09.rev Momentum Spread Effects in Bending and Focusing*

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"Accelerator Physics"
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Outline

Review equations to understand impact of axial momentum deviations

Review S1: Particle Equations of Motion
S1A: Introduction: The Lorentz Force Equation

The *Lorentz force equation* of a charged particle is given by (MKS Units):

$$\frac{d}{dt}\mathbf{p}_i(t) = q_i\left[\mathbf{E}(\mathbf{x}_i,t) + \mathbf{v}_i(t)\times\mathbf{B}(\mathbf{x}_i,t)\right]$$

$m_i,\ q_i$ .... particle mass, charge

$i =$ particle index

$t =$ time

$\mathbf{x}_i(t)$ .... particle coordinate

$\mathbf{p}_i(t) = m_i\gamma_i(t)\mathbf{v}_i(t)$ .... particle momentum

$\mathbf{v}_i(t) = \frac{d}{dt}\mathbf{x}_i(t) = c\vec{\beta}_i(t)$ .... particle velocity

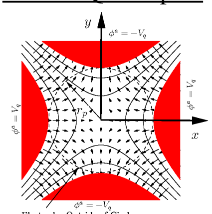
$\gamma_i(t) = \frac{1}{\sqrt{1-\beta_i^2(t)}}$ .... particle gamma factor

	Total		Applied		Self
Electric Field:	$\mathbf{E}(\mathbf{x},t)$	$=$	$\mathbf{E}^a(\mathbf{x},t)$	$+$	$\mathbf{E}^s(\mathbf{x},t)$
Magnetic Field:	$\mathbf{B}(\mathbf{x},t)$	$=$	$\mathbf{B}^a(\mathbf{x},t)$	$+$	$\mathbf{B}^s(\mathbf{x},t)$

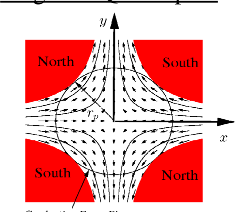
S1B: Applied Fields used to Focus, Bend, and Accelerate Beam

Transverse optics for focusing:

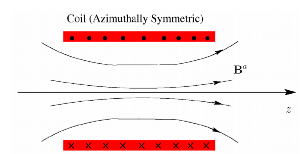
Electric Quadrupole



Magnetic Quadrupole

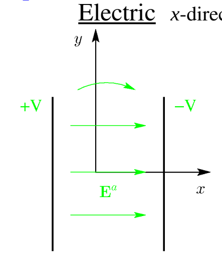


Solenoid

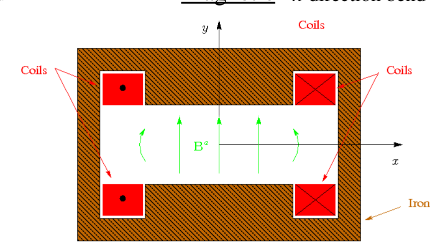


Dipole Bends:

Electric x-direction bend

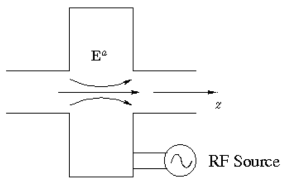


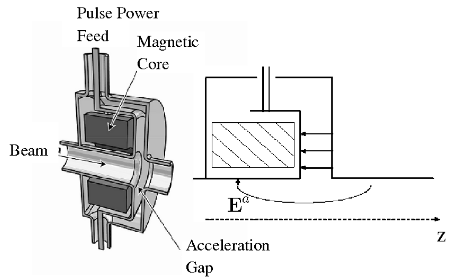
Magnetic x-direction bend



Longitudinal Acceleration:

RF Cavity — **Induction Cell**

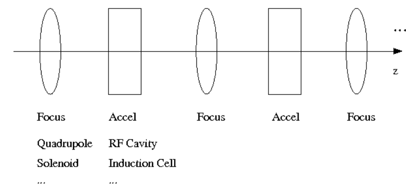




Acceleration influences transverse dynamics – not possible to fully decouple

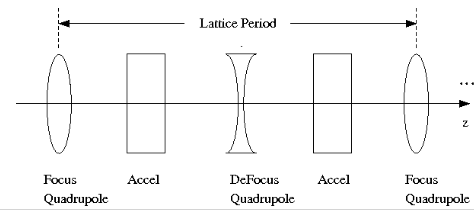
S1C: Machine Lattice

Applied field structures are often arraigned in a regular (periodic) lattice for beam transport/acceleration:

