate the effect on betatron tunes (particle oscillations per lap) in a ring based on integrals of the unpreturbed betatron function: see Wiedemann, *Particle Accelerator Physics*
- For magnetic quadrupole lattices further detailed analysis (see Steffen, *High Energy Beam Optics*) it can be shown that:
  - Impossible to make an achromatic focus in any quadrupole system. Here achromatic means if

$$\eta(s_i) = 0 = \eta'(s_i)$$

  there is some achromatic point $s = s_f$ post optics with

$$\eta(s_f) = 0 = \eta'(s_f)$$

- More detailed analysis of the chromatic correction to particle orbits in rings show that a properly oriented nonlinear sextupole inserted into the periodic ring lattice with correct azimuthal orientation at a large dispersion points can to leading order compensate for chromatic corrections. See Wille, *The Physics of Particle Accelerators* for details.
  - Correction introduces nonlinear terms for large amplitude
  - Correction often distributed around ring for practical reasons

Appendix A: Green Function for Driven Hill's Equation

Following Wiedemann (Particle Accelerator Physics, 1993, pp 106) first, consider more general *Driven Hill's Equation*

$$x'' + \kappa(s)x = p(s)$$

The corresponding homogeneous equation:

$$x'' + \kappa(s)x = 0$$

has principal solutions

$$x(s) = C_1\mathcal{C}(s|s_i) + C_2\mathcal{S}(s|s_i) \qquad C_1, C_2 = \text{constants}$$

where

Cosine-Like Solution	Sine-Like Solution
$\mathcal{C}'' + \kappa(s)\mathcal{C} = 0$	$\mathcal{S}'' + \kappa(s)\mathcal{S} = 0$
$\mathcal{C}(s = s_i) = 1$	$\mathcal{S}(s = s_i) = 0$
$\mathcal{C}'(s = s_i) = 0$	$\mathcal{S}'(s = s_i) = 1$

Recall that the homogeneous solutions have the Wronskian symmetry:

- See S5C

$$W(s) = \mathcal{C}(s)\mathcal{S}'(s) - \mathcal{C}'(s)\mathcal{S}(s) = 1 \qquad \mathcal{C}(s) \equiv \mathcal{C}(s|s_i) \quad \text{etc.}$$

A particular solution to the *Driven Hill's Equation* can be constructed using a Greens' function method:

$$x(s) = \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})p(\tilde{s})$$
$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

Demonstrate this works by first taking derivatives:

$$x = \mathcal{S}(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s})$$

$$x' = \mathcal{S}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s}) + p(s)\left[\mathcal{S}(s)\mathcal{C}(s) - \mathcal{S}(s)\mathcal{C}(s)\right]$$

$$= \mathcal{S}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s})$$

$$x'' = \mathcal{S}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s}) + p(s)\left[\mathcal{S}'(s)\mathcal{C}(s) - \mathcal{C}'(s)\mathcal{S}(s)\right]$$

$$= p(s) \;+\; \mathcal{S}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \;-\; \mathcal{C}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s})$$

Insert these results for $x,\ x''$ in the *Driven Hill's Equation*:

Definition of Principal Orbit Functions

$$x'' + \kappa(s)x = p(s) + \underbrace{[\mathcal{S}'' + \kappa\mathcal{S}]}_{0}\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) - \underbrace{[\mathcal{C}'' + \kappa\mathcal{C}]}_{0}\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s})$$
$$= p(s)$$

Thereby proving we have a valid particular solution. The general solution to the *Driven Hill's Equation* is then:

$$x(s) = x(s_i)\mathcal{C}(s|s_i) \;+\; x'(s_i)\mathcal{S}(s|s_i) \;+\; \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})p(\tilde{s})$$
$$G(s,\tilde{s}) = \mathcal{S}(s|s_i)\mathcal{C}(\tilde{s}|s_i) - \mathcal{C}(s|s_i)\mathcal{S}(\tilde{s}|s_i)$$

- Choose constants $C_1,\ C_2$ consistent with particle initial conditions at $s = s_i$

Appendix B: Uniqueness of the Dispersion Function in a Periodic (Ring) Lattice

Consider the equation for the dispersion function in a periodic lattice

$$D'' + \kappa_x D = \frac{1}{R} \qquad \begin{matrix} \kappa_x(s+L_p) = \kappa_x(s) \\ R(s+L_p) = R(s) \end{matrix}$$

