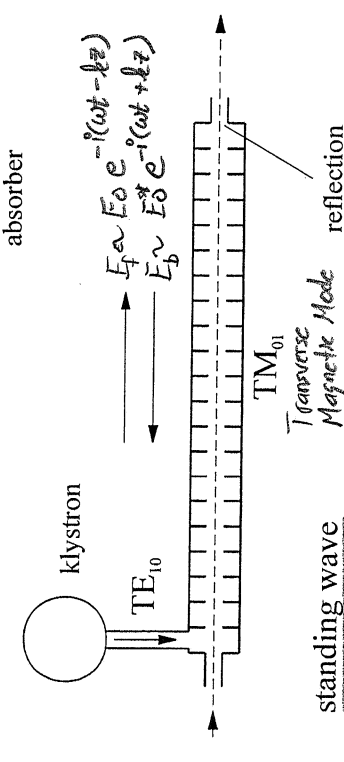
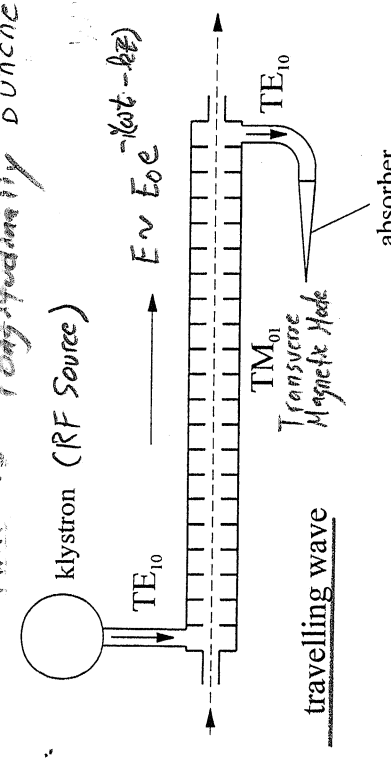


Longitudinal Physics: Beam Acceleration

Different technologies can be employed for beam acceleration

RF: Radio Frequency EM Waves
 Tuned to resonate with beam.
 that is longitudinally bunched.



$$E + E_b \Rightarrow 2E_0 \cos(kz) e^{-i\omega t}$$

$$E_a = E_0 e^{-i(\omega t - kz)}$$

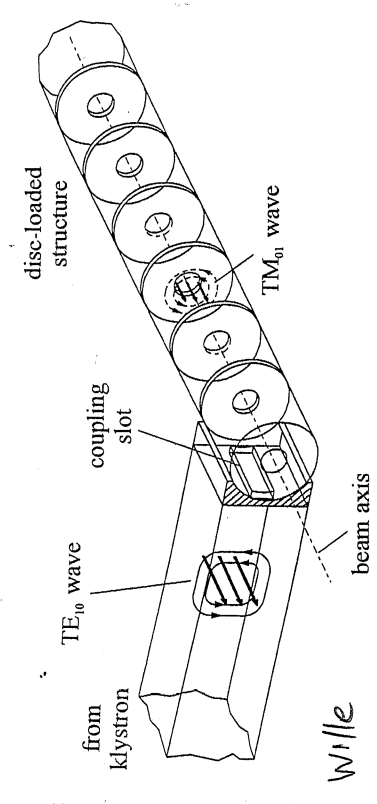
$$E_b = E_0 e^{+i(\omega t + kz)}$$

Wille

Fig. 5.9 The two modes of operation of the linac structure. The upper diagram shows the more commonly used travelling wave mode in which an absorber is installed at the end of the structure to prevent reflections. In the second case the wave is reflected virtually without losses, resulting in a standing wave.

- Two basic schemes!
- 1) Travelling Wave: e⁻ machines common
 - 2) Standing Wave: most common
 Cavities coupled or individually controlled.

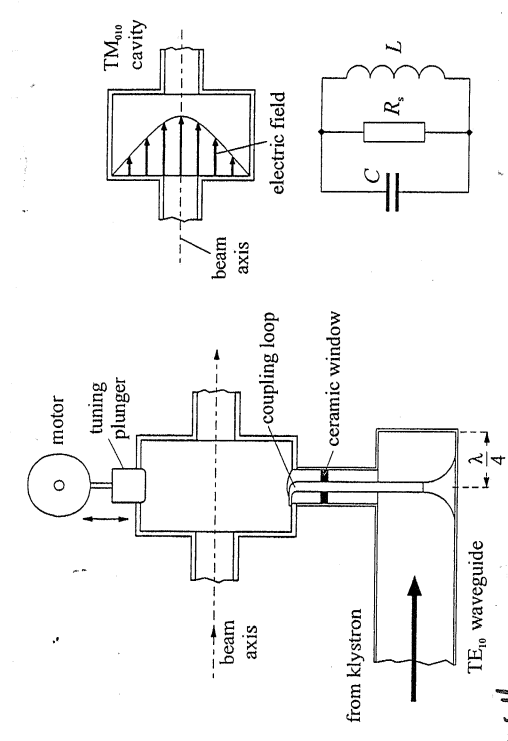
Travelling Wave



Wille

Fig. 5.8 Coupling of the TE₁₀ waveguide to the linac structure. The transfer of the wave is achieved without reflections via an appropriately sized coupling slot.

Standing Wave



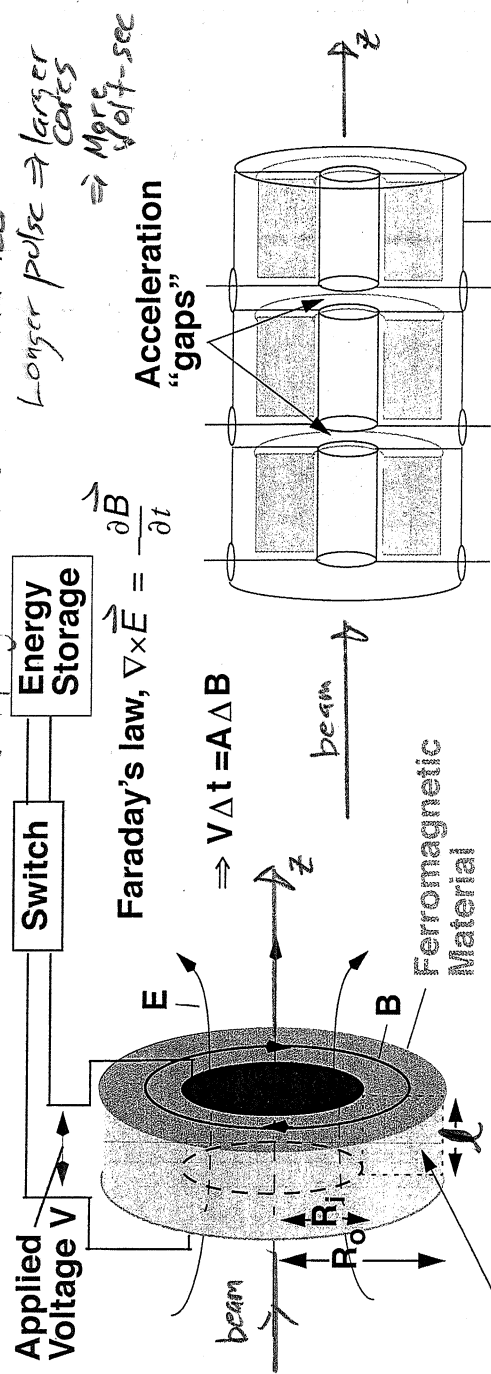
Wille

Fig. 5.4 Design of a single-cell accelerating structure using the TM₀₁₀ mode. The exact resonant frequency is adjusted using a tuning plunger. The resonator is excited by an inductive coupling loop.

Induction Acceleration

Beam coupled inductively to a pulsed power source. Operates like a 1:1 transformer. Ferromagnetic core must have sufficient "capacity" (Volt-seconds) to keep voltage from collapsing over pulse duration of beam. Beam pulse can be as long as voltage can be maintained.

Schematic

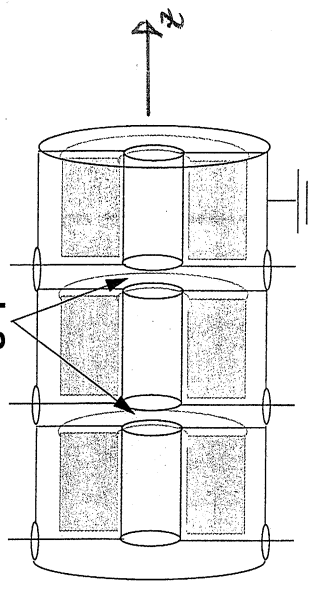


Faraday's law, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\Rightarrow V \Delta t = A \Delta B$

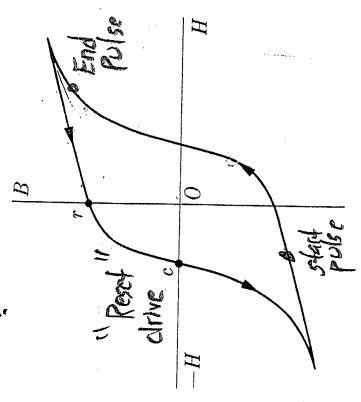
Cross-sectional area A
 $A = (R_o - R_i) \ell$

"Acceleration gaps"

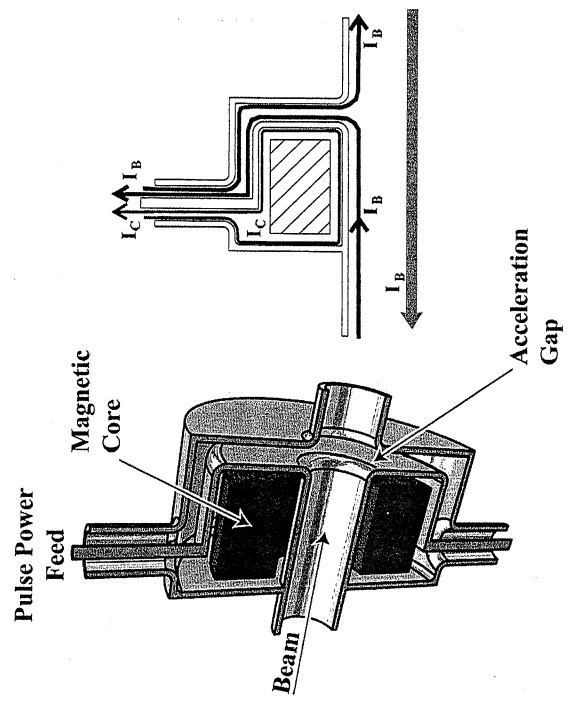


Longer pulse \Rightarrow larger cores \Rightarrow More Volt-sec

"Reset" to same point on B(H) curve each pulse.



More Realistic Geometry



$$\frac{dV}{dz} = (R_o - R_i) \Delta B \left(\frac{\text{Radial Packing}}{\text{Frac}} \right) \left(\frac{\text{Axial Packing}}{\text{Frac}} \right)$$

$\sim 1m \times 25T \times 0.8 \times 0.8 \sim 1.6 \frac{\text{Volt-sec}}{m}$

- * Losses in material heat core + reset time \Rightarrow Challenging for Rings or CW = Continuous Wave applications
- * Easy to shape pulse. Good for low rep rate, high intensity.
- * Conceptually simple / appealing, and can be efficient, but pulse power control also can be challenge.

Electrostatic Acceleration

see Livingston and Blewett, "Particle Accelerators" for more info.

Use DC high voltage electric field to accelerate charged particles falling through a potential well. Beam can be continuous or pulsed.

* $\Delta E = q \Delta V$, $\Delta V = \text{change in E.S. potential}$
 $\Delta E = \text{kinetic energy}$

Concept

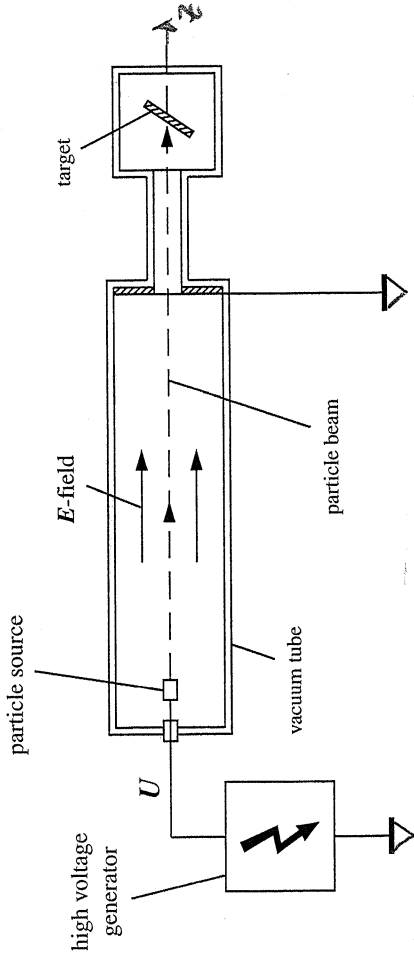
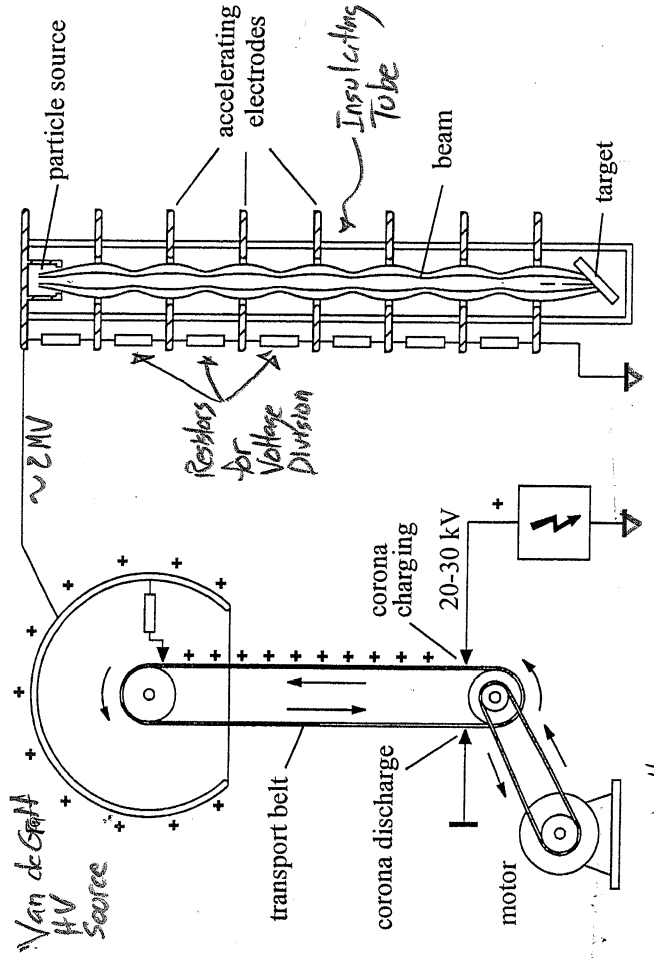


Fig. 1.3 General principle of the electrostatic accelerator.

Wille

closer to Reality!

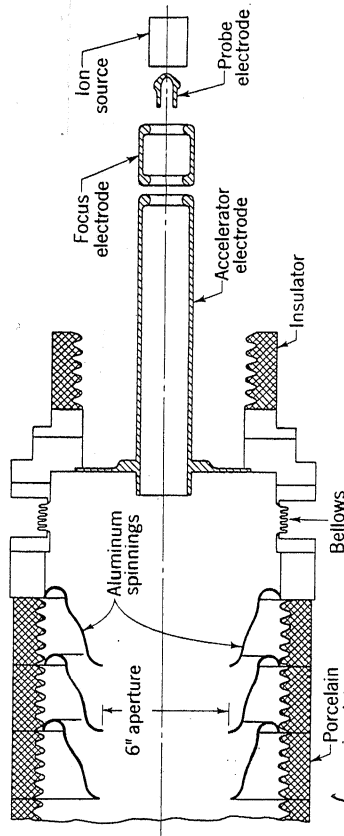


Wille Fig. 1.7 The Van de Graaff accelerator.

Need DC or long pulse supplies to work!

- Van de Graaff (static electro mechanical)
- Cockcroft-Walton (AC to DC voltage mult)
- Marx Generator (long pulse)

More Realistic Geometry



LIVINGSTON & BLOCH

Fig. 3-13. Positive-ion source, focusing electrodes, and accelerating-tube structure for the Brookhaven 4-Mv generator.²²

- * Gratings and voltage division limit local fields to inhibit electrical breakdown.
- * Insulators structured to inhibit avalanche breakdown.
- * Careful attention to details
 - No sharp metal corners near large potential diff.
 - Metal / Insulator junctions.

Best Efforts result in only few MV max.