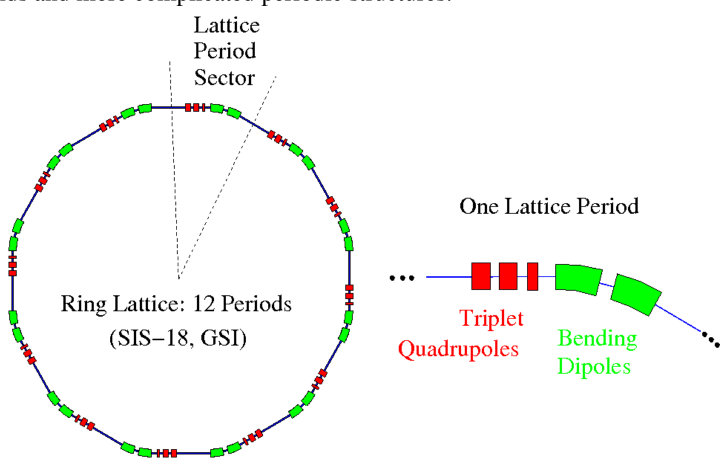


- Sometimes functions like bending/focusing are combined into a single element

Example – Linear FODO lattice (symmetric quadrupole doublet)



Lattices for rings and some beam insertion/extraction sections also incorporate bends and more complicated periodic structures:



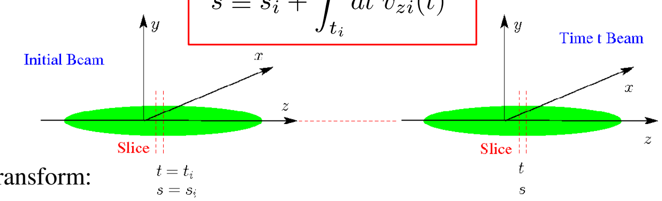
- Elements to insert beam into and out of ring further complicate lattice
- Acceleration cells also present
  (typically several RF cavities at one or more location)

S1E: Equations of Motion in *s* and the Paraxial Approximation

In transverse accelerator dynamics, it is convenient to employ the axial coordinate (s) of a particle in the accelerator as the independent variable:

- Need fields at lattice location of particle to integrate equations for particle trajectories

$$s \equiv s_i + \int_{t_i}^{t} d\tilde{t}\; v_{zi}(\tilde{t})$$



Transform:

$$v_{zi} = \frac{ds}{dt} \quad \Longrightarrow \quad v_{xi} = \frac{dx_i}{dt} = \frac{ds}{dt}\frac{dx_i}{ds} = v_{zi}\frac{dx_i}{ds} = (\beta_b c + \delta v_{zi})\frac{dx_i}{ds}$$

Neglect

$$\simeq \beta_b c \frac{dx_i}{ds}$$

Denote:

$$' \equiv \frac{d}{ds} \qquad v_{xi} = \frac{dx_i}{dt} \simeq \beta_b c x_i' \qquad v_{yi} = \frac{dy_i}{dt} \simeq \beta_b c y_i'$$

Neglecting term consistent with assumption of small longitudinal momentum spread (paraxial approximation)

- Procedure becomes more complicated when bends present: see S1H

S1G: Summary: Paraxial Transverse Particle Eqns of Motion

$$\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}^a_\perp + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}}\times\mathbf{B}^a_\perp + \frac{qB^a_z}{m\gamma_b\beta_b c}\mathbf{x}'_\perp\times\hat{\mathbf{z}}$$
$$- \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}_\perp}\phi$$

$\mathbf{E}^a$ = Applied Electric Field

$\mathbf{B}^a$ = Applied Magnetic Field

$$' \equiv \frac{d}{ds} \qquad \gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}}$$

$$\nabla^2\phi = \frac{\partial}{\partial\mathbf{x}}\cdot\frac{\partial}{\partial\mathbf{x}}\phi = -\frac{\rho}{\epsilon_0}$$
+ Boundary Conditions on ϕ

Drop particle *i* subscripts (in most cases) henceforth to simplify notation

Neglects axial energy spread, bending, and electromagnetic radiation

$\gamma-$ factors different in applied and self-field terms:

In $-\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}}\phi$, contributions to γ_b^3 :

$\gamma_b \Longrightarrow$ Kinematics

$\gamma_b^2 \Longrightarrow$ Self-Magnetic Field Corrections (leading order)

Transverse equations of motion in component form

Write out transverse particle equations of motion in explicit component form:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' = \frac{q}{m\gamma_b\beta_b^2c^2}E^a_x - \frac{q}{m\gamma_b\beta_b c}B^a_y + \frac{q}{m\gamma_b\beta_b c}B^a_z y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' = \frac{q}{m\gamma_b\beta_b^2c^2}E^a_y + \frac{q}{m\gamma_b\beta_b c}B^a_x - \frac{q}{m\gamma_b\beta_b c}B^a_z x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