It is required that the solution for a periodic (ring) lattice has the periodicity of the lattice:

$$\mathrm{D(s+L}_p) = D(s)$$

Assume that there are two unique solutions to *D* and label them as D_j. Each must satisfy:

$$D_j'' + \kappa_x D_j = \frac{1}{R} \qquad \mathrm{D}_j(s+L_p) = D_j(s) \qquad \mathrm{j} = 1, \quad 2$$

Subtracting the two equations shows that $D_1 - D_2$ satisfies Hill's equation:

$$(D_1 - D_2)'' + \kappa_x (D_1 - D_2) = 0$$

The solution can be expressed in terms of the usual principal orbit functions of Hill's Equation in matrix form as:

$$\begin{bmatrix} D_1 - D_2 \\ (D_1 - D_2)' \end{bmatrix}_s = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \cdot \begin{bmatrix} D_1 - D_2 \\ (D_1 - D_2)' \end{bmatrix}_{s_i}$$

Because C and S do not, in general, have the periodicity of the lattice, we must have:

$$D_1(s_i) = D_2(s_i)$$
$$D_1'(s_i) = D_2'(s_i)$$

which implies a zero solution for $D_1 - D_2$ and:

$$D_1(s) = D_2(s) \implies D \text{ us unique for a periodic lattice}$$

The proof fails for $\sigma_{0x}/(2\pi = \text{integer}$, however, this exceptional case should never correspond to a lattice choice because it would result in unstable particle orbits.

An alternative proof based on the eigenvalue structure of the 3x3 transfer matrices for *D* can be found in "Accelerator Physics" by SY Lee.

- Proof helps further clarify the structure of *D*

Appendix C: Transfer Matrix of a Negative Bend

Input from C.Y. Wong, MSU

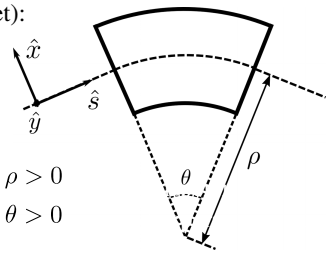
For a clockwise bend (derived in the problem set):

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_f = \mathbf{M}_B \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_i$$

$$\mathbf{M}_B = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

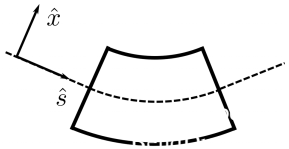
$\rho > 0$

$\theta > 0$



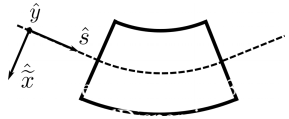
This definition of the x,y,s coordinates is right-handed

The transfer matrix for a negative (anti-clockwise) bend is obtained by making the transformation $\rho \to -\rho,\ \theta \to -\theta$



$$\mathbf{M}_{-B} = \begin{pmatrix} \cos|\theta| & |\rho|\sin|\theta| & -|\rho|(1-\cos|\theta|) \\ -\frac{\sin|\theta|}{|\rho|} & \cos|\theta| & -\sin|\theta| \\ 0 & 0 & 1 \end{pmatrix}$$

If one finds the result counterintuitive, it can be derived as follows:



Define $\tilde{x} = -x$

(The new set of coordinates is not right-handed, but this does not affect the reasoning)

The dispersion functions in the two coordinate systems are related by

$$\begin{pmatrix} \tilde{D} \\ \tilde{D}' \\ 1 \end{pmatrix} = \mathbf{R} \begin{pmatrix} D \\ D \\ 1 \end{pmatrix} \qquad \text{where} \qquad \mathbf{R} = \mathbf{R}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The anti-clockwise bend is effectively clockwise in the primed coordinate system:

$$\begin{pmatrix} \tilde{D} \\ \tilde{D}' \\ 1 \end{pmatrix}_f = \mathbf{M}_B \begin{pmatrix} \tilde{D} \\ \tilde{D}' \\ 1 \end{pmatrix}_i \implies \mathbf{R} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_f = \mathbf{M}_B \mathbf{R} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_i$$

Transfer matrix of anti-clockwise bend in normal coordinates:

$$\mathbf{M}_{-B} = \mathbf{R}^{-1}\mathbf{M}_B\mathbf{R} = \begin{pmatrix} \cos|\theta| & |\rho|\sin|\theta| & -|\rho|(1-\cos|\theta|) \\ -\frac{\sin|\theta|}{|\rho|} & \cos|\theta| & -\sin|\theta| \\ 0 & 0 & 1 \end{pmatrix}$$

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

Prof. Steven M. Lund
Facility for Rare Isotope Beams
Michigan State University
640 South Shaw Lane
East Lansing, MI 48824

lund@frib.msu.edu
(517) 908 – 7291 office
(510) 459 - 4045 mobile

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