Breakdown Scaling

Voltage Holding found to scale as
(Handbook Accel. Phys., A. Faltens)

$$V_{max} \approx 100 \text{ kV} \begin{cases} \left(\frac{d}{1 \text{ cm}}\right) & d \leq 1 \text{ cm} \\ \left(\frac{d}{1 \text{ cm}}\right)^{1/2} & d > 1 \text{ cm} \end{cases}$$

d = characteristic distance

Scaling can be degraded:

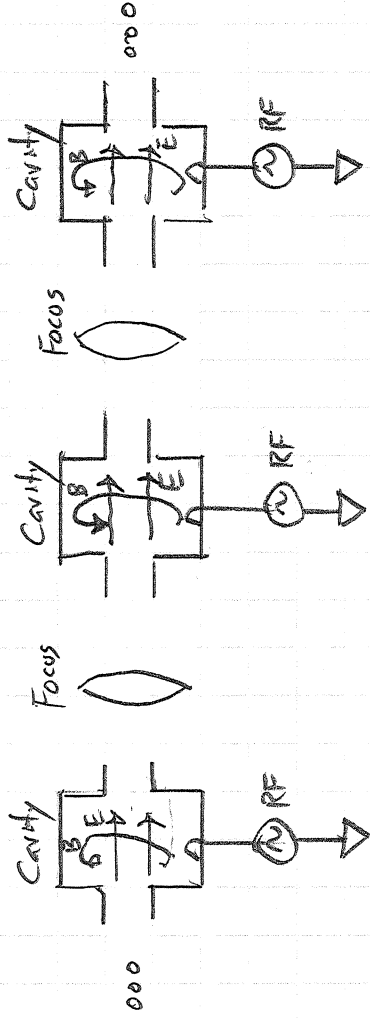
- * Under "typical" near injector vacuum conditions $\sim 10^{-7}$ Torr
 - Poor vacuum can degrade.
- * Assumes steps taken to minimize local peak field.
 - Rounded edges
 - Smooth conductors
- * Lost particles on conductors or insulators can trigger breakdown.

RF Acceleration

We will concentrate primarily on RF acceleration, primarily from the perspective of an RF Linac using resonant cavities. But before proceeding to outline how these works here we frame a range of potential concepts to place in context.

In RF concepts the beam must be longitudinally bunched with bunches maintaining proper synchronism with an oscillating RF wave.

Linear Accelerator with RF Cavities



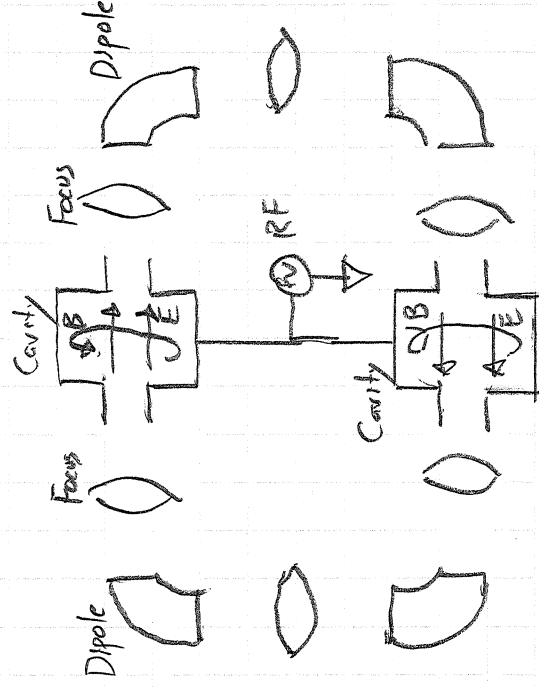
* Cavities placed where particle transit between cavities is phased for energy gain + longitudinal focusing

* Transverse focusing provided by optics between cavities.

* RF sources drive cavities with proper phase control.

- Heavy ions with low β may require individual phase control
- Cavities may also be coupled with established phase relationship.

Circular Accelerator with RF Cavities



* Cavity phase control (possibly in some high harmonic) setup for energy gain + longitudinal focusing consistent with particle time transit around ring

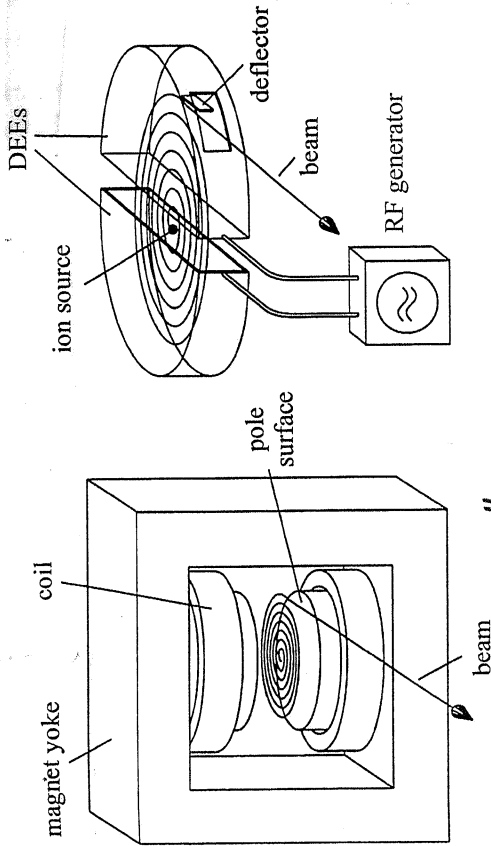
- Path length with p + slip factor must be accounted for

* Focus + Bending between cavities

* One or few RF cavities at positions in ring. Cavities have related phase.

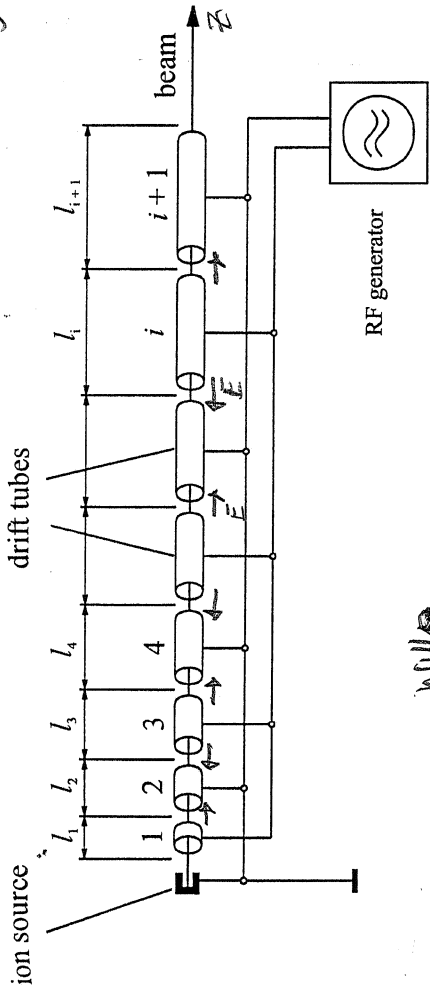
Range of RF Concepts : Very Broad! Only Schematic Outline here.

Cyclotron See Livingston and Blenett, "Particle Accelerators", chapter 6 for more info.



Wille
Fig. 1.12 The cyclotron.

Wideröe Linac See historical discussions in many accelerator books: Wiedemann, Conte & McKay, Wangler, "..."



Wille
Fig. 1.9 Wideröe linear accelerator.

Non-Relativistic:

$$\omega = \frac{qBz}{m} = \text{const}$$

$$\frac{1}{f} = \frac{Bz}{(BP)}$$

↑ increases
with energy gain
till particle
spirals out to
deflector.

As particle becomes
relativistic, synchronism
will be broken.

- * Not much focusing possible
- * Continuous train/bunches possible
- * Relatively simple.

Already discussed
1st lectures.

Non-Relativistic

$$W_i = \frac{1}{2} m v_i^2$$

Kinetic Energy

Gap Separation $l_i = \frac{2\pi v_i T_A}{z}$ π Phase advance between gaps

$$= \frac{Bz C T_A}{z} = \frac{\beta_i \lambda_A}{z} \quad \lambda_A \approx c T_A$$

for resonance acceleration

Wideroe Linac continued

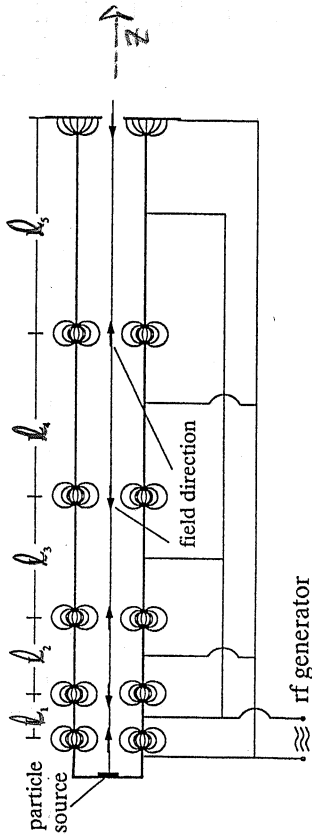


Fig. 2.5. Wideroe linac structure (schematic)

Wiedemann

Alvarez Linac or Drift Tube Linac (DTL)

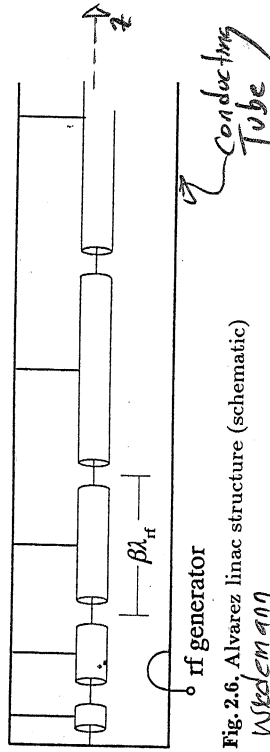


Fig. 2.6. Alvarez linac structure (schematic)

Wiedemann

For more info see:

- Wiedemann, Wille, Wangler,
- Conte & Mackay, Edwards & Syphers

- ★ Tubes shield particles from deaccelerating (wrong phase) RF till they get to the next gap.
- ★ Resonance condition established by adjusting the tube length.

- ★ Structure is lossy; radiates power.
⇒ Enclose in tube to make cavity ⇒ Alvarez structure.

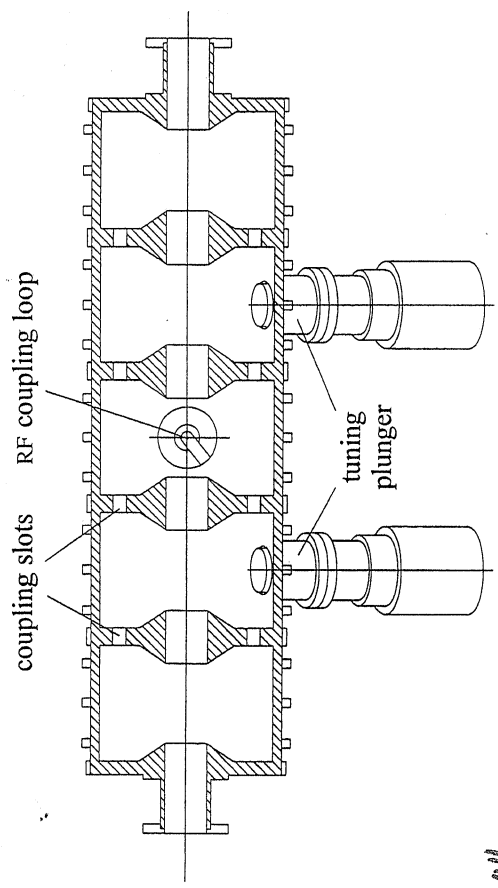
- ★ Tube to contain radiation boosts efficiency and allows higher freq. RF

Can use $l_1 = \frac{\beta \lambda}{2}$ or $l_1 = \beta \lambda \pi$
 ↑ phase advance 2π phase advance.

due to this.

- ★ Still common pre-accelerator for protons/ions from injection to a few hundred MeV
 $\beta \approx 0.04 \sim 0.4$
- ★ Not used for electrons since β is typically too high from injector

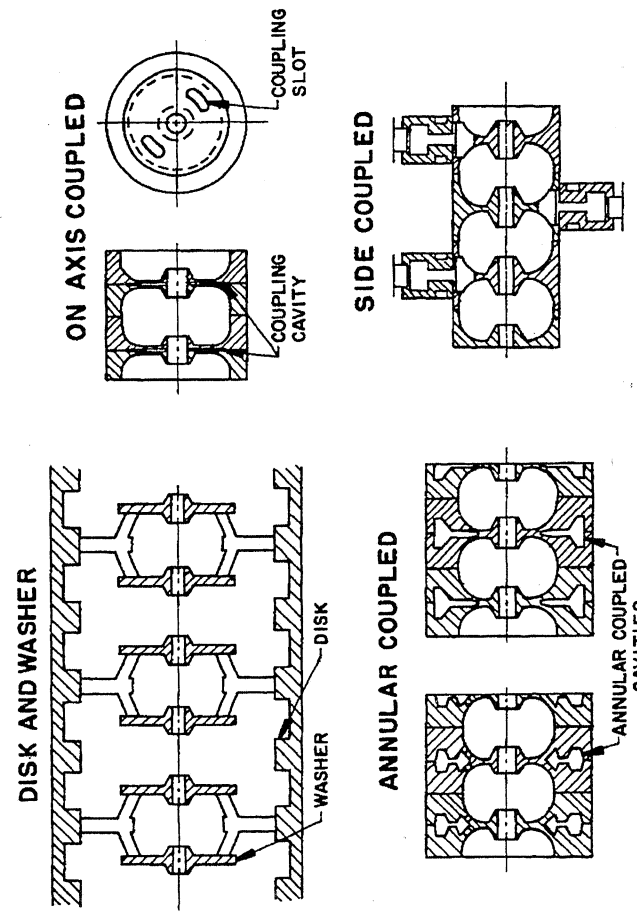
Coupled Cavity Linac See Wangler, "RF Linear Accelerators" for more info.



Wangler
 Fig. 5.5 Layout of a five-cell accelerating structure. The power feed is coupled to the middle cell and two tuning plungers are sufficient for the entire structure.

Banks of RF cavities are coupled together to maintain relative RF phase control needed.

- Very common for high β particles.
- Simplifies RF drive and saves cost.
- Many possible geometries.

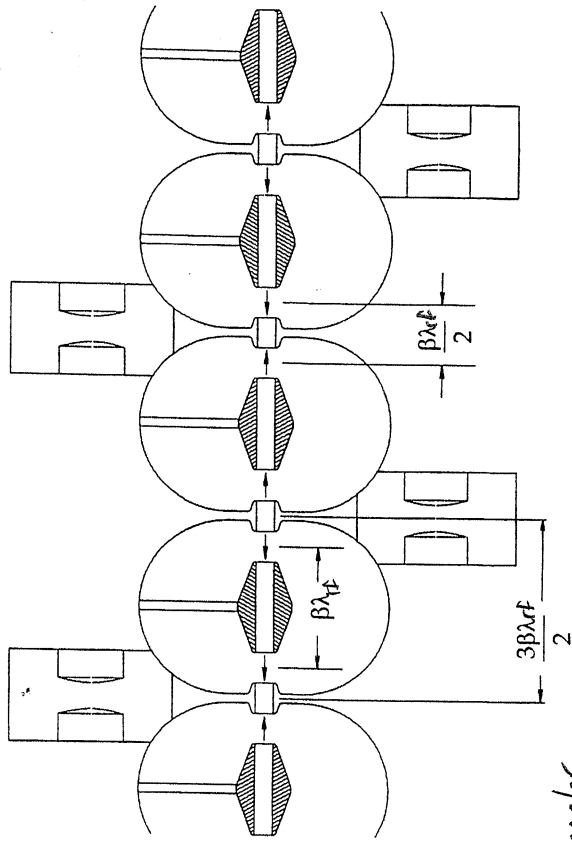


Wangler
 Figure 4.17 Four examples of coupled-cavity linacs.

* Coupling cavities sometimes in beam line and other times moved off-axis for more efficient packing.

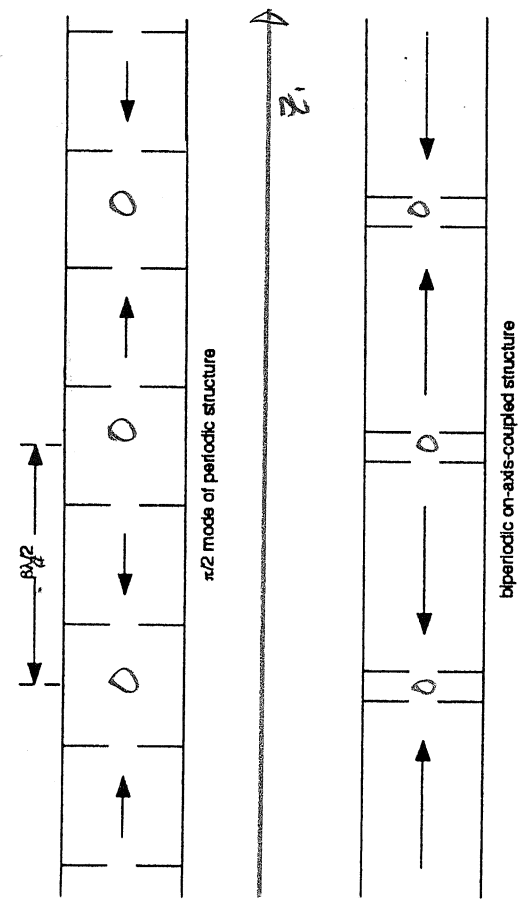
* Usually transverse focusing interspersed between banks of coupled RF cavities.