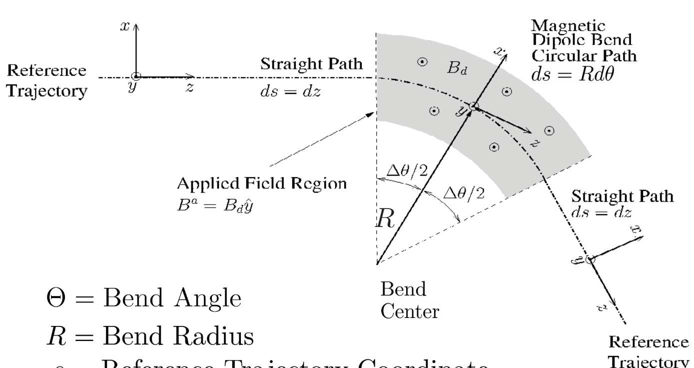
Equations previously derived under assumptions:

- No bends (fixed *x-y-z* coordinate system with no local bends)
- Paraxial equations ($x'^2, y'^2 \ll 1$)
- No dispersive effects (β_b same all particles), acceleration allowed ($\beta_b \neq$ const)
- Electrostatic and leading-order (in β_b) self-magnetic interactions

S1I: Bent Coordinate System and Particle Equations of Motion with Dipole Bends and Axial Momentum Spread

The previous equations of motion can be applied to dipole bends provided the *x,y,z* coordinate system is fixed. It can prove more convenient to employ coordinates that follow the beam in a bend.

- Orthogonal system employed called Frenet-Serret coordinates



Θ = Bend Angle

R = Bend Radius

s = Reference Trajectory Coordinate

In this perspective, dipoles are adjusted given the design momentum of the reference particle to bend the orbit through a radius *R*.

- Bends usually only in one plane (say *x*)
  - Implemented by a dipole applied field: E^a_x or B^a_y
- Easy to apply material analogously for *y*-plane bends, if necessary

Denote:

$$p_0 = m\gamma_b\beta_b c = \text{design momentum}$$

Then a magnetic *x*-bend through a radius *R* is specified by:

$$\mathbf{B}^a = B^a_y\hat{\mathbf{y}} = \text{const in bend}$$
$$\frac{1}{R} = \frac{qB^a_y}{p_0}$$

Analogous formula for Electric Bend will be derived in problem set

The particle rigidity is defined as ($[B\rho]$ read as one symbol called “B-Rho”):

$$[B\rho] \equiv \frac{p_0}{q} = \frac{m\gamma_b\beta_b c}{q}$$

is often applied to express the bend result as:

$$\frac{1}{R} = \frac{B^a_y}{[B\rho]}$$

Comments on bends:

- R can be positive or negative depending on sign of $B_y^a/[B\rho]$
- For straight sections, $R \rightarrow \infty$ (or equivalently, $B_y^a = 0$)
- Lattices often made from discrete element dipoles and straight sections with separated function optics
  - Bends can provide "edge focusing"
  - Sometimes elements for bending/focusing are combined
- For a ring, dipoles strengths are tuned with particle rigidity/momentum so the reference orbit makes a closed path lap through the circular machine
  - Dipoles adjusted as particles gain energy to maintain closed path
  - In a Synchrotron dipoles and focusing elements are adjusted together to maintain focusing and bending properties as the particles gain energy. This is the origin of the name "Synchrotron."
- Total bending strength of a ring in Tesla-meters limits the ultimately achievable particle energy/momentum in the ring

For a magnetic field over a path length S , the beam will be bent through an angle:

$$\Theta = \frac{S}{R} = \frac{SB_y^a}{[B\rho]}$$

To make a ring, the bends must deflect the beam through a total angle of 2π :

- Neglect any energy gain changing the rigidity over one lap

$$2\pi = \sum_{i,\text{Dipoles}} \Theta_i = \sum_i \frac{S_i}{R_i} = \sum_i \frac{S_i B_{y,i}^a}{[B\rho]}$$

For a symmetric ring, N dipoles are all the same, giving for the bend field:

- Typically choose parameters for dipole field as high as technology allows for a compact ring

$$B_y^a = 2\pi \frac{[B\rho]}{NS}$$

For a symmetric ring of total circumference C with straight sections of length L between the bends:

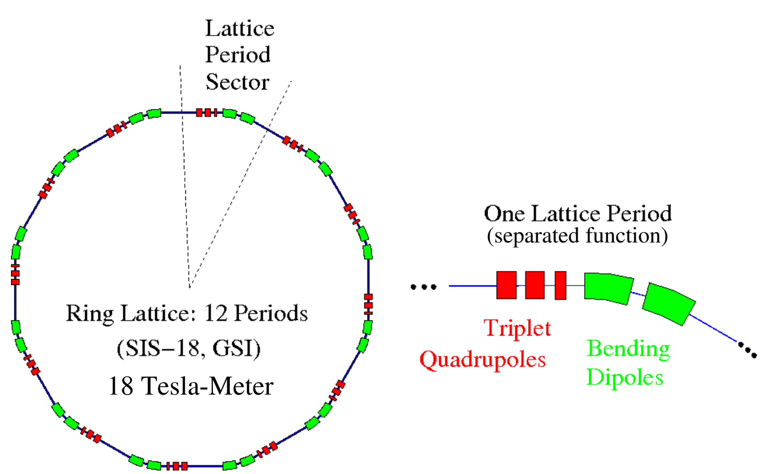
- Features of straight sections typically dictated by needs of focusing, acceleration, and dispersion control

$$C = NS + NL$$

Example: Typical separated function lattice in a Synchrotron

Focus Elements in Red

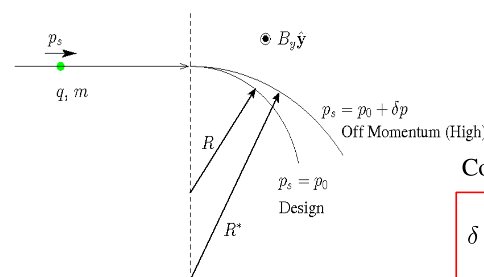
Bending Elements in Green



For "off-momentum" errors:

$$p_s = p_0 + \delta p$$
$$p_0 = m\gamma_b\beta_b c = \text{design momentum}$$
$$\delta p = \text{off- momentum}$$

This will modify the particle equations of motion, particularly in cases where there are bends since particles with different momenta will be bent at different radii



Common notation:

$$\delta \equiv \frac{\delta p}{p_0} = \text{Fractional Momentum Error}$$

- Not usual to have acceleration in bends
  - Dipole bends and quadrupole focusing are sometimes combined

Derivatives in accelerator Frenet-Serret Coordinates

Summarize results only needed to transform the Maxwell equations, write field derivatives, etc.

- Reference: Chao and Tigner, *Handbook of Accelerator Physics and Engineering*

$$\Psi(x,y,s) = \text{Scalar}$$
$$\mathbf{V}(x,y,s) = V_x(x,y,s)\hat{\mathbf{x}} + V_y(x,y,s)\hat{\mathbf{y}} + V_s(x,y,s)\hat{\mathbf{s}} = \text{Vector}$$

Vector Dot and Cross-Products: $(\mathbf{V}_1,\ \mathbf{V}_2\ \ \text{Two Vectors})$

$$\mathbf{V}_1\cdot\mathbf{V}_2 = V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1s}V_{2s}$$

$$\mathbf{V}_1\times\mathbf{V}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{s}} \\ V_{1x} & V_{1y} & V_{1s} \\ V_{2x} & V_{2y} & V_{2s} \end{vmatrix} = (V_{1x}V_{2s} - V_{1s}V_{2x})\hat{\mathbf{x}} + (V_{1s}V_{2x} - V_{1x}V_{2s})\hat{\mathbf{y}} + (V_{1x}V_{2y} - V_{1y}V_{2x})\hat{\mathbf{s}}$$

Elements:

$$d^2x_\perp = dxdy$$
$$d^3x_\perp = \left(1 + \frac{x}{R}\right)dxdyds$$
$$d\vec{\ell} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{s}}\left(1+\frac{x}{R}\right)ds$$

Gradient:

$$\nabla\Psi = \hat{\mathbf{x}}\frac{\partial\Psi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Psi'}{\partial y} + \hat{\mathbf{s}}\frac{1}{1+x/R}\frac{\partial\Psi}{\partial s}$$

Divergence:

$$\nabla\cdot\mathbf{V} = \frac{1}{1+x/R}\frac{\partial}{\partial x}\left[(1+x/R)V_x\right] + \frac{\partial V_y}{\partial y} + \frac{1}{1+x/R}\frac{\partial V_s}{\partial s}$$

Curl:

$$\nabla\times\mathbf{V} = \hat{\mathbf{x}}\left(\frac{\partial V_s}{\partial y} - \frac{1}{1+x/R}\frac{\partial V_y}{\partial s}\right) + \hat{\mathbf{y}}\frac{1}{1+x/R}\left(\frac{\partial V_x}{\partial s} - \frac{\partial}{\partial x}\left[(1+x/R)V_s\right]\right) + \hat{\mathbf{s}}(1+x/R)\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Laplacian:

$$\nabla^2\Psi = \frac{1}{1+x/R}\frac{\partial}{\partial x}\left[\left(1+\frac{x}{R}\right)\frac{\partial\Psi}{\partial x}\right] + \frac{\partial^2\Psi}{\partial y^2} + \frac{1}{1+x/R}\frac{\partial}{\partial s}\left[\frac{1}{1+x/R}\frac{\partial\Psi}{\partial s}\right]$$

Transverse particle equations of motion including bends and "off-momentum" effects

- See texts such as Edwards and Syphers for guidance on derivation steps
- Full derivation is beyond needs/scope of this class

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \left[\frac{1}{R^2(s)}\frac{1-\delta}{1+\delta}\right]x = \frac{\delta}{1+\delta}\frac{1}{R(s)} + \frac{q}{m\gamma_b\beta_b^2c^2}\frac{E_x^a}{(1+\delta)^2} - \frac{q}{m\gamma_b\beta_bc}\frac{B_y^a}{1+\delta} + \frac{q}{m\gamma_b\beta_bc}\frac{B_s^a}{1+\delta}y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{1}{1+\delta}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' = \frac{q}{m\gamma_b\beta_b^2c^2}\frac{E_y^a}{(1+\delta)^2} + \frac{q}{m\gamma_b\beta_bc}\frac{B_x^a}{1+\delta} - \frac{q}{m\gamma_b\beta_bc}\frac{B_s^a}{1+\delta}x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{1}{1+\delta}\frac{\partial\phi}{\partial y}$$

$$p_0 = m\gamma_b\beta_bc = \text{Design Momentum} \qquad \frac{1}{R(s)} = \frac{B_y^a(s)|_{\text{Dipole}}}{[B\rho]} \qquad [B\rho] = \frac{p_0}{q}$$
$$\delta \equiv \frac{\delta p}{p_0} = \text{Fractional Momentum Error}$$

Comments:

- Design bends only in *x* and $B_y^a,\ E_x^a$ contain *no* dipole terms (design orbit)
  - Dipole components set via the design bend radius *R*(*s*)
- Equations contain only low-order terms in momentum spread δ

Comments continued:

- Equations are often applied linearized in δ
- Achromatic focusing lattices are often designed using equations with momentum spread to obtain focal points independent of δ to some order
  x and *y* equations differ significantly due to bends modifying the *x*-equation when *R*(*s*) is finite
- It will be shown in the problems that for electric bends:

$$\frac{1}{R(s)} = \frac{E_x^a(s)}{\beta_bc[B\rho]}$$

- Applied fields for focusing: $\mathbf{E}_\perp^a,\ \mathbf{B}_\perp^a,\ B_s^a$
  must be expressed in the bent *x,y,s* system of the reference orbit
  - Includes error fields in dipoles
- Self fields may also need to be solved taking into account bend terms
  - Often can be neglected in Poisson's Equation

$$\left\{\frac{1}{1+x/R}\frac{\partial}{\partial x}\left[\left(1+\frac{x}{R}\right)\frac{\partial}{\partial x}\right] + \frac{\partial^2}{\partial y^2} + \frac{1}{1+x/R}\frac{\partial}{\partial s}\left[\frac{1}{1+x/R}\frac{\partial}{\partial s}\right]\right\}\phi = -\frac{\rho}{\epsilon_0}$$

if $R\to\infty$ reduces to familiar: $\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial s^2}\right\}\phi = -\frac{\rho}{\epsilon_0}$

S2: Transverse Particle Equations of Motion in Linear Applied Focusing Channels

S2A: Introduction

Write out transverse particle equations of motion in explicit component form:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' = \frac{q}{m\gamma_b\beta_b^2c^2}E_x^a - \frac{q}{m\gamma_b\beta_b c}B_y^a + \frac{q}{m\gamma_b\beta_b c}B_z^a y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' = \frac{q}{m\gamma_b\beta_b^2c^2}E_y^a + \frac{q}{m\gamma_b\beta_b c}B_x^a - \frac{q}{m\gamma_b\beta_b c}B_z^a x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

Equations previously derived under assumptions:
- No bends (fixed *x-y-z* coordinate system with no local bends)
- Paraxial equations ($x'^2, y'^2 \ll 1$)
- No dispersive effects (β_b same all particles), acceleration allowed ($\beta_b \neq \text{const}$)
- Electrostatic and leading-order (in β_b) self-magnetic interactions

The applied focusing fields

Electric:	$E_x^a,\ E_y^a$
Magnetic:	$B_x^a,\ B_y^a,\ B_z^a$

must be specified as a function of *s* and the transverse particle coordinates *x* and *y* to complete the description
- Consistent change in axial velocity ($\beta_b c$) due to E_z^a must be evaluated
  - Typically due to RF cavities and/or induction cells
- Restrict analysis to fields from applied focusing structures