Coupled Cavity Linac For other examples

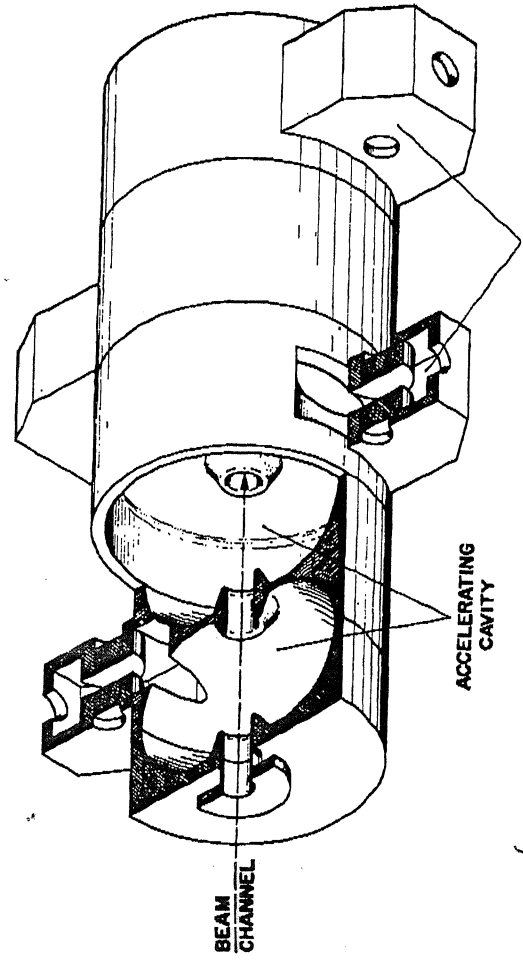


Wangler
 Figure 12.2 Coupled-cavity drift-tube linac (CCDTL) structure with a single drift tube in each accelerating cavity.

Example Phase relations of E field between cavities



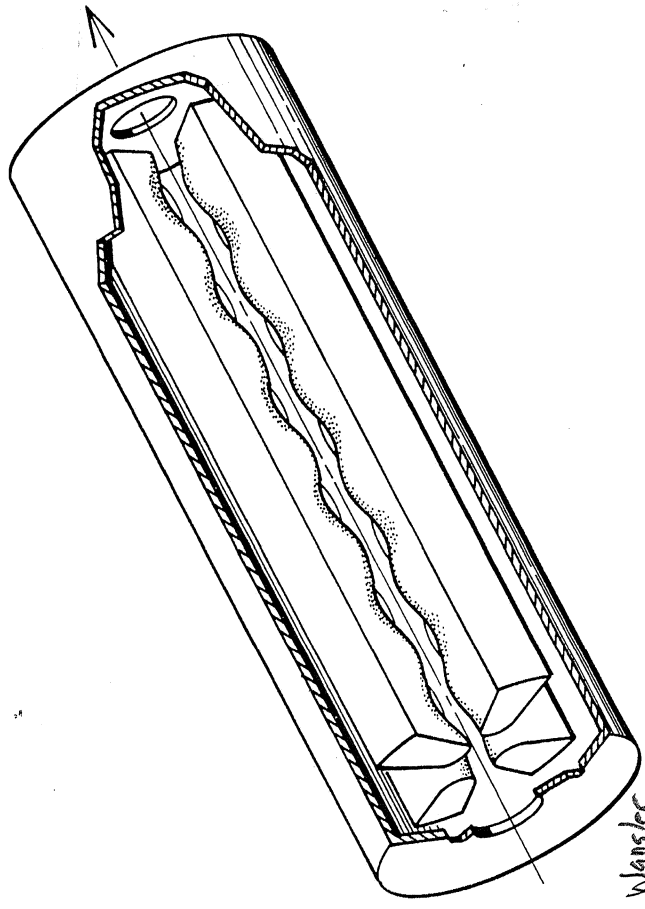
Wangler
 Figure 4.15 $\pi/2$ -like-mode operation of a cavity resonator chain. From top to bottom are shown a periodic structure in $\pi/2$ mode, a biperiodic on-axis coupled-cavity structure in $\pi/2$ mode, and a biperiodic side-coupled cavity in $\pi/2$ mode.



Wangler
 Figure 4.11 Side-coupled linac structure as an example of a coupled-cavity linac structure. The cavities on the beam axis are the accelerating cavities. The cavities on the side are nominally unexcited and stabilize the accelerating-cavity fields against perturbations from fabrication errors and beam loading.

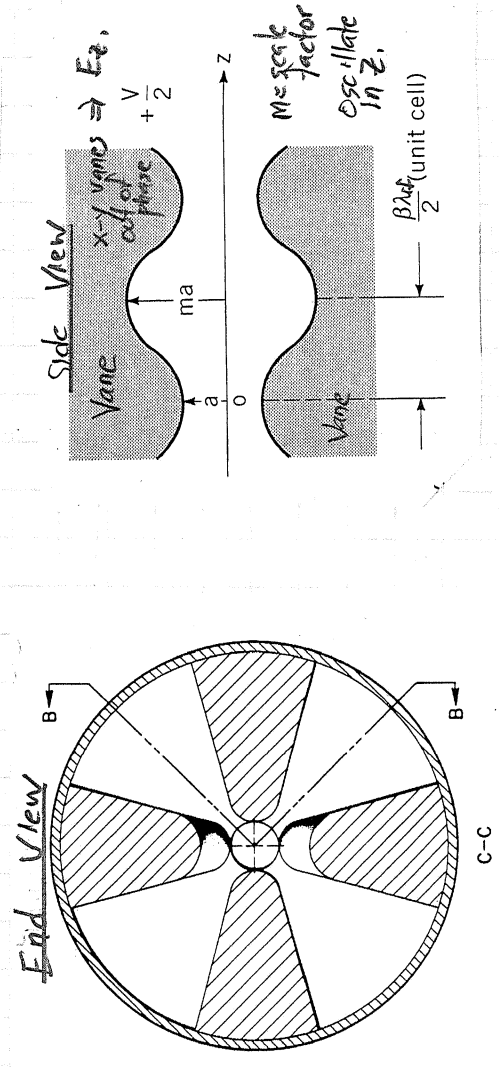
Radio Frequency Quadrupole (RFQ)

see Wangler, "RF Linear Accelerators" for more info.



Wangler

Figure 1.7 The radio-frequency quadrupole (RFQ), used for acceleration of low-velocity ions, consists of four vanes mounted within a cylindrical cavity. The cavity is excited in a quadrupole mode in which the RF electric field is concentrated near the vane tips to produce an electric transverse restoring force for particles that are off-axis. The modulation of the vane tips produces a longitudinal electric-field component that accelerates the beam along the axis.



★ Electric quadrupole mode excited in cavity with four quadrupole symmetry vane electrodes.

- Vanes concentrate \perp E field to provide strong transverse quadrupole (electric) focusing
- Longitudinal ripples of vanes provides \parallel E field for longitudinal acceleration and focusing.

★ Works for $\beta \approx 0.01 - 0.06$

★ Can be setup to bunch and accelerate a DC injected beam from source to match info required bunch structure of RF accelerator.

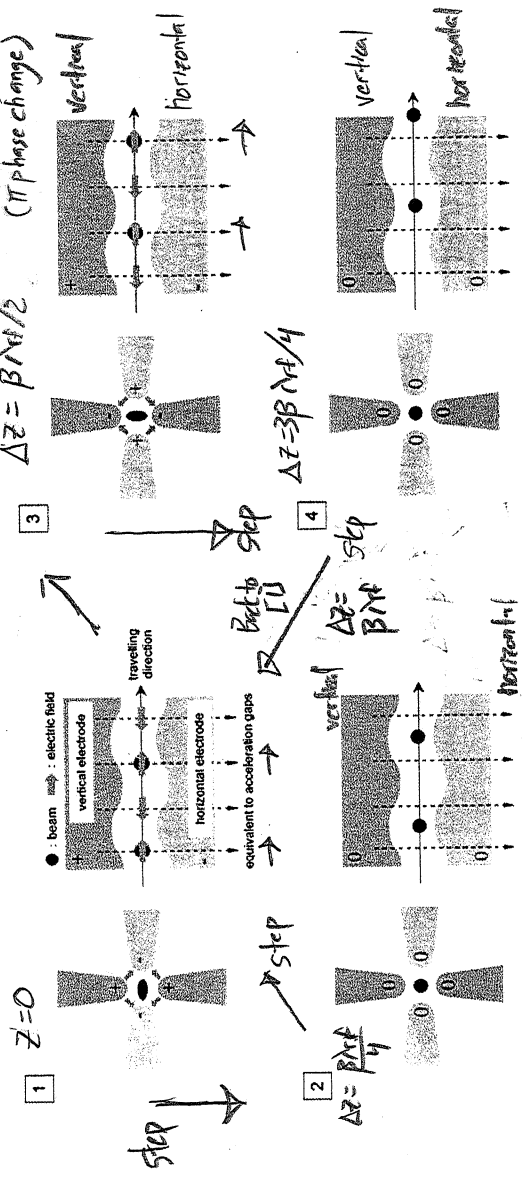
- Structure can be tapered to enhance bunching or acceleration. Periodic β that varies with energy gain.

★ Common choice for front ends. - Including FRIB.

★ Not used for electrons since low β structure.

Schematic on how an RFQ works! M. Syphers, USPAS Notes.

The Radio Frequency Quadrupole (RFQ)



Many variants: 4-vanes, 4-rods, Al/Cu, large/small
 Typical energy range — up to few MeV (protons, ions typically; also electrons)

1	de Focus x Focus y	Accel. in z + phase focus from field variation
2	Null	
3	Focus x de Focus y	Accel. in z + phase focus from field variation
4	Null	AG Focus-Defocus cycle

Where + Electrode closer $\Rightarrow \phi$ on axis +
 Where - Electrode closer $\Rightarrow \phi$ on axis -

Comment: An RFQ essentially employs AG electric transverse focusing which is strong for low Velocity (β) particles.

$$x'' + \frac{(x\beta)'}{(\beta\beta)} x' + R x = 0$$

$$R = \begin{cases} \frac{B'}{(\beta\beta)} & \text{Magnetic} \\ \frac{E'}{(\beta\beta)(\beta\beta)} & \text{Electric} \end{cases}$$

extra factor $\beta^{1/4}$ denominator

$$B' = \frac{\partial B_x}{\partial x} = \frac{\partial B_x}{\partial y} = \text{Magnetic Quad Gradient}$$

$$E' = -\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = \text{Electric Quad Gradient}$$

Traveling Wave Linac

see Wangler, "RF Linear Accelerators" Chapter 3

for more info.

Why not use a simple waveguide TM mode to have a longitudinal E_z resonant with beam for acceleration?

Phase velocity of waveguide modes $> c$ so cannot maintain resonance.

But can add periodic structure in waveguide to slow wave and maintain resonance.

- Structure essentially sets up small coupled cavities with part reflections.

- periodic lattice of disks filling cylindrical waveguide commonly used. Example: SLAC electron linac.

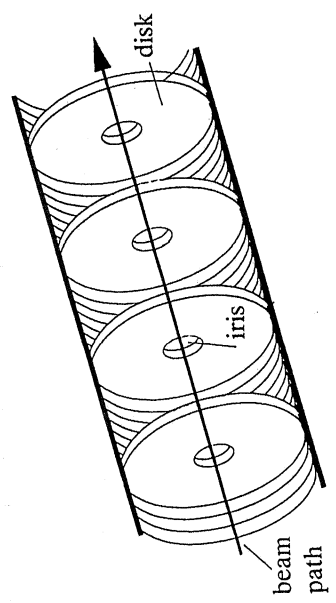


Fig. 2.8. Disk loaded accelerating structure for an electron linear accelerator (schematic) Wiedemann

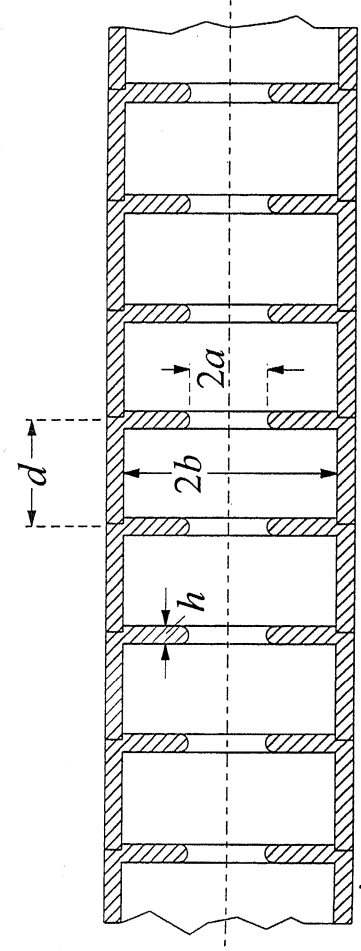


Fig. 5.6 Cross-section through a typical linac structure. The phase velocity of the RF wave is reduced to the particle velocity by the insertion of irises.

Some aspects will be discussed more later:

- Waveguide modes.
- Traveling wave field to calculate energy gain.

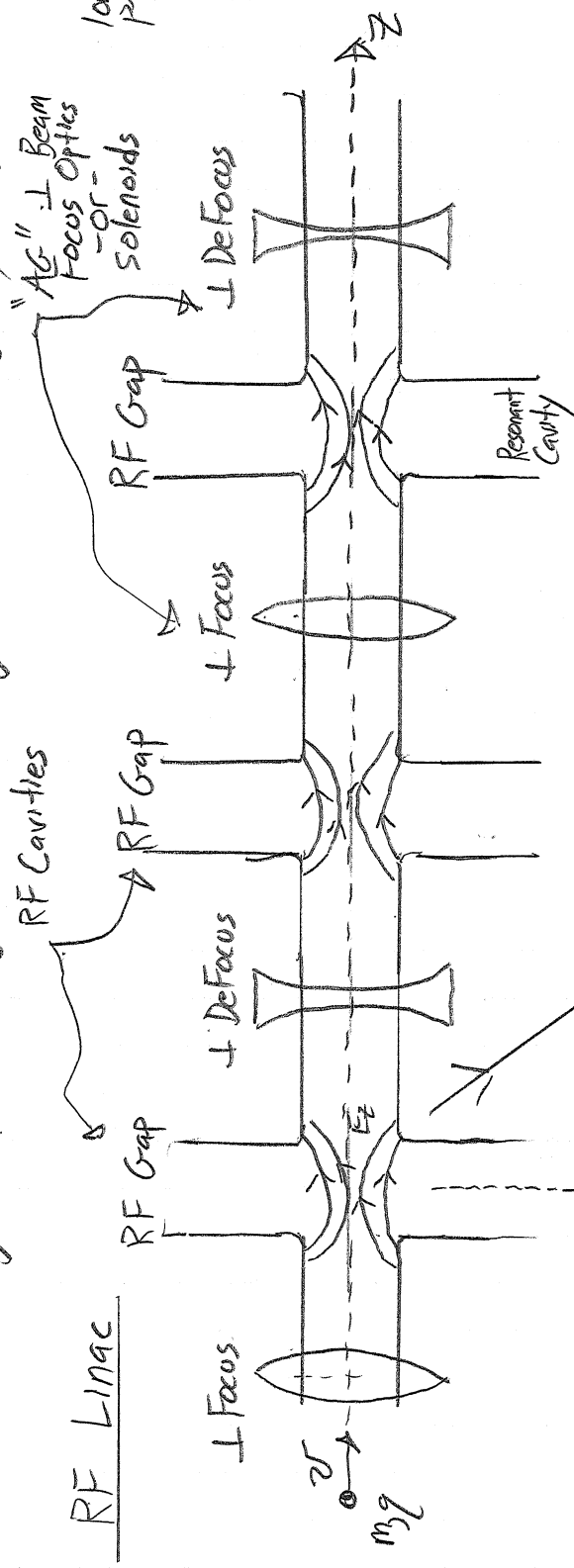
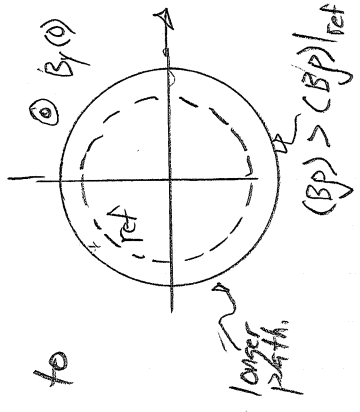
RF Linear Acceleration

Will follow Wangler's "RF Linear Accelerators"

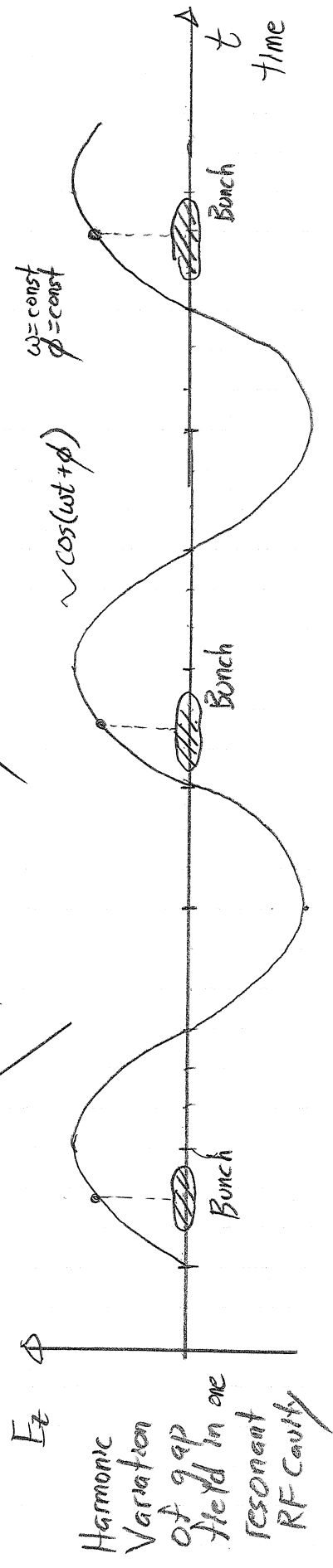
+ info from other sources cited, and Bernard 4 Lund USPAS

We will first cover RF linacs and then modify the formulation to a form appropriate for rings.

* Rings require modification of synchronism conditions due to longer path length for larger particle rigidity (BP)



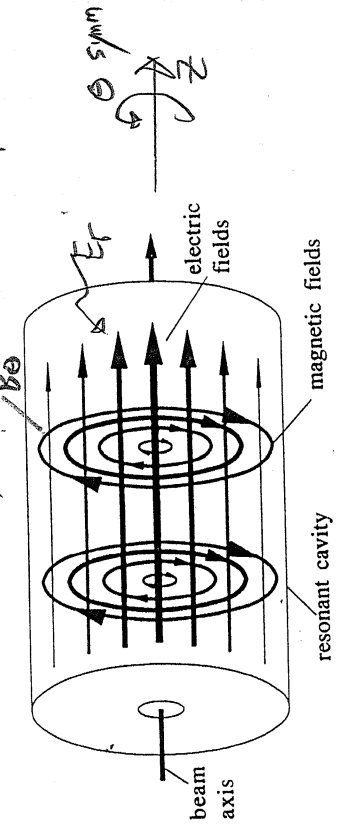
Gap Field in Cavity: all RF "buckets" filled



Harmonic Variation of gap field in one resonant RF cavity

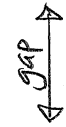
RF Cavity Fields
"Pill Box" Cavity

Wangler, §102
Cotte & Mackay, Chapter 9
Wiedemann, §20.2
Wille, Chapter 5

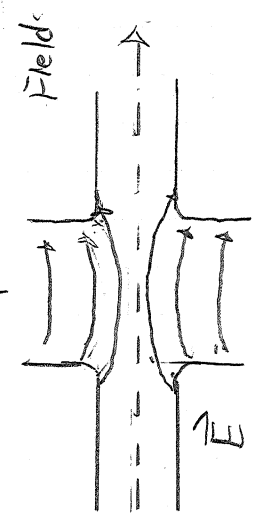
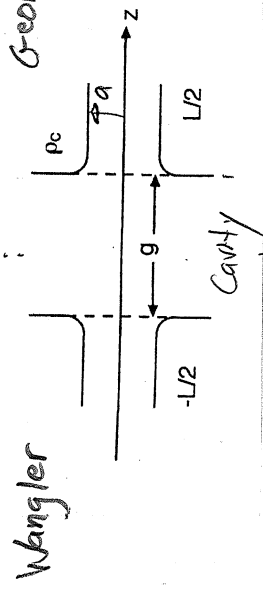


Wiedemann

Structures within the cavity often concentrate the field in an "RF Gap"



Geometry



Allowed
Due to finite
beam hole +
gap structures

Simplist Case:

Cavity excited harmonically with a lowest order transverse magnetic mode that primarily generates a longitudinal E_z for beam acceleration when particles transit at the right phase.

TM₀₁₀ mode shown

- * Cavities may be coupled (high β) or independently driven (low β) with appropriate phase control.
- * More details on cavities later.

Harmonic TM₀₁₀ Fields

$\phi = \text{const}$ RF phase
 $\omega = \text{const}$ RF angular velocity

$E_z(r, z, t) = E_z(r, z) \cos(\omega t + \phi)$

$B_\theta(r, z, t) = B_\theta(r, z) \sin(\omega t + \phi)$ \times $\frac{\text{out of page}}{\text{by } 1/2}$

$E_r(r, z, t) = E_r(r, z) \cos(\omega t + \phi)$

Will discuss cavity fields more later, but for moment motivate form Ok

Within cavity (vacuum region)

- 1) $\nabla \cdot \vec{E} = \frac{\rho_{ext}}{\epsilon_0} = 0$
 - 2) $\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$
 - 3) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- $\nabla \cdot \vec{B} = 0$

- $E_z(\rho, z, t) = E_z(\rho, z) \cos(\omega t + \phi)$
- $B_\theta(\rho, z, t) = B_\theta(\rho, z) \sin(\omega t + \phi)$
- $E_r(\rho, z, t) = E_r(\rho, z) \cos(\omega t + \phi)$

1) $\nabla \cdot \vec{E} = \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_r) + \frac{\partial E_z}{\partial z} \right] \cos(\omega t + \phi) = 0$

$\Rightarrow \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_r) + \frac{\partial E_z}{\partial z} = 0$ (1)

2) $\nabla \times \vec{B} = \left[-\frac{\partial B_\theta}{\partial z} \hat{r} + 0 \hat{\theta} + \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\theta) \hat{z} \right] \sin(\omega t + \phi)$

$= \left[\frac{1}{c^2} \frac{\partial E_r}{\partial t} \hat{r} + E_z(\rho, z) \hat{z} \right] \sin(\omega t + \phi)$

3) $\hat{z}: \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\theta) = -\frac{\omega E_z(\rho, z)}{c^2}$

3) $\nabla \times \vec{E} = \left[0 \hat{r} + \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_r) \hat{\theta} + 0 \hat{z} \right] \cos(\omega t + \phi)$

$= -\frac{\partial B_\theta}{\partial t} = -\omega B_\theta \cos(\omega t + \phi)$

4) $\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_r) - \frac{\partial E_z}{\partial z} = -\omega B_\theta \sin(\omega t + \phi)$

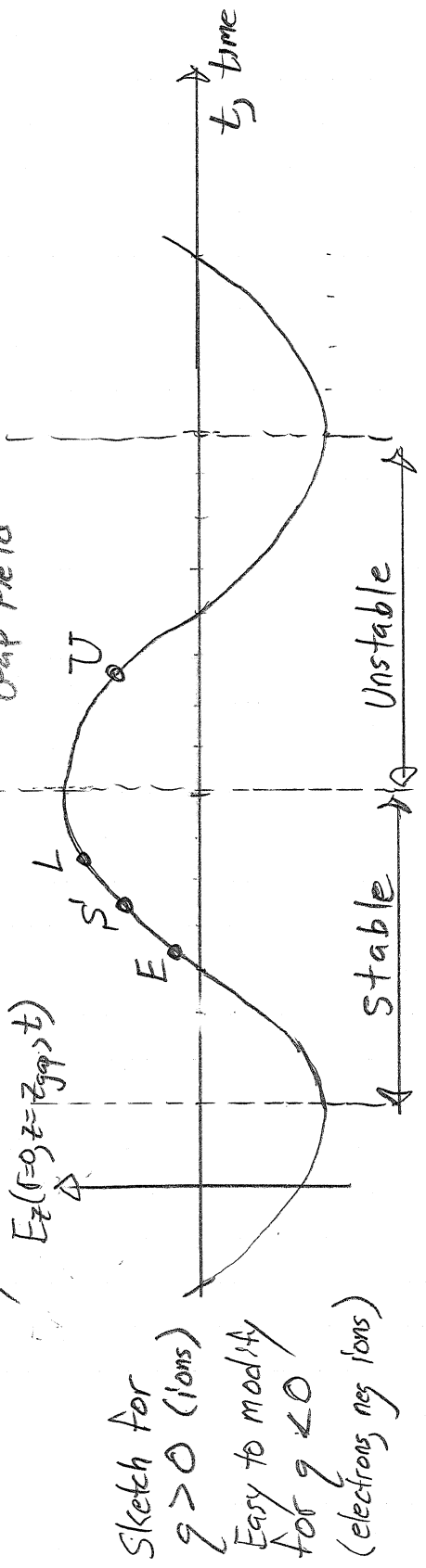
4) $\nabla \cdot \vec{B} = 0 = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\theta) = 0$ Satisfied

Cavity Field Constraints

- 1) $\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_r) + \frac{\partial E_z}{\partial z} = 0$
- 2) $\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_r) = \frac{\omega E_z}{c^2}$
- 3) $\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\theta) = -\frac{\omega E_z}{c^2}$
- 4) $\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_r) - \frac{\partial E_z}{\partial z} = -\omega B_\theta \sin(\omega t + \phi)$

Phase Stability: Basic Idea.

see Wangler, § 1.3.



Sketch for $q > 0$ (ions)
 Easy to modify for $q < 0$ (electrons, neg ions)

Stable on rising E_z -field $dE_z/dt > 0$

S' : "Synchronous" Particle: Will reach next gap at design time to same position on RF wave.

E : Early: More energetic particle arrives early
 E_z lower \Rightarrow less energy gain \Rightarrow smaller ω increase
 \Rightarrow moves toward S' at next gap.

L : Late: Less energetic particle arrives late
 E_z higher \Rightarrow more energy gain \Rightarrow larger ω increase
 \Rightarrow moves toward S' at next gap.

Unstable on falling E_z -field $dE_z/dt < 0$

Cases reversed: early 'late' will move away from any design particle choice at next gap.

Particle Dynamics in Gap

see Wangler's Chapter 2

RF Gap Fields — TM₀₁₀ - like excitation

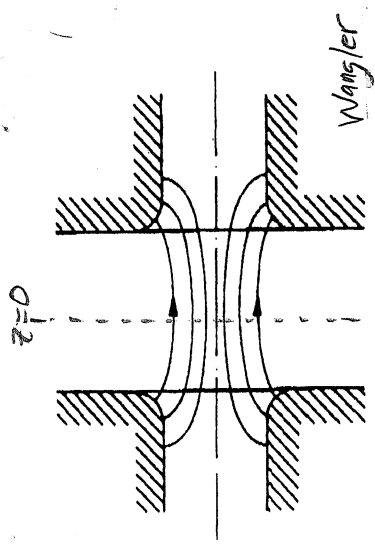


Figure 1.9 Electric-field lines in an accelerating gap.

$$E_z(r, z, t) = E_z(r, z) \cos(\omega t + \phi)$$

$$B_\theta(r, z, t) = B_\theta(r, z) \sin(\omega t + \phi)$$

$$E_r(r, z, t) = E_r(r, z) \cos(\omega t + \phi)$$

Lorentz Force

$$\frac{d\vec{p}}{dt} = q \vec{E} + q \vec{v} \times \vec{B}$$

$$= q E_r \hat{r} + q E_z \hat{z} - q v_\theta B_\theta \hat{r} + q v_r B_\theta \hat{z}$$

$$\hat{z}: \frac{dp_z}{dt} = q E_z(r, z) \cos(\omega t + \phi) + q v_r B_\theta(r, z) \sin(\omega t + \phi)$$

$$\hat{r}: \frac{dp_r}{dt} = q E_r(r, z) \cos(\omega t + \phi) - q v_\theta B_\theta(r, z) \sin(\omega t + \phi) + F_r$$

$$\hat{\theta}: \frac{dp_\theta}{dt} = 0$$

Focusing Optics

From \perp focusing elements

17/ Estimate the kinetic energy gain of a particle in the gap from the on-axis ($r=0$) fields.

$$E_z(r=0, z, t) = E(0, z) \cos(\omega t(z) + \phi)$$

$$B_\theta(r=0, z, t) = 0$$

will find later B_θ at $r=0$ cavity \Rightarrow be small on axis of gap of small radial extent in cavity
 Insert in equations of motion

$$t(z) = \int \frac{dz}{v(z)} + \text{const}$$

* Ref: Particle at center of c-gap ($z=0$) at time $t=0$.

* $v(z) \approx v \ll |c|$ Paraxial Approx

$$t(z) = \int_0^z \frac{dz'}{v(z')}$$

Note: At time $t=0$, ϕ is the phase of the E-field relative to the peak value

$$E_z(r=0, z, t=0) = E(0, z) \cos \phi$$

In one gap examined, But will now vary ϕ in other gaps to keep this relation true. } Usually use near same value ϕ for all cavities for reference particle.

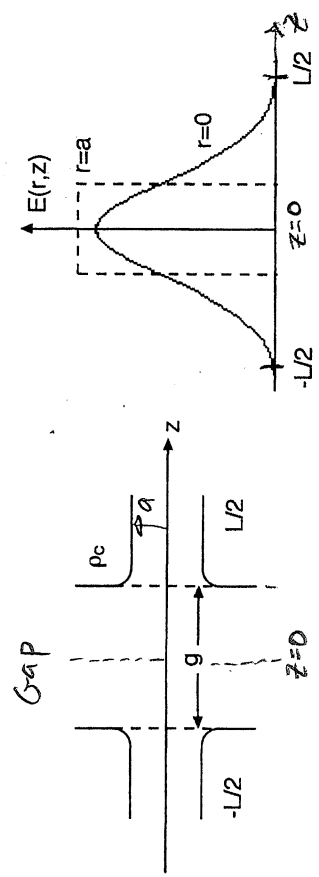


Figure 2.1 Gap geometry and field distribution.

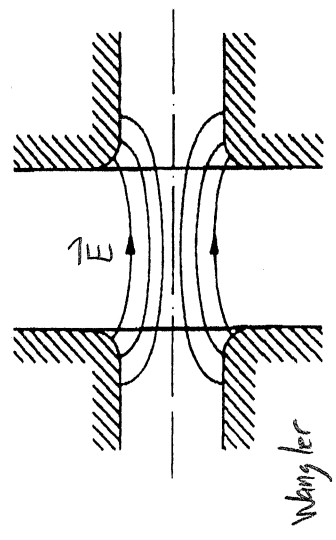


Figure 1.9 Electric-field lines in an accelerating gap.

Wangler

Wangler

Kinetic Energy Gain. See Wangler Chapter 2

$$W = (\gamma - 1) mc^2$$

= Particle Kinetic Energy

* Use $W =$ kinetic energy in longit. dynamics / dynamics to be consistent with usual notation.

Denote:

$$|\Delta W| = KE \text{ gain through gap.}$$

$$\text{Denote } E_z(r=z, z) = E(z, z)$$

Comment
Use capital W for KE to later distinguish from another variable w .

$$|\Delta W| = \int_{\text{gap}} \vec{E} \cdot d\vec{l} = \int_{-L/2}^{L/2} E_z(z, z) dz = \int_{-L/2}^{L/2} E(z, z) \cos[\omega t(z) + \phi] dz$$

$$= \int_{-L/2}^{L/2} E(z, z) \{ \cos(\omega t(z)) \cos \phi - \sin(\omega t(z)) \sin \phi \} dz$$

↳ some axial length large enough to contain πeH

$$\Delta W = q V_0 T \cos \phi$$

$$V_0 = \int_{-L/2}^{L/2} E(z, z) dz = \text{RF Voltage } [V_0] = eV$$

$$T = \frac{\int_{-L/2}^{L/2} E(z, z) \cos(\omega t(z)) dz - \tan \phi \frac{\int_{-L/2}^{L/2} E(z, z) \sin(\omega t(z)) dz}{\int_{-L/2}^{L/2} E(z, z) dz}}{\int_{-L/2}^{L/2} E(z, z) dz}$$

= Transit-Time Factor $[T] = 1$

Sometimes denote $V_0 = E_0 L$ to define avg field E_0 over gap field extent L
* Important: Specify L used here or ambiguous!

$$\Delta W = q E_0 L T \cos \phi = \text{Panofsky Equation}$$

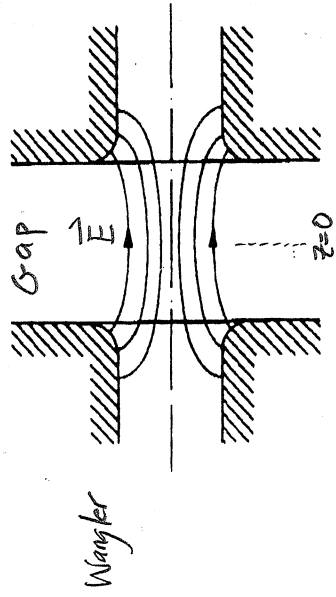
Panofsky eqn is deceptively simple appearing; contains much physics via T .

Transit Time Wangler, §2.2

Much physics contained within the transit time factor T .

* Time variation of field in gap always reduces energy gain relative to static case; for any RF phase ϕ .

- T provides normalized measure of reduction; $T=1 \Rightarrow$ static



$E(z)$ even function of z in typical gap.

$\Rightarrow \int_{-L/2}^{L/2} E(z) \sin(\omega t(z)) dz \approx 0$
 if change in v within gap negligible; $v = \text{const}$

Figure 1.9 Electric-field lines in an accelerating gap.

If $v \approx \text{const}$ in gap:

$$t(z) = \int_0^z \frac{dz}{v} = \frac{z}{v} \Rightarrow \omega t(z) = \omega z = \frac{2\pi z}{\beta c \lambda_f} = \frac{2\pi z}{\beta c \lambda_f}$$

$$= \frac{2\pi z}{\beta \lambda_f} \quad \lambda_f \equiv c \lambda = \text{RF Wavelength}$$

Using this

$$T \equiv \frac{\int_{-L/2}^{L/2} E(z) \cos(\omega t(z)) dz}{\int_{-L/2}^{L/2} E(z) dz} - \tan \phi \frac{\int_{-L/2}^{L/2} E(z) \sin(\omega t(z)) dz}{\int_{-L/2}^{L/2} E(z) dz}$$

$$T = \frac{\int_{-L/2}^{L/2} E(z) \cos\left(\frac{2\pi z}{\beta \lambda_f}\right) dz}{\int_{-L/2}^{L/2} E(z) dz} - \tan \phi \frac{\int_{-L/2}^{L/2} E(z) \sin\left(\frac{2\pi z}{\beta \lambda_f}\right) dz}{\int_{-L/2}^{L/2} E(z) dz}$$

1991

Transit time:

$$T = \frac{\int_{-1/2}^{1/2} E(0,z) \cos\left(\frac{z\pi z}{\beta \Delta t}\right) dz}{\int_{-1/2}^{1/2} E(0,z) dz} - \tan\phi \frac{\int_{-1/2}^{1/2} E(0,z) \sin\left(\frac{z\pi z}{\beta \Delta t}\right) dz}{\int_{-1/2}^{1/2} E(0,z) dz}$$

For "usual" cases of a symmetric field in the gap:

$$\int_{-1/2}^{1/2} E(0,z) \sin\left(\frac{z\pi z}{\beta \Delta t}\right) dz \approx 0$$

and the transit time reduces to

$$T = \frac{\int_{-1/2}^{1/2} E(0,z) \cos\left(\frac{z\pi z}{\beta \Delta t}\right) dz}{\int_{-1/2}^{1/2} E(0,z) dz}$$

* Most "usual" situation and many books define transit-time T using this formula.

- Go back to original definition in cases where tails

field to illustrate T

Take a simple approximation for the gap

Constant field in gap, zero outside.

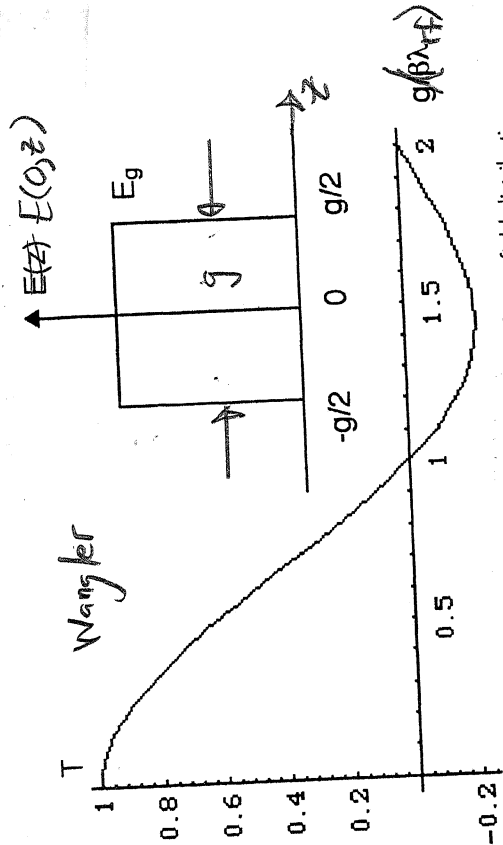


Figure 2.2 Transit-time factor for square-wave electric-field distribution.

Take! $E(0,z) = E_g = \text{const}$
 over $L = g$ and zero otherwise

Then!

$$T = \frac{\int_{-L/2}^{L/2} E(0,z) \cos\left(\frac{2\pi z}{\beta\lambda_{rf}}\right) dz}{E_g \int_{-g/2}^{g/2} \cos\left(\frac{2\pi z}{\beta\lambda_{rf}}\right) dz}$$

$$T = \frac{\sin\left(\frac{\pi g}{\beta\lambda_{rf}}\right)}{\left(\frac{\pi g}{\beta\lambda_{rf}}\right)} = \text{sinc}\left(\frac{\pi g}{\beta\lambda_{rf}}\right)$$

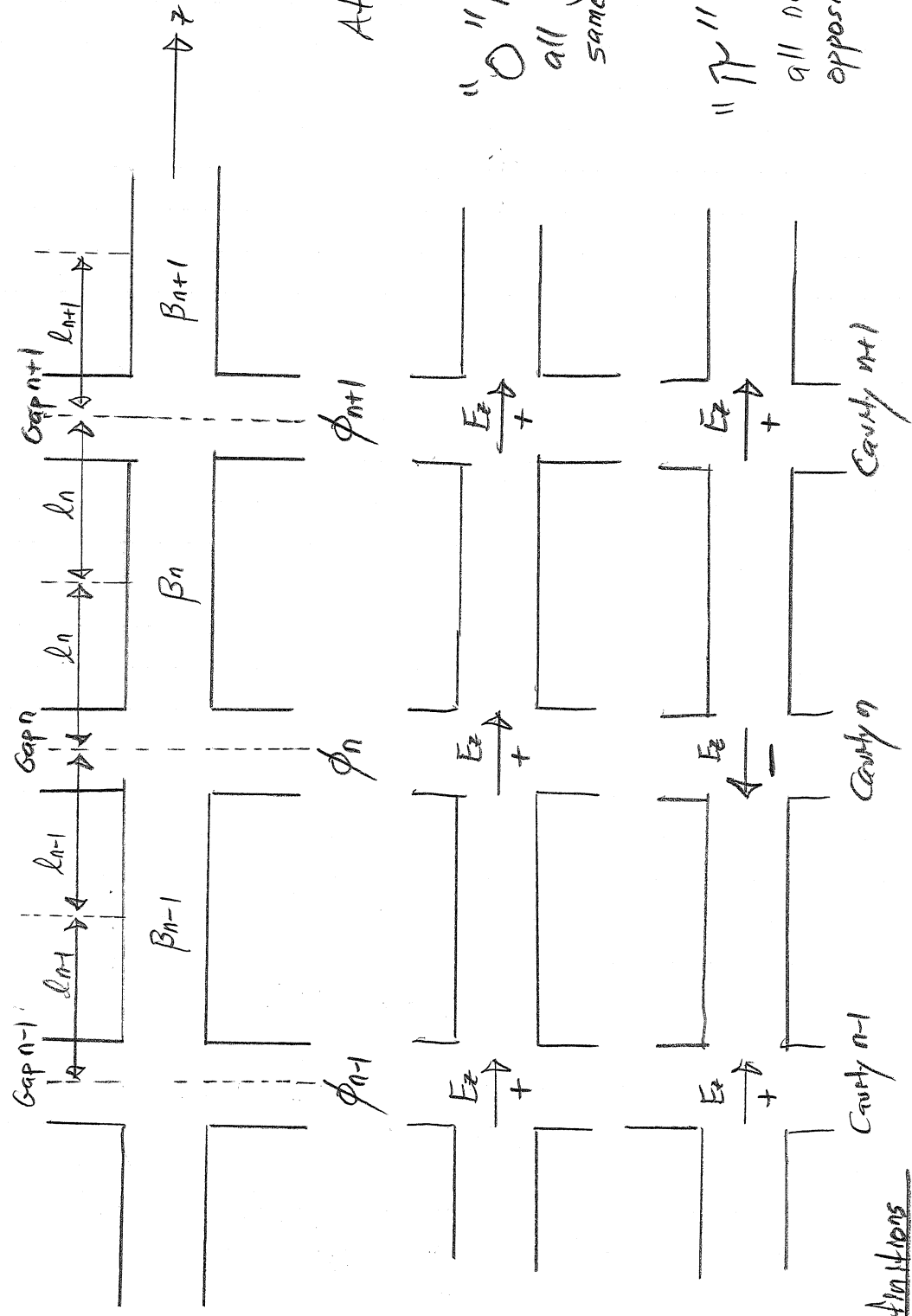
Numerous expressions for T can be found in literature for a variety of cavities under a range of approximations and idealizations. For examples, see Wangler. Some cavities have 2 or more gaps that may be lumped into T.

* $T \rightarrow 1$ when $g \ll \beta\lambda_{rf}$

- Want short gap relative to $\beta\lambda_{rf}$ for efficient use of RF cavity accelerating potential.
- For electrons or very energetic protons/ions $\beta \approx 1$ and want $g \ll \lambda_{rf}$. Approximation $v \approx \text{const}$ in gap very good for $\beta \approx 1$.

Difference Equations for longitudinal motion in a standing-wave linac
 Now have parts needed to analyze the longitudinal dynamics!

See Wangler's Chapter 6
 Lund and Barwood
 USPAS Notes



At time t_1

"O" Mode
 all gap E_z
 same direction

"Pi" Mode
 all neighboring gap E_z
 opposite direction.

$\beta_{SD}, W_{SD}, \phi_{SD}$
 = synchronous
 (design)
 particle values

Definitions

Z_n = distance from z-center
 nth gap to n+1'th gap

W_n = kinetic energy at
 end nth gap (const between gaps)

ϕ_n = RF phase at z-center
 each gap.

Particle Phase

Transit time between gaps: $\Delta t|_{n-1 \rightarrow n} = \frac{(ZL_{n-1})}{\beta_{n-1} \cdot c}$

$\omega \Delta t|_{n-1 \rightarrow n} =$ Advance RE particle transmits between gaps

For an arbitrary particle:

$$\phi_n = \phi_{n-1} + \frac{\omega(ZL_{n-1})}{\beta_{n-1} c} + \begin{cases} 0 & \text{O-Mode} \\ \pi & \text{\pi-Mode} \end{cases}$$

For the synchronous particle:

$$\phi_{sn} = \phi_{sn-1} + \frac{\omega(ZL_{n-1})}{\beta_{sn-1} c} + \begin{cases} 0 & \text{O-Mode} \\ \pi & \text{\pi-Mode} \end{cases}$$

for both cases so $\phi_{sn} = \phi_{sn-1} \text{ modulo } 2\pi$

Factor = 2π

$$\Rightarrow (ZL_{n-1}) \frac{\omega}{\beta_{sn-1} c} = \begin{cases} 2\pi & \text{O-Mode} \\ \pi & \text{\pi-Mode} \end{cases}$$

But $\frac{\omega}{c} = \frac{2\pi}{\lambda_{RF}} = \frac{2\pi}{\lambda_{rf}}$

$$\Rightarrow (ZL_{n-1}) = \lambda_{rf} \beta_{sn-1} \begin{cases} 1 & \text{O-Mode} \\ 1/2 & \text{\pi-Mode} \end{cases}$$

Use this to eliminate the inter-gap length (ZL_{n-1}):

$$\phi_n = \phi_{n-1} + \left(\frac{\omega \lambda_{rf}}{c} \right) \frac{\beta_{sn-1}}{\beta_{n-1}} \cdot \begin{cases} 1 & \text{O-Mode} \\ 1/2 & \text{\pi-Mode} \end{cases} + \begin{cases} 0 & \text{O-Mode} \\ \pi & \text{\pi-Mode} \end{cases}$$

$$\phi_n = \phi_{n-1} + 2\pi \frac{\beta_{S,n-1}}{\beta_{n-1}} \cdot \begin{cases} 1 & \text{O-Mode} \\ 1/2 & \text{\pi-Mode} \end{cases} + \begin{cases} 0 & \text{O-Mode} \\ \pi & \text{\pi-Mode} \end{cases}$$

For synchronous particle, $\phi_n \rightarrow \phi_{S,n}$; $\beta_n \rightarrow \beta_{S,n}$ etc.

$$\phi_{S,n} = \phi_{S,n-1} + 2\pi \frac{\beta_{S,n-1}}{\beta_{S,n-1}} \cdot \begin{cases} 1 & \text{O-Mode} \\ 1/2 & \text{\pi-Mode} \end{cases} + \begin{cases} 0 & \text{O-Mode} \\ \pi & \text{\pi-Mode} \end{cases}$$

Subtract to measure phase change relative to the synchronous particle. going from the n-1'th gap to the n'th gap as:

$$(\phi_n - \phi_{S,n}) - (\phi_{n-1} - \phi_{S,n-1}) = 2\pi \beta_{S,n-1} \left[\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{S,n-1}} \right] \cdot \begin{cases} 1 & \text{O-Mode} \\ 1/2 & \text{\pi-Mode} \end{cases}$$

$$N \equiv \begin{cases} 1 & \text{O-Mode} \\ 1/2 & \text{\pi-Mode} \end{cases}$$

Giving

$$(\phi_n - \phi_{S,n}) - (\phi_{n-1} - \phi_{S,n-1}) = 2\pi N \beta_{S,n-1} \left[\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{S,n-1}} \right]$$

Phase change rel. to sync particle n-1'th gap to n'th gap.

$$\frac{\Delta \beta_n}{\Delta W_n} = \frac{\beta_n - \beta_{S,n}}{W_n - W_{S,n}}$$

But denoting $\Delta(\text{Measure}) = (\text{Measure}) - (\text{Measure})_{\text{synchronous}}$

$$\beta_{S,n} \left[\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{S,n-1}} \right] = - \left[1 - \frac{\beta_{S,n-1}}{\beta_{n-1}} \right] = - \frac{\Delta \beta_{n-1}}{\beta_{S,n-1} + \Delta \beta_{n-1}} \approx - \frac{\Delta \beta_{n-1}}{\beta_{S,n-1}} \quad \text{A)}$$

$$\Delta W = (\gamma - 1) m c^2 \quad \Delta W = \Delta \gamma m c^2 = (1 - \beta^2)^{-3/2} \beta \Delta \beta m c^2 = \gamma^3 \beta m c^2 \Delta \beta$$

$$\Delta W = \gamma^3 \beta m c^2 \Delta \beta$$

$$\gamma = (1 - \beta^2)^{-1/2}$$

Label ΔW at the $n-1$ th gap constant using $\Delta W \approx \gamma^3 \beta^2 mc^2 \Delta \beta$ (B)

$$\Delta W_{n-1} = W_{n-1} - W_{s,n-1} \approx \gamma_{s,n-1}^3 \beta_{s,n-1}^2 mc^2 \Delta \beta_{n-1}$$

Using these, Equation * for the phase becomes:

$$(\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) = 2\pi N \left[\frac{1}{\beta_{s,n}} - \frac{1}{\beta_{s,n-1}} \right]$$

Use A) = $-\frac{\Delta \beta_{n-1}}{\beta_{s,n-1}}$

$$\Delta \phi_n - \Delta \phi_{n-1} = -2\pi N \frac{\Delta \beta_{n-1}}{\beta_{s,n-1}}$$

$$= -2\pi N \frac{\left(\frac{\Delta W_{n-1}}{mc^2} \right)}{\gamma_{s,n-1}^3 \beta_{s,n-1}^2}$$

-or-

$$\Delta \phi_n - \Delta \phi_{n-1} = \frac{-2\pi N}{\gamma_{s,n-1}^3 \beta_{s,n-1}^2} \left(\frac{\Delta W_{n-1}}{mc^2} \right) \quad (1)$$

$$\frac{\Delta \phi_n}{\Delta W_n} = \frac{\phi_n - \phi_{s,n}}{W_n - W_{s,n}}$$

Next apply Panofsky's equation $\Delta W = \gamma E_0 L T \cos \phi$ to the n th gap

$$W_n - W_{n-1} = \gamma E_0 n L_n T_n(\beta_n) \cos \phi_n \quad (C)$$

$$T_n = T_n(\beta_n)$$

func of β_n
* Taking picture of β const in gap here.

For the synchronous particle: $W_n \rightarrow W_{s,n}$ etc giving:

$$W_{s,n} - W_{s,n-1} = \gamma E_0 n L_n T_n(\beta_{s,n}) \cos \phi_{s,n} \quad (D)$$

Subtract C) and D) for an energy gain equation

$$(W_n - W_{s,n}) - (W_{n-1} - W_{s,n-1}) = 2 E_0 L_0 [T_n(\beta_n) \cos \phi_n - T_n(\beta_{s,n}) \cos \phi_{s,n}]$$

But, expect that $T_n(\beta_{s,n}) \approx T_n(\beta_n)$

* Little variation in T for small changes in β for usual applications.

This gives:

$$\Delta W_n - \Delta W_{n-1} = 2 E_0 L_0 T_n(\beta_n) [\cos \phi_n - \cos \phi_{s,n}] \quad (2)$$

Summary D) and 2) + energy gain equation for synchronous particle form a closed system describing the particle evolution in phase-energy phase space.

- * Nonlinearly coupled difference equations
- * Solve numerically for initial values of $\phi_n, \Delta W_n$
- * Advance synchronous particle also to calculate $\delta_{s,n}, T_n(\beta_{s,n})$.

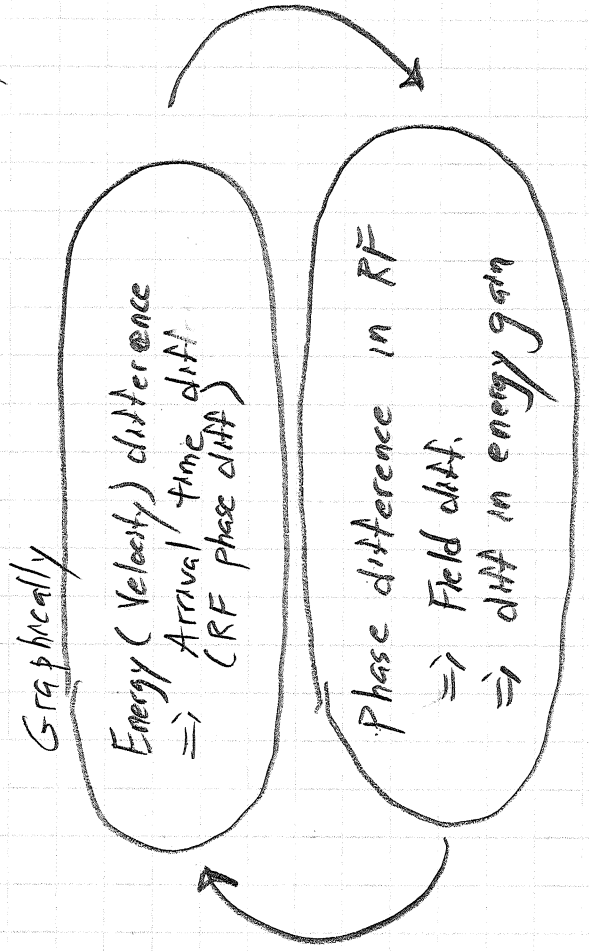
$$(\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) = -\frac{2\pi N}{\beta_{s,n}^3 \beta_{s,n-1}^2} \frac{\Delta \bar{W}_{n-1}}{mc^2} \sim \frac{\text{phase dev.}}{\text{energy dev.}}$$

$$\Delta \bar{W}_n - \Delta \bar{W}_{n-1} = g E_{0,n} L_n \ln(\beta_{s,n}) [\cos \phi_n - \cos \phi_{s,n}] \sim \text{energy dev.}$$

$$\bar{W}_{s,n} - \bar{W}_{s,n-1} = g E_{0,n} L_n T(\beta_{s,n}) \cos \phi_{s,n} \sim \text{sync. energy gain}$$

* Easy to solve (*) on computer to study phase stability about L^* synchronous particle in terms of evolution of ϕ and ΔW .

- Solve for specified initial values $\phi_0, \Delta W_0$
 * Also analyze later in "continuous" approx when cavity changes small.



Avg Accel Gradient
On the equation for ΔW_n :

$$E_{on} \cdot L_n = V_{on} = \int_{-L_n/2}^{L_n/2} E(z) dz$$

n^{th} gap

Need to connect L_n to L_{n-1}
Poisson's Eqn

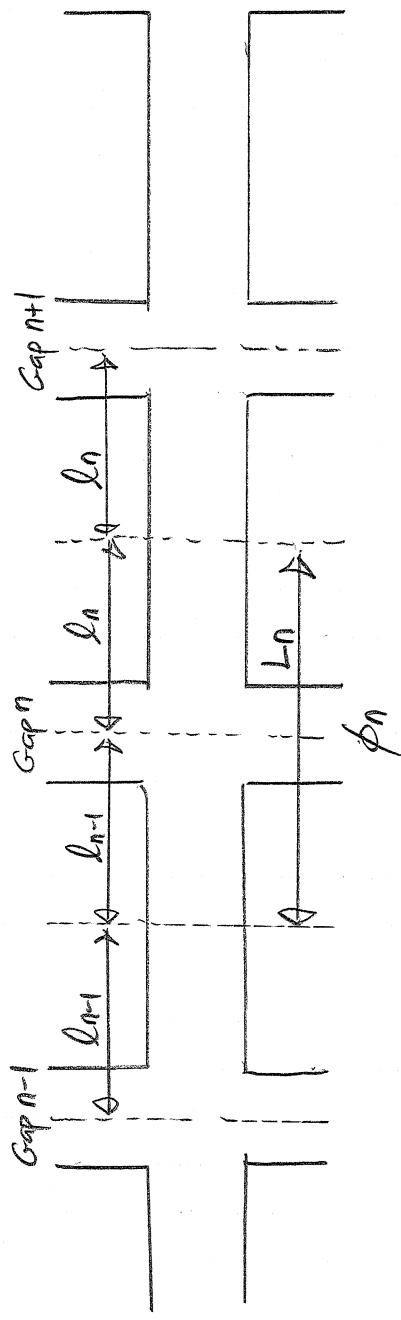
Derivation diff eqns

V_{on} = Accel potential n^{th} gap
 L_n = Length containing full gap fringe field.

E_{on} = Avg. E-Field over gap extent defined by $Z_{gap} \pm L_n/2$.

Sometimes, one defines E_{on} over a length L_n about the n^{th} gap mid-way between neighboring gaps upstream and downstream.

* Convenient to define avg gradient over "cell" in a periodic or quasi-periodic lattice.



But (per)

$$= \beta_{s,n-1} \lambda rf \begin{cases} 1 & \text{O-Mode} \\ 1/2 & \text{\pi-Mode} \end{cases} \Rightarrow L_{n-1} = \frac{N \beta_{s,n-1} \lambda rf}{2}$$

$$L_n = \frac{N \beta_{s,n} \lambda rf}{2}$$

$$\Rightarrow L_n = L_{n-1} + L_n = N (\beta_{s,n-1} + \beta_{s,n}) \frac{\lambda rf}{2}$$

This should safely contain the gap fringe extent and define E_{on} naturally as the average gradient in the cell.

Continuous Differential Equations to Model Longitudinal Dynamics

See: Wangler §6.3
Lund and Bernard USPAS notes.

Derived "kick" difference equations to model longitudinal dynamics about the synchronous particle:

$$\begin{aligned} (\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) &= -\frac{2\pi N}{\beta_{s,n-1}} \frac{\Delta W_{n-1}}{\beta_{s,n-1}^2 m c^2} & \Delta W_n &= W_n - W_{s,n} \\ \Delta W_n - \Delta W_{n-1} &= g E_0 n \ln T_n(\beta_{s,n}) [\cos \phi_n - \cos \phi_{s,n}] \end{aligned}$$

For small gap-to-gap changes replace discrete kicks by a continuous variation of field.

$$\begin{aligned} (\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) &\longrightarrow \frac{d(\phi - \phi_s)}{dn} & \phi_n &\longrightarrow \phi \\ \Delta W_n - \Delta W_{n-1} &\longrightarrow \frac{d\Delta W}{dn} & \phi_{s,n} &\longrightarrow \phi_s \\ & & \Delta W_n &\longrightarrow \Delta W \end{aligned}$$

* Treat n as continuous.

Convert from gap index n to axial coordinate s as an independent variable

$$\begin{aligned} n &= \frac{(S - S_n)}{N \beta_s \lambda r f} \\ &\equiv \frac{S}{N \beta_s \lambda r f} \end{aligned}$$

S_n = axial position of nth gap along reference trajectory
For notational simplicity

$$\Rightarrow \frac{d}{dn} = N \beta_s \lambda r f \frac{d}{ds}$$

* Using sync. particle to define coord. s.

Then, the difference eqns

$$(\phi_n - \phi_{s,n}) - (\phi_{n-1} - \phi_{s,n-1}) = -\frac{2\pi N}{\beta_{s,n-1}} \frac{\Delta W_{n-1}}{\beta_{s,n-1}^2 m c^2}$$

$$\Delta W_n - \Delta W_{n-1} = \int E_0 L_n \cdot T_n(\beta_{s,n}) [\cos \phi_n - \cos \phi_{s,n}]$$

Becomes

$$N \beta_{s,n} \frac{d}{ds} (\phi - \phi_s) = -\frac{2\pi N \Delta W}{\beta_{s,n}^2 m c^2}$$

$$N \beta_{s,n} \frac{d}{ds} \Delta W = \int E_0(s) T(\beta_{s,s}) L(s) [\cos \phi - \cos \phi_s]$$

1)

* Take $L_0 = L(s)$ with (usually) $L(s) = \text{const.}$

Also, the synchronous particle equation must also be integrated for the gain in energy for the ϕ_s, β_s factors etc.

$$-W_n - W_{n-1} = \int E_0 L_n \cdot T_n(\beta_{s,n}) \cos \phi_{s,n}$$

Becomes

$$N \beta_{s,n} \frac{d}{ds} W_s = \int E_0(s) T(\beta_{s,s}) L(s) \cos \phi_s \quad 2)$$

1) and 2) can be analyzed for the longitudinal dynamics of a particle evolving through many small cavity "kicks" smeared out into a continuously acting force.

* Should work well to understand and in many applications (especially rings).

For simplicity, denote

$$E_0(s) = E_0 = \text{const}$$

$$T(\beta_s, s) = T = \text{const}$$

$$L(s) \equiv L = \text{const}$$

Constants in a periodic lattice.

$$\Rightarrow (\gamma_s \beta_s)^3 \frac{d}{ds} (\phi - \phi_s) = -\frac{2\pi}{\lambda T} \left(\frac{\Delta W}{mc^2} \right)$$

$$\Delta W = W - W_s$$

$$\frac{d}{ds} \Delta W = \gamma E_0 T \left(\frac{L}{N \beta_s \lambda T} \right) (\cos \phi - \cos \phi_s)$$

$$L = N(\beta_{sn-1} + \beta_{sn}) \frac{\Delta T}{2} \Rightarrow \frac{L}{N \beta_s \lambda T} = 1$$

Giving

$$\boxed{(\gamma_s \beta_s)^3 \frac{d}{ds} (\phi - \phi_s) = -\frac{2\pi}{\lambda T} \left(\frac{\Delta W}{mc^2} \right)}$$

$$\boxed{\frac{d}{ds} \Delta W = \gamma E_0 T (\cos \phi - \cos \phi_s)}$$

These can be combined to eliminate W-Ws as:

$$\boxed{\frac{d}{ds} \left[(\gamma_s \beta_s)^3 \frac{d}{ds} (\phi - \phi_s) \right] = -\frac{2\pi}{\lambda T} \gamma \frac{E_0 T}{mc^2} (\cos \phi - \cos \phi_s)}$$

* 2nd order nonlinear equation for evolution of $\phi(s)$

provided we take L to be the cell spacing; in this context E_0 is the avg. gradient over the cell length.

Notes: * N has been eliminated. Same formulae for O-Mode and π -Mode * Nonlinear equations

Small Amplitude Phase Excursions

see Wigner §6.6, Lund and Barraud, OSIPAS notes

$$\frac{d}{ds} \left(\gamma_s \beta_s \right)^3 \frac{d \Delta \phi}{ds} = -\frac{2\pi}{\lambda r} \frac{2 E_0 I}{m c^2} \left[\cos(\phi_s + \Delta \phi) - \cos \phi_s \right]$$

Nonlinear Phase evolution Equation

$$\Delta \phi = \phi - \phi_s$$

Assume:

$\gamma_s \beta_s \sim$ varies slowly \Rightarrow pull through $\frac{d}{ds}$

$|\Delta \phi| \ll 1 \Rightarrow$ small phase excursions about synchronous particle.

Then:

$$\cos(\phi_s + \Delta \phi) = \cos \phi_s \cos \Delta \phi - \sin \phi_s \sin \Delta \phi + \mathcal{O}(\Delta \phi^2)$$

$$\approx \cos \phi_s - \sin \phi_s \Delta \phi + \mathcal{O}(\Delta \phi^2)$$

To obtain:

$$\frac{d^2 \Delta \phi}{ds^2} + k_s^2 \Delta \phi = 0$$

$$k_s \equiv \sqrt{\frac{2\pi}{\lambda r} \frac{2 E_0 I}{m c^2} \frac{\sin(-\phi_s)}{\beta_s \gamma_s^3}}$$

= Synchrotron Wavenumber

Linear equation for small phase excursions about synchronous particle.

This implies for:

$$-\pi < \phi_s < 0 \Rightarrow k_s^2 > 0$$

Small amplitude oscillations about synchronous particle stable

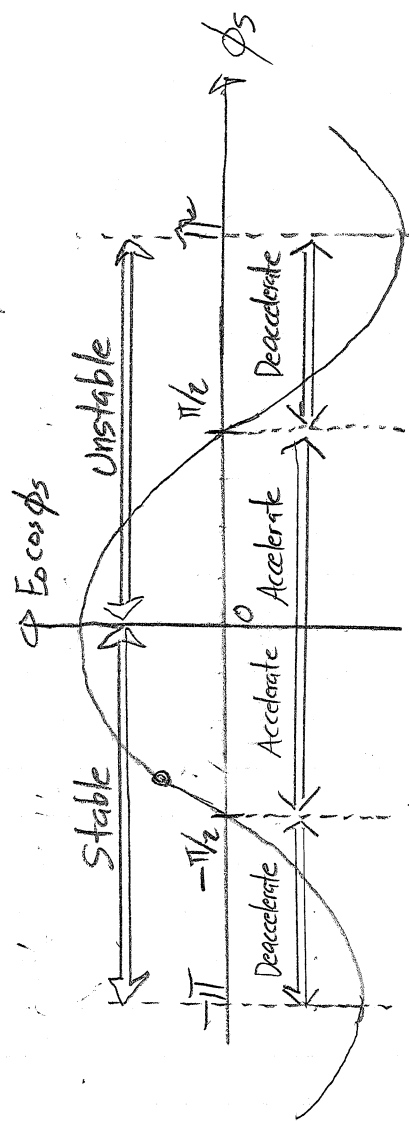
$$0 < \phi_s < \pi \Rightarrow k_s^2 < 0$$

Small amplitude oscillations about synchronous particle unstable

Recall the phase is defined relative to the RF wave peak

$$\frac{dW_s}{ds} \sim qE_0 \cos \phi_s$$

$$k_s = \sqrt{\frac{2\pi}{\lambda} \frac{qE_0 T_s m(-\phi_s)}{MC^2 (k_s \beta_s)^3}}$$



Stable range \Rightarrow particle arrives at gap in rising field } consistent with
 Unstable range \Rightarrow particle arrives at gap in falling field. } qualitative expectation
 Particle accelerates and is stable for $-\frac{\pi}{2} < \phi_s < 0$

* A commonly taken value of ϕ_s to accelerate with a reasonable phase width for stability (k_s large) is to take:

$$\phi_s \approx -\frac{\pi}{6} = -30^\circ \Rightarrow \text{Compromise: Accel strength + phase width focusing}$$

* If RF is used for beam bunching rather than acceleration, the strength of k_s is maximized by taking

$$\phi_s = -\frac{\pi}{2} \Rightarrow \text{Bunching: Max focus strength. But no acceleration.}$$

* If E₀T and ϕ_s remain nearly constant in acceleration:

$$k_s \sim \frac{1}{(k_s \beta_s)^{3/2}}$$

Showing that synchrotron oscillations will slow down (weaker focusing) as the beam accelerates.

- Good intuitive sense: energetic particle more "rigid"

The corresponding angular frequency to k_s is

$$\omega_s \equiv k_s(\beta c) \quad \text{Synchrotron angular Freq.}$$

$$k_s = \sqrt{\frac{2\pi}{\lambda r} \frac{q E_0 T \sin(\phi_s)}{mc^2 (\gamma \beta)^3}}$$

Relative to the RF freq: $\beta \approx \beta_s$

$$\frac{\omega_s}{\omega} = \frac{f_s}{f_{rf}} = \sqrt{\frac{2\pi}{\lambda r} \frac{\beta_s^2}{\omega^2} \frac{q E_0 T \sin(\phi_s)}{mc^2 (\gamma \beta_s)^3}}$$

But $\frac{c}{\omega} = \frac{c T r}{2\pi} = \frac{\lambda r}{2\pi}$

$$\Rightarrow \frac{\omega_s}{\omega} = \frac{f_s}{f_{rf}} = \sqrt{\frac{1}{2\pi (\gamma \beta_s)^3} \left(\frac{q E_0 T \lambda r}{mc^2} \right) \sin(\phi_s)}$$

From this expect:

* $f_s \ll f_{rf}$ as beam becomes more relativistic.

The linear synchrotron equation of motion can be solved for $k_s = \text{const.}$

$$d^2 \Delta\phi / ds^2 + k_s^2 \Delta\phi = 0$$

Solution:

$$\Delta\phi(s) = \Delta\phi_0 \cos[k_s(s-s_1)] + \frac{\Delta\phi_0'}{k_s} \sin[k_s(s-s_1)]$$

Initial condition

$$\Delta\phi(s=s_1) = \Delta\phi_0'$$

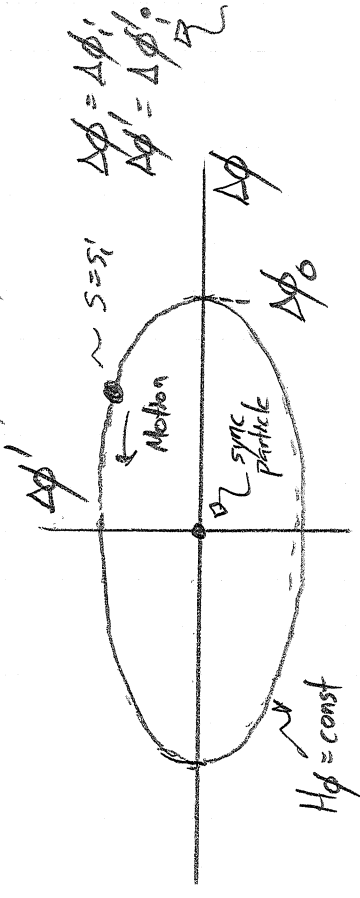
$$d\Delta\phi / ds (s=s_1) = \Delta\phi_0'$$

Conservation of Hamiltonian H:

$$H_\phi = \frac{1}{2} (\Delta\phi)' ^2 + \frac{1}{2} k_s^2 (\Delta\phi)^2 = \frac{1}{2} (\Delta\phi_0')^2 + \frac{1}{2} k_s^2 (\Delta\phi_0)^2 = \text{const.}$$

$$1 = \frac{d}{ds}$$

Phase-space in $\Delta\phi - \Delta\phi'$ is an ellipse



When $k_s (s-s_1) = 2\pi$ particle cycles around ellipse.

Denote for convenience:

$$\Delta\phi_0 = \text{Max Phase excursion} \Rightarrow H_\phi = \frac{1}{2} k_s^2 \Delta\phi_0^2$$

Then the Hamiltonian conservation is expressed as:

$$H_\phi = \frac{1}{2} (\Delta\phi')^2 + \frac{1}{2} k_s^2 (\Delta\phi)^2 = \frac{1}{2} k_s^2 (\Delta\phi_0)^2 = \text{const}$$

Using this and:

$$(k_s \beta_s)^3 \Delta\phi' = -\frac{2\pi}{\lambda v} \frac{\Delta W}{mc^2} \equiv -\frac{2\pi}{\lambda v} \frac{W}{mc^2}$$

→ $\frac{1}{2} (\Delta\phi')^2 = \frac{1}{2} \frac{2\pi}{\lambda v} W^2$

The ellipse becomes

$$\left(\frac{W}{W_0}\right)^2 + \left(\frac{\Delta\phi'}{\Delta\phi_0}\right)^2 = 1$$

$$W_0 = \frac{2\pi}{\lambda v} \frac{1}{2} k_s \beta_s^3 \Delta\phi_0^2$$

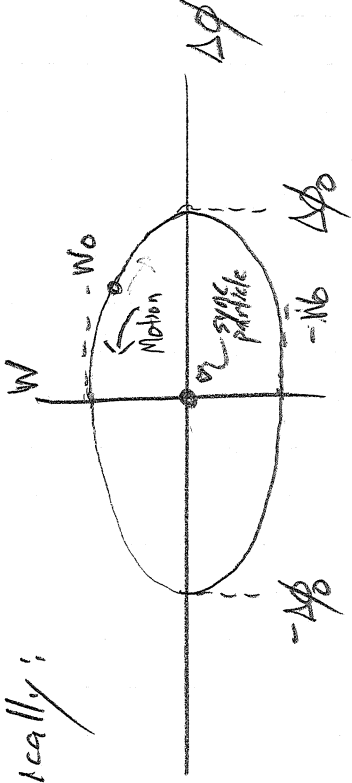
$$= \frac{1}{2} \text{width norm energy deviation}$$

$$= \frac{1}{2} \frac{2\pi}{\lambda v} \frac{1}{2} k_s \beta_s^3 \Delta\phi_0^2$$

$$= \frac{1}{2} \frac{2\pi}{\lambda v} \frac{1}{2} k_s \beta_s^3 \left(\frac{2 E_0 T \Delta t}{mc^2} \right) \sin(\phi_s - \phi_0) \Delta\phi_0^2$$

Notation Caution:

small W \swarrow \nwarrow capital W
 $W \equiv \frac{\Delta W}{mc^2}$



Graphically:

The phase-space area of the ellipse is:

$$\begin{aligned} \text{Area} &= \pi \left(\frac{1}{2} \text{width} \right) \left(\frac{1}{2} \text{height} \right) \\ &= \pi \Delta\phi_0 W_0 \\ &= \sqrt{\frac{\pi}{2}} (\delta\beta)^2 \left(\frac{2E_0 \beta^2}{mc^2} \right) \sin(\phi_s), \Delta\phi_s^2 \end{aligned}$$

Many choices of longitudinal coordinates are employed to study longitudinal dynamics. Some include:

(coord, momentum)

Phase-energy: (ϕ, \bar{W}) , $\bar{W} = (\gamma-1)mc^2 = \text{Kinetic Energy}$
 or $(\Delta\phi, \Delta\bar{W})$ etc.

Position - momentum: (z, p_z)

or $(\Delta z, \Delta p_z)$

Time - energy (t, \bar{W})

or $(\Delta t, -\Delta\bar{W})$

- o
- o
- o
- o

Proper sets of canonical variables (perhaps rescaled by constants like mc^2) should be employed to measure phase-space areas. Canonical transforms can be applied to connect to other variable choices.

Nonlinear Phase-Space Structure of RF Bucket.

See Wangler, § 6.4
Lind and Barnard, USPAS notes

Cannot use small phase excursion approximation to analyse.
Return to nonlinear coupled equations:

$$\begin{aligned} (\alpha_s \beta_s)^3 \frac{d \Delta \phi}{ds} &= -\frac{2\pi}{\lambda \Delta} \frac{\Delta W}{mc^2} & \Delta W &= W - W_s \\ \frac{d \Delta W}{ds} &= q E_0 T [\cos \phi - \cos \phi_s] & \Delta \phi &= \phi - \phi_s \end{aligned}$$

Denote:

$$W \equiv \frac{\Delta W}{mc^2} \quad A \equiv \frac{2\pi}{\lambda \Delta (\alpha_s \beta_s)^3} \quad B \equiv \frac{q E_0 T}{mc^2}$$

$$\phi = \phi_s + \Delta \phi \Rightarrow \Delta \phi' = \phi' \quad \text{since we take } \phi_s = \text{const (simplifying)} \Rightarrow \frac{d}{ds}$$

Then the nonlinear equations can be expressed as:

$$\begin{aligned} \phi' &= -A W \\ W' &= B [\cos \phi - \cos \phi_s] \end{aligned}$$

Assume that A and B vary weakly in s
* Likely need for continuous approx anyway

$$\Rightarrow \phi'' = -A W' = -AB [\cos \phi - \cos \phi_s]$$

$$\phi'' = -AB (\cos \phi - \cos \phi_s)$$

Multiply by ϕ' and integrate:

$$\phi' \phi'' = -AB(\cos\phi - \cos\phi_s) \phi'$$

$$\int \phi' \phi'' ds = -AB \int (\cos\phi - \cos\phi_s) \phi' ds \Rightarrow$$

$$\frac{1}{2} \int \frac{d}{ds} \phi'^2 ds = -AB \int (\cos\phi - \cos\phi_s) d\phi$$

Now use $\phi' = -AW$ and divide by A :

$$\frac{AW^2}{2} + B(\sin\phi - \phi \cos\phi_s) = \text{const} \equiv H\phi$$

$H\phi =$ Synchrotron Hamiltonian.

$$\frac{\phi'^2}{2} + AB(\sin\phi - \phi \cos\phi_s) = \text{const.}$$

Analogy: $\frac{AW^2}{2} \Rightarrow$ Interpret as "Kinetic Energy"
 $B(\sin\phi - \phi \cos\phi_s) \Rightarrow$ Interpret as "Potential Energy"

To exploit this analogy, denote:

$$V(\phi) \equiv B(\sin\phi - \phi \cos\phi_s) \Rightarrow H\phi = \frac{AW^2}{2} + V(\phi) = \text{const}$$

$$\frac{\partial V(\phi)}{\partial \phi} = B(\cos\phi - \cos\phi_s) \sim \text{Focus Strength}$$

$$\frac{\partial^2 V(\phi)}{\partial \phi^2} = -B \sin\phi \sim \text{Concavity}$$

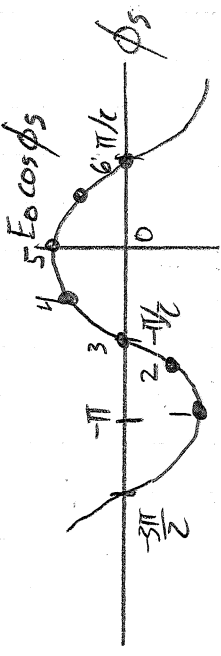
Want for stability about synchronous particle:

$$\frac{\partial^2 V(\phi)}{\partial \phi^2} \Big|_{\phi=\phi_s} > 0 \Rightarrow -B \sin\phi_s > 0 \Rightarrow$$

$$\phi_s < \pi \text{ for } T > 0$$

Same result obtained in small phase excursion limit as is should be expected.

$$\phi_s < \pi \text{ for stability}$$



Plots of

$$V(\phi) = B(\sin \phi - \phi \cos \phi_s)$$

$$B = \frac{qE_0 T}{mc^2} \geq 0$$

$\phi_s =$ various values

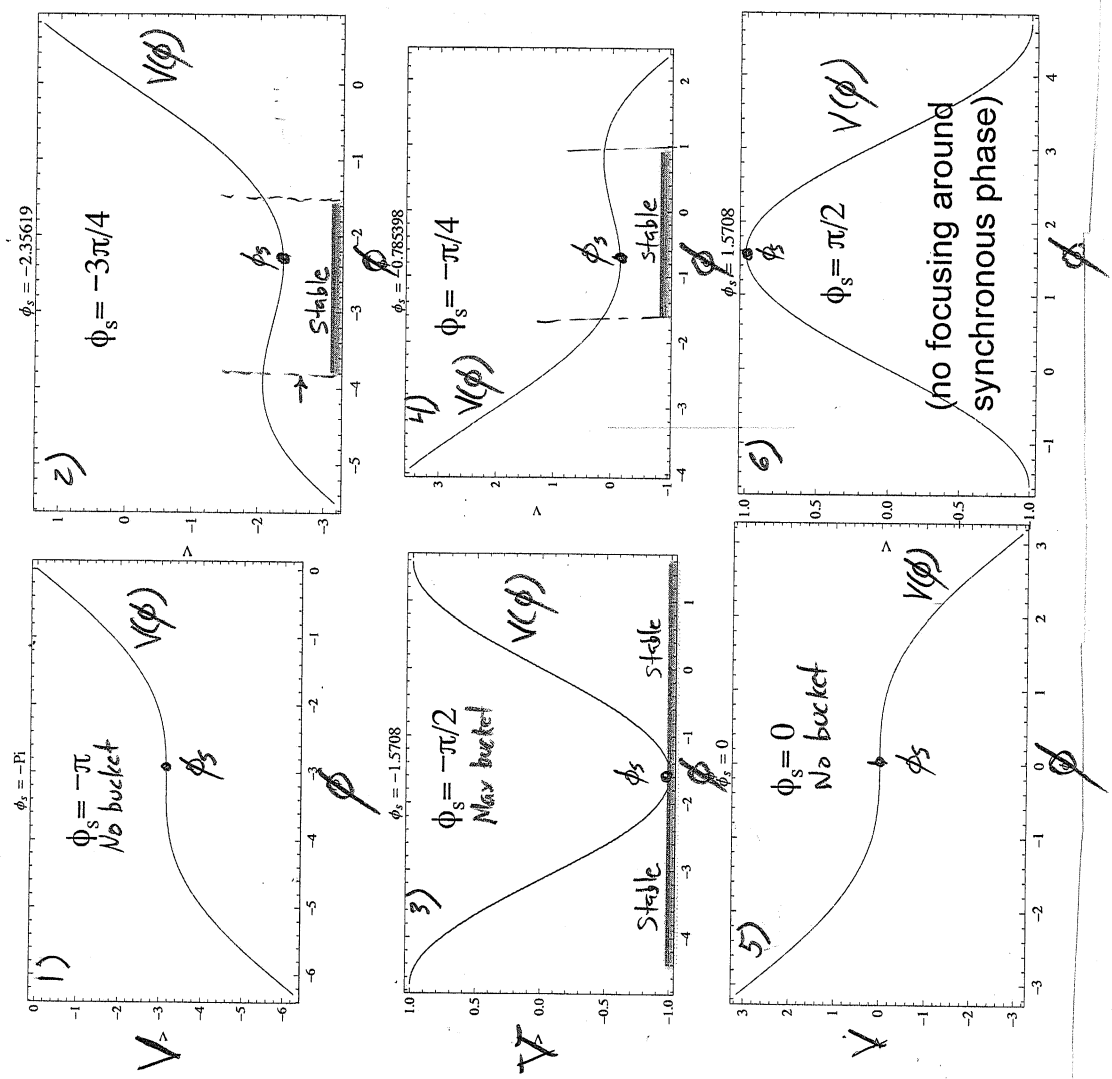
$-\pi < \phi_s < 0$; Stable

$-\pi/2 < \phi_s < \pi/2$; Accel

$\pi/2 < \phi_s < 0$ Accel and Focus.

$$\phi_s \approx -30^\circ = -\pi/6$$

typical value



$$H_{\phi} = \frac{A\omega^2}{2} + V(\phi)$$

$$V(\phi) = B(\sin\phi - \phi \cos\phi_s)$$

$$A = \frac{2\pi}{NA} \frac{1}{(\Delta\omega)^3} > 0$$

$$B = \frac{2E_0T}{c^2 m c^2} > 0 \quad (\text{forward accel})$$

$T > 0$

$$H_{\phi}(w=0, \phi = \phi_s) = \text{Stable Fixed Point} \quad \phi_s < 0$$

$$H_{\phi}(w=0, \phi = -\phi_s) = \text{Unstable Fixed Point}$$

Denote

$$H_{\phi}(w=0, \phi = \phi_s) = H_{\phi}(w=0, \phi = -\phi_s) \equiv H_{\phi}(-\phi_s) \\ = B[-\sin\phi_s + \phi_s \cos\phi_s]$$

Separatrix defining RF "Fish" satisfy:

$$H_{\phi} = H_{\phi}(-\phi_s)$$

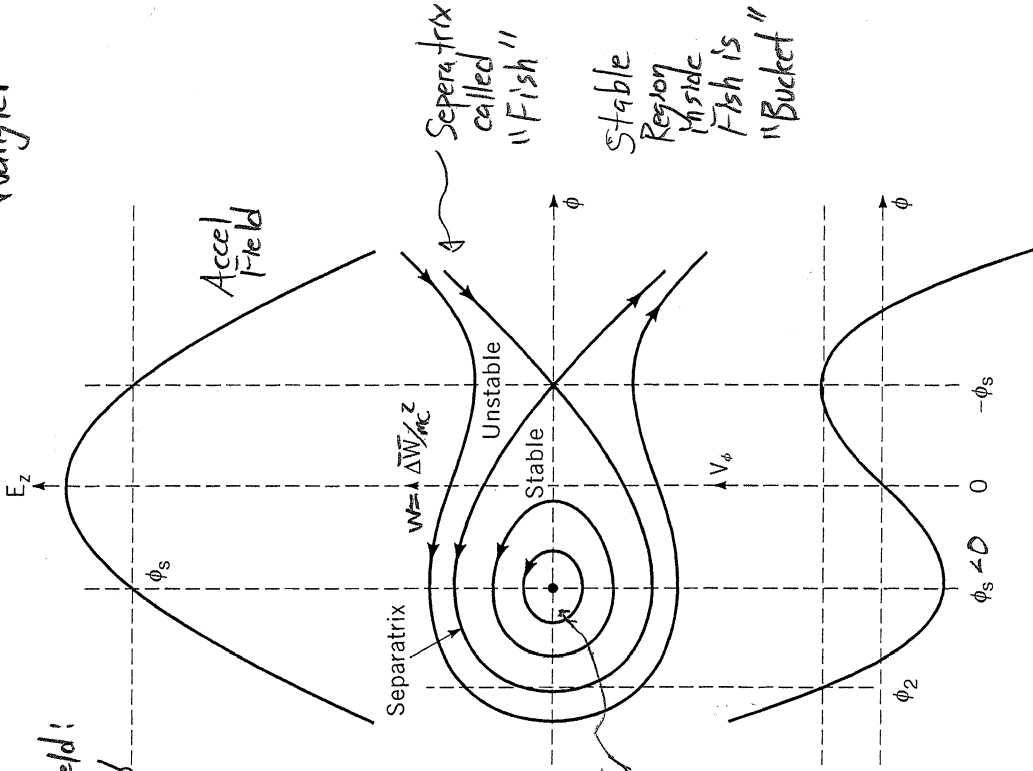
$$\Rightarrow \frac{A\omega^2}{2} + B[\sin\phi - \phi \cos\phi_s] = B[-\sin\phi_s + \phi_s \cos\phi_s]$$

Separatrix Constant

Gives stable Bucket

Wangler

Accel Field:
 $E_z = E_0 \cos\phi$



Phase Space

For small excursions about sync particle: elliptical P.S. \Rightarrow linear motion

Potential

$V(\phi)$

Wangler

Figure 6.3. At the top, the accelerating field is shown as a cosine function of the phase; the synchronous phase ϕ_s is shown as a negative number, which lies earlier than the crest where the field is rising in time. The middle plot shows some longitudinal phase-space trajectories, including the separatrix, the limiting stable trajectory, which passes through the unstable fixed point at $\Delta W = 0$, and $\phi = -\phi_s$. The stable fixed point lies at $\Delta W = 0$ and $\phi = \phi_s$, where the longitudinal potential well has its minimum, as shown in the bottom plot.

The total phase width of the separatrix about the synchronous particle is:

$$\Psi = \text{Phase width} = |\phi_s| + |\phi_2| = -\phi_s - \phi_2$$

$\phi = -\phi_s$ Right X point
 $\phi_2 < 0$ Left Turning point
 $\phi_2 < 0$ also

From the separatrix eqn:

$$H_\phi(\phi = \phi_2, w=0) = H_\phi(-\phi_s) \quad B[\sin \phi_2 - \phi_2 \cos \phi_s] = -B[-\sin \phi_s - \phi_s \cos \phi_s]$$

$$\Rightarrow \sin \phi_2 - \phi_2 \cos \phi_s = -[\sin \phi_s - \phi_s \cos \phi_s] \quad *$$

* Can be solved numerically for ϕ_2 to calculate the phase width Ψ for a given value of ϕ_s .

Approximately:

$$\phi_2 = -\phi_s - \psi$$

$$\sin \phi_2 = -\sin(\phi_s + \psi) = -(\sin \phi_s \cos \psi + \sin \psi \cos \phi_s)$$

} substitute in separatrix eqn *

$$\sin \phi_s \cos \psi + \sin \psi \cos \phi_s - \phi_s \cos \phi_s - \psi \cos \phi_s = \sin \phi_s - \phi_s \cos \phi_s$$

$$\tan \phi_s = \frac{\sin \psi - \psi}{1 - \cos \psi} \Rightarrow \frac{(\sin \psi - \psi) \cos \phi_s}{\psi - \psi^3/6 + \dots - \psi} = \frac{(-1 - \cos \psi) \sin \phi_s}{\psi - \psi^3/6 - \psi^2} \approx \frac{\psi - \psi^3/6 - \psi^2}{\psi^2}$$

$$\psi \approx -3 \tan \phi_s$$

Numerical checks show works well up to $|\phi_s| \approx \pi$. ; even though approx is "poor".

For case of $\phi_s = -\pi/2$ (Max Focus Case)

separatrix eqn * $\sin\phi_2 - \phi_2 \cos\phi_s = -[\sin\phi_s - \phi_s \cos\phi_s]$

Gives: $\sin\phi_2 - \phi_2 \cos(\pi/2) = -[\sin(\pi/2) + (\pi/2) \cos(\pi/2)]$

$\sin\phi_2 = 1 \Rightarrow \phi_2 = -3\pi/2 = -270^\circ$

$\Rightarrow \psi = -\phi_s - \phi_2 = 2\pi = 360^\circ$ Focuses for full RF phase width ϕ

This choice will give no acceleration

but will be most efficient for beam bunching. Note also that the synchrotron wavenumber $k_s = \sqrt{\frac{2\pi}{\lambda_{rf}} \frac{2E_0 T \sin(-\phi_s)}{c(\beta_s \beta_3)^3}}$ is largest for $\phi_s = -\pi/2$.

To estimate the vertical $1/2$ -width in w of the separatrix for arb ϕ_s :

$w = w_{max}, \phi = -\phi_s$ in separatrix eqn:

$\Rightarrow H\phi = H\phi(-\phi_s)$

$\frac{A w_{max}^2}{2} + B [\sin\phi_s - \phi_s \cos\phi_s] = -B [\sin\phi_s - \phi_s \cos\phi_s]$

$w_{max} = \sqrt{4 \frac{B}{A} [\phi_s \cos\phi_s - \sin\phi_s]}$

$w_{max} = \frac{\Delta W_{max}}{MC} = \sqrt{\frac{2g E_0 T (\beta_s \beta_3)^3}{c \pi MC^2}} \lambda_{rf} (\phi_s \cos\phi_s - \sin\phi_s)$

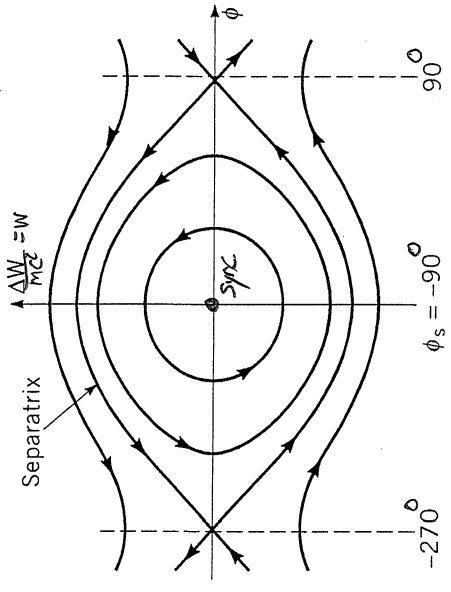


Figure 6.4. Separatrix for $\phi_s = -90^\circ$ (no acceleration).

$A = \frac{2\pi}{\lambda_{rf}} \frac{1}{(\beta_s \beta_3)^3}$

$B = \frac{2E_0 T}{c MC^2}$

$w_{min} = -w_{max}$

$w\text{-width} = 2w_{max}$

Approximating crudely the $\phi - W$ phase-space area of the bucket is:

$$\text{Area-Bucket} = \int_{\text{Bucket}} d\phi dW \approx \pi (W_{\text{max}}) (\psi/2)$$

$$\psi \approx -3 \text{ bands}$$

$$W_{\text{max}} = \sqrt{\frac{2q E_0 T (\gamma_s \beta_s)^3}{4\pi m c^2}} \Delta t (\phi_s \cos \phi_s - \sin \phi_s)$$

Approx as an ellipse with area $\pi \times (x\text{-radius}) \times (y\text{-radius})$

$$\text{Area Bucket} \approx \frac{3\pi}{2} \tan(\phi_s) \sqrt{\frac{2q E_0 T (\gamma_s \beta_s)^3}{4\pi m c^2}} \Delta t (\sin(\phi_s) - \phi_s \cos \phi_s)$$

This provides an estimate of the phase-space area that can be accelerated.

Comments:

Relativistic
(electrons or very energetic protons/ions)

$$\beta_s \approx 1 \Rightarrow$$

Field errors ($E_0 T \approx E_0; T \approx 1$) do not change synchronous condition, but shift final energy.

Non-Relativistic
(low energy e^- , protons or ions)

$$\beta_s \approx 1/2 \Rightarrow$$

Field errors ($E_0 T$) cause shift to a new synchronous phase.

Adiabatic Phase Damping Ref: Wangler, "RF Linear Accelerators" Secs. 5.12, 6.7. 41/

If parameters (focus well) of an oscillator are changed slowly relative to the period of the oscillation, then expect an adiabatic invariant:

"Action" = $\oint p dq$ = const.

↓
cycle

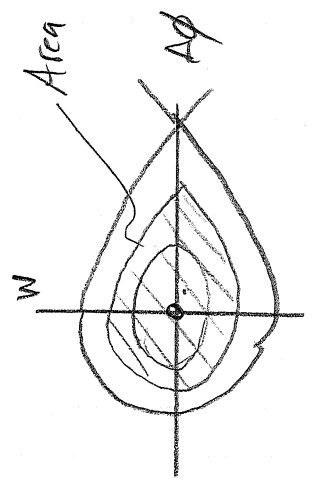
See Landau & Lifshitz
"Mechanics", 3rd Edition,
PS 154.

* True for any number of parameters varying simultaneously

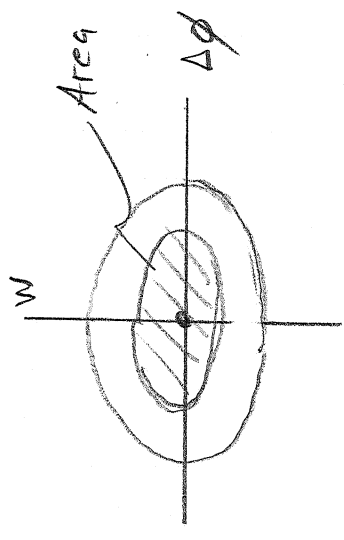
* For synchrotron motion synchrotron wavenumber sets the scale to measure slowness for validity.

$$k_s = \sqrt{\frac{2\pi}{\lambda_{rf}} \frac{2 E_0 T \sin(-\phi_s)}{(R_s \beta_s)^3 m c^2}}$$

Nonlinear RF



Linear RF



This result tells us that the longitudinal phase-space area (or emittance) will be conserved as the focusing parameters (say due to acceleration) vary slowly on the synchrotron oscillation period.

Reminder: $k_s = \sqrt{\frac{2\pi}{\lambda_{rf}} \frac{2 E_0 T \sin(-\phi_s)}{m c^2 (R_s \beta_s)^3}}$ = Synchrotron Wavenumber

For the case of linear motion with small phase excursions about the synchronous particle:

"Action" = $\pi \Delta\phi_0 W_0 = \text{const}$

$$= \pi (\Delta\phi_0)^2 \sqrt{\frac{(\gamma_s \beta_s)^3}{2\pi} \left(\frac{2 E_0 T \Delta f}{m c^2} \right) \sin(-\phi_s)}$$

$\Delta\phi_0 =$ phase $\frac{1}{2}$ -width linear orbit.

$W_0 = \frac{\Delta W}{m c^2} =$ corresponding normalized energy deviation of linear orbit.

$\Delta\phi_0 = \frac{\text{const}}{\left[\frac{(\gamma_s \beta_s)^3}{2\pi} \left(\frac{2 E_0 T \Delta f}{m c^2} \right) \sin(-\phi_s) \right]^{1/4}}$ (rescaled const)

If we take $\phi_s \approx \text{const}$
 $E_0 T \Delta f \approx \text{const}$

$$\Delta\phi_0 = \frac{\text{const}}{(\gamma_s \beta_s)^{3/4}}$$

and then

$$W_0 = \text{const} (\gamma_s \beta_s)^{3/4}$$

* called "phase damping" with adiabatic accel.

* Energy deviation grows with adiabatic accel. for const phase-space area.

$$= \left(\frac{\gamma_s \beta_s}{2\pi} \right)^3 \left(\frac{2 E_0 T \Delta f}{m c^2} \right) \sin(\phi_s) \times \Delta\phi_0$$

see pg 33 33

$$W = \frac{\Delta W}{m c^2}$$

Recall

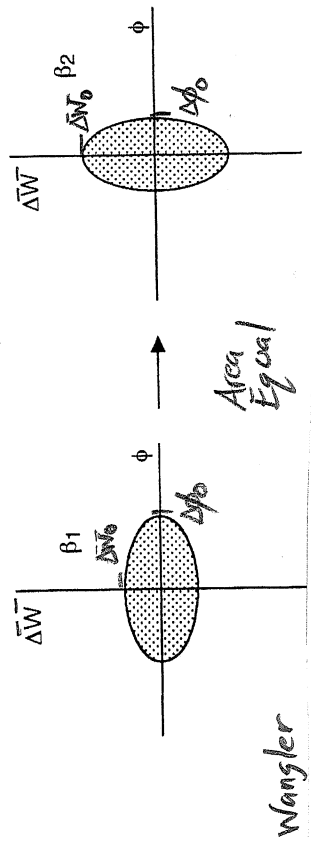
$$\Rightarrow \Delta W_0 = \text{const} (\gamma_s \beta_s)^{3/4}$$

or equivalently: $\Delta\phi_0|_i = \text{initial value of } \Delta\phi_0$

$$\frac{\Delta\phi_0}{\Delta\phi_0|_i} = \left(\frac{(\chi_s \beta_s)|_i}{(\chi_s \beta_s)|_f} \right)^{3/4}$$

$$\frac{\Delta W_0|_f}{\Delta W_0|_i} = \left(\frac{(\chi_s \beta_s)|_i}{(\chi_s \beta_s)|_f} \right)^{3/4}$$

Graphically:



For RF high energy synchronons, $\chi_s \beta_s$ will vary slowly over many laps and the adiabatic approximation can be well satisfied. For RF linacs, $\chi_s \beta_s$ may change too rapidly for validity of the adiabatic approximation.

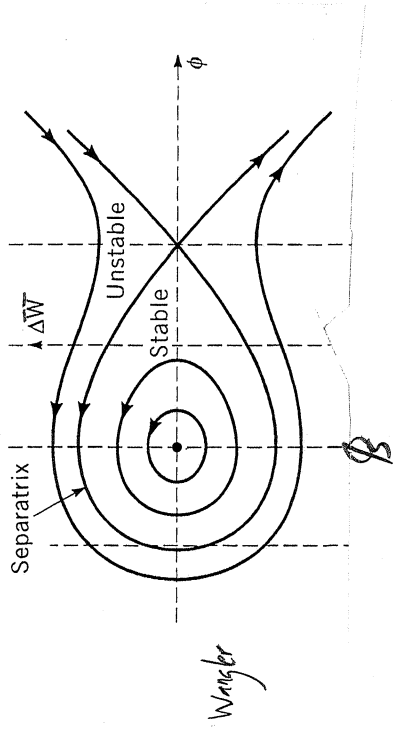
For FRIB linac segment #1: $\beta_s \cdot \text{Length} \sim (2\pi)(\sim 10) \Rightarrow 10$ oscillations

Ref: Q. Zhao

When $\chi B_s \neq \text{const}$, $H_\parallel \neq \text{const}$ and the RF "fish" structure becomes distorted to a more characteristic "golf-club" shape.

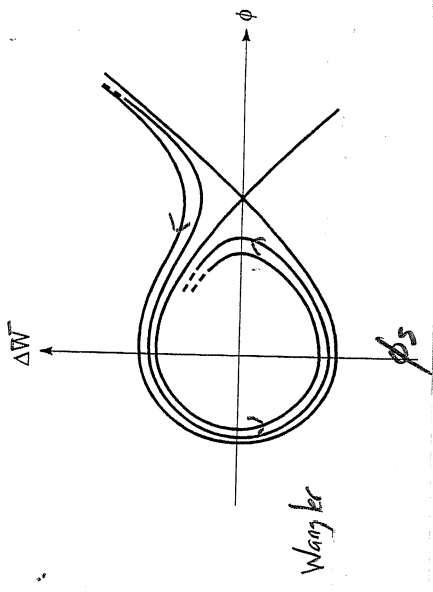
- * Density in phase-space of non-interacting particles governed by Hamiltonian is invariant even if Hamiltonian H is non-constant by Liouville's Theorem.
- \Rightarrow Phase volume enclosed by surface of fixed density is constant.
- \Rightarrow Shape can distort due to acceleration.

$\chi B_s = \text{const}$



* Use $H_\parallel = \text{const}$ to analyze

$\chi B_s \neq \text{const}$ accelerating



* Use difference equations to analyze general case.

* Untrapped for $H_\parallel = \text{const}$ can move within for bounded orbit.

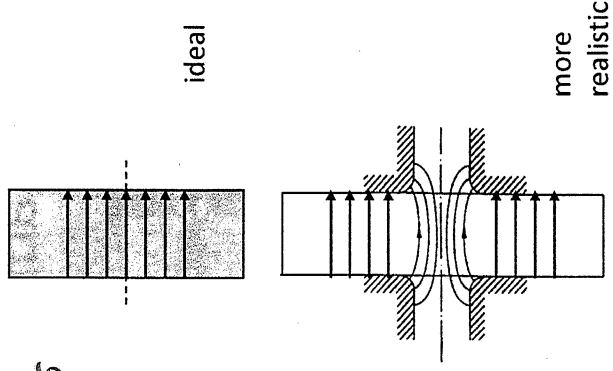
Transverse RF Defocusing

Qualitative:

RF Defocusing

Syphers
OSPAS

- When a particle enters a cavity off center, the field lines will have an inward component; and they will have an outward component upon exit from the cavity.
- However, the strength of the field is changing — typically, increasing — during transit.
- Thus, the outward “kick” due to the field will be greater than the inward kick — defocusing effect
- This “RF defocusing” is more important at lower energies



see T. Wangler, RF Linear Accelerators

$$\frac{1}{f} = \frac{\Delta x'}{x} \approx \pi \frac{eV_{\text{eff}}}{mc^2} T \cos \phi_s \frac{\lambda}{\lambda(\beta\gamma)^2}$$

⇒ Ideal pillbox cavity has no radial E-field. E_r to lead to transverse focusing / defocusing.

⇒ When aperture added to cavity to allow beam to enter / exit this produces an E_r and transverse focusing / defocusing now possible.

Defocusing kick generally larger due to exit field gaining strength due to variation during transit. Part offset due to velocity gain within gap (E_{in} > E_{out} lens effect).

Field rising for stability longitudinally

Transverse RF Defocusing Ref: Wangler "RF Linear Accelerators" § 7.3
 Conte and Mackay, "Intro to the Physics of Particle Accelerators" Chapter 9

The field structure of an RF gap can also lead to transverse (radial) beam defocusing. Here we present a simple analysis to calculate the radial impulse a particle experiences when traversing the gap.

Qualitative Picture