Intense beam accelerators and transport lattices are designed to optimize *linear* applied focusing forces with terms:

Electric:
$$E_x^a \simeq (\text{function of } s) \times (x \text{ or } y)$$
$$E_y^a \simeq (\text{function of } s) \times (x \text{ or } y)$$

Magnetic:
$$B_x^a \simeq (\text{function of } s) \times (x \text{ or } y)$$
$$B_y^a \simeq (\text{function of } s) \times (x \text{ or } y)$$
$$B_z^a \simeq (\text{function of } s)$$

Common situations that realize these linear applied focusing forms will be overviewed:

- Continuous Focusing (see: S2B)
- Quadrupole Focusing
  - Electric (see: S2C)
  - Magnetic (see: S2D)
- Solenoidal Focusing (see: S2E)

Other situations that will not be covered (typically more nonlinear optics):
- Einzel Lens (see: J.J. Barnard, Intro Lectures)
- Plasma Lens
- Wire guiding

Why design around linear applied fields ?
- Linear oscillators have well understood physics allowing formalism to be developed that can guide design
- Linear fields are "lower order" so it should be possible for a given source amplitude to generate field terms with greater strength than for "higher order" nonlinear fields

Solenoid equations of motion:
- Insert field components into equations of motion and collect terms

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' - \frac{B'_{z0}(s)}{2[B\rho]}y - \frac{B_{z0}(s)}{[B\rho]}y' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \frac{B'_{z0}(s)}{2[B\rho]}x + \frac{B_{z0}(s)}{[B\rho]}x' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$[B\rho] \equiv \frac{\gamma_b\beta_b mc}{q} = \text{Rigidity} \qquad \frac{B_{z0}(s)}{[B\rho]} = \frac{\omega_c(s)}{\gamma_b\beta_b c}$$

$$\omega_c(s) = \frac{qB_{z0}(s)}{m} = \text{Cyclotron Frequency (in applied axial magnetic field)}$$

- Equations are linearly cross-coupled in the applied field terms
  - *x* equation depends on *y*, *y'*
  - *y* equation depends on *x*, *x'*

If the beam space-charge is *axisymmetric*:

$$\frac{\partial \phi}{\partial \mathbf{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial \mathbf{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\mathbf{x}_\perp}{r}$$

then the space-charge term also decouples under the Larmor transformation and the equations of motion can be expressed in fully uncoupled form:

$$\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa(s)\tilde{x} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial r}\frac{\tilde{x}}{r}$$

$$\tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa(s)\tilde{y} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial r}\frac{\tilde{y}}{r}$$

$$\kappa(s) = k_L^2(s) \equiv \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2$$

Will demonstrate this in problems for the simple case of:

$B_{z0}(s) = \text{const}$

- Because Larmor frame equations are in the same form as continuous and quadrupole focusing with a different κ, for solenoidal focusing we implicitly work in the Larmor frame and simplify notation by dropping the tildes:

$$\tilde{\mathbf{x}}_\perp \rightarrow \mathbf{x}_\perp$$

/// Aside: Notation:

A common theme of this class will be to introduce new effects and generalizations while keeping formulations looking as similar as possible to the the most simple representations given. When doing so, we will often use "tildes" to denote transformed variables to stress that the new coordinates have, in fact, a more complicated form that must be interpreted in the context of the analysis being carried out. Some examples:

- Larmor frame transformations for Solenoidal focusing
  See: Appendix B
- Normalized variables for analysis of accelerating systems
  See: S10
- Coordinates expressed relative to the beam centroid
  See: S.M. Lund, lectures on Transverse Centroid and Envelope Model
- Variables used to analyze Einzel lenses
  See: J.J. Barnard, Introductory Lectures

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S2F: Summary of Transverse Particle Equations of Motion

In linear applied focusing channels, without momentum spread or radiation, the particle equations of motion in both the *x*- and *y*-planes expressed as:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial x}\phi$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial y}\phi$$

$\kappa_x(s) =$ x-focusing function of lattice

$\kappa_y(s) =$ y-focusing function of lattice

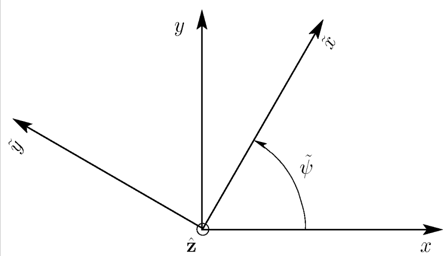
Common focusing functions:

Continuous: $\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$

Quadrupole (Electric or Magnetic):
$\kappa_x(s) = -\kappa_y(s) = \kappa(s)$

Solenoidal (equations must be interpreted in Larmor Frame: see Appendix B):
$\kappa_x(s) = \kappa_y(s) = \kappa(s)$

It can be shown (see: Appendix B) that the linear cross-coupling in the applied field can be removed by an s-varying transformation to a rotating "Larmor" frame:



$$\tilde{x} = \quad x\cos\tilde{\psi}(s) + y\sin\tilde{\psi}(s)$$

$$\tilde{y} = -x\sin\tilde{\psi}(s) + y\cos\tilde{\psi}(s)$$

$$\tilde{\psi}(s) = -\int_{s_i}^{s} d\bar{s}\; k_L(\bar{s})$$

$$k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]} = \frac{\omega_c(s)}{2\gamma_b\beta_b c}$$

= Larmor wave number

$s = s_i$ defines initial condition

$\tilde{\dots}$ used to denote rotating frame variables

Although the equations have the same form, the couplings to the fields are different which leads to different regimes of applicability for the various focusing technologies with their associated technology limits:

Focusing:

Continuous:

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

Good qualitative guide (see later material/lecture)
BUT not physically realizable (see S2B)

Quadrupole:

$$\kappa_x(s) = -\kappa_y(s) = \begin{cases} \frac{G(s)}{\beta_b c [B\rho]}, & \text{Electric} \\ \frac{G(s)}{c[B\rho]}, & \text{Magnetic} \end{cases} \qquad [B\rho] = \frac{m\gamma_b\beta_b c}{q}$$

G is the field gradient which for linear applied fields is:

$$G(s) = \begin{cases} -\frac{\partial E_x^a}{\partial x} = \frac{\partial E_y^a}{\partial y} = \frac{2V_q}{r_p^2}, & \text{Electric} \\ \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x} = \frac{B_p}{r_p}, & \text{Magnetic} \end{cases}$$

Solenoid:

$$\kappa_x(s) = \kappa_y(s) = k_L^2(s) = \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2 \qquad \omega_c(s) = \frac{qB_{z0}(s)}{m}$$

It is instructive to review the structure of solutions of the transverse particle equations of motion in the absence of:

Space-charge: $\frac{\partial \phi}{\partial x} \sim \frac{\partial \phi}{\partial y} \sim 0$

Acceleration: $\gamma_b\beta_b \simeq \text{const} \qquad \Longrightarrow \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq 0$

In this simple limit, the x and y-equations are of the same Hill's Equation form:

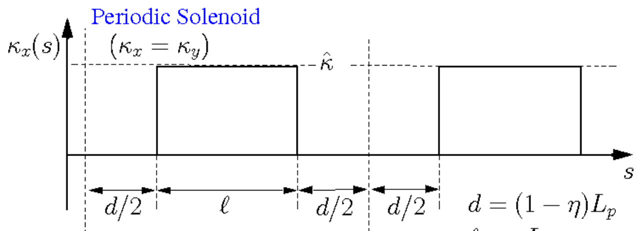
$$x'' + \kappa_x(s)x = 0$$
$$y'' + \kappa_y(s)y = 0$$

- These equations are central to transverse dynamics in conventional accelerator physics (weak space-charge and acceleration)
  - Will study how solutions change with space-charge in later lectures

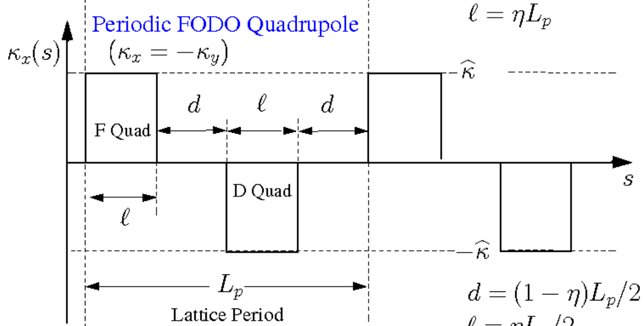
In many cases beam transport lattices are designed where the applied focusing functions are periodic:

$$\kappa_x(s + L_p) = \kappa_x(s) \qquad L_p = \text{Lattice Period}$$
$$\kappa_y(s + L_p) = \kappa_y(s)$$

Common, simple examples of periodic lattices:



$d = (1-\eta)L_p$
$\ell = \eta L_p$



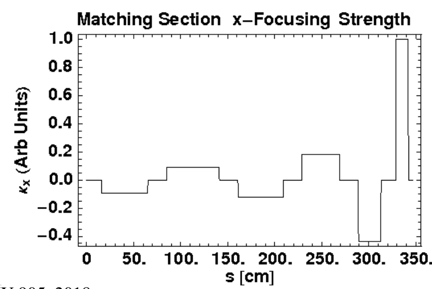
$d = (1-\eta)L_p/2$
$\ell = \eta L_p/2$

However, the focusing functions need not be periodic:
- Often take periodic or continuous in this class for simplicity of interpretation

Focusing functions can vary strongly in many common situations:
- Matching and transition sections
- Strong acceleration
- Significantly different elements can occur within periods of lattices in rings
  - "Panofsky" type (wide aperture along one plane) quadrupoles for beam insertion and extraction in a ring

Example of Non-Periodic Focusing Functions: Beam Matching Section

Maintains alternating-gradient structure but not quasi-periodic



Example corresponds to High Current Experiment Matching Section (hard edge equivalent) at LBNL (2002)

S5: Linear Transverse Particle Equations of Motion without Space-Charge, Acceleration, and Momentum Spread

S5A: Hill's Equation

Neglect:

- Space-charge effects: $\partial \phi / \partial \mathbf{x} \simeq 0$
- Nonlinear applied focusing and bends: $\mathbf{E}^a$, $\mathbf{B}^a$ have only linear focus terms
- Acceleration: $\gamma_b \beta_b \simeq$ const
- Momentum spread effects: $v_{zi} \simeq \beta_b c$

Then the transverse particle equations of motion reduce to Hill's Equation:

$$x''(s) + \kappa(s) x(s) = 0$$

x=$\perp$ particle coordinate
(i.e., x or y or possibly combinations of coordinates)

$s =$ Axial coordinate of reference particle

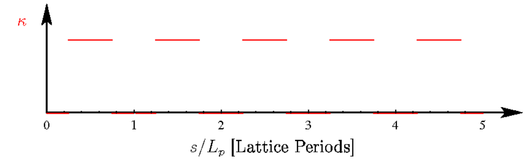
$' = \frac{d}{ds}$

$\kappa(s) =$ Lattice focusing function (linear fields)

For a periodic lattice:

$$\kappa(s + L_p) = \kappa(s)$$
$$L_p = \text{Lattice Period}$$

/// Example: Hard-Edge Periodic Focusing Function



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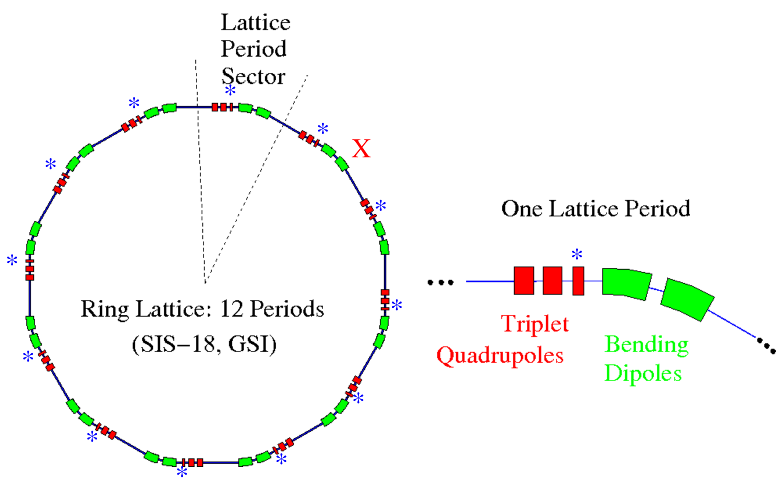
For a ring (i.e., circular accelerator), one also has the "superperiod" condition:

$$\kappa(s + \mathcal{C}) = \kappa(s)$$
$$\mathcal{C} = \mathcal{N} L_p = \text{Ring Circumfrance}$$
$$\mathcal{N} = \text{Superperiod Number}$$

- Distinction matters when there are (field) construction errors in the ring
  - Repeat with superperiod but not lattice period
  - See lectures on: Particle Resonances

/// Example: Period and Superperiod distinctions for errors in a ring

* Magnet with systematic defect will be felt every lattice period
X Magnet with random (fabrication) defect felt once per lap



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For Hill's Equation, we covered:

$$x''(s) + \kappa(s) x(s) = 0$$

- Phase amplitude method to solve particle orbit
  - Phase advance of oscillations and relation to stability
  - W and betatron function being a special unique function of a periodic lattice
- Quadratic Courant-Snyder invariant
  - Implication on understanding how bundle of particle evolves in lattice
  - Relation to max particle extent with the betatron function
- Use of normalized coordinates to map accelerated beam orbit with to a coasting beam orbit with
  - Normalized emittance to measure beam phase space with acceleration
- Resonance instabilities

What happens now to particle orbits when the axial momentum/Rigidity is not the design value?

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/msu/phy905_2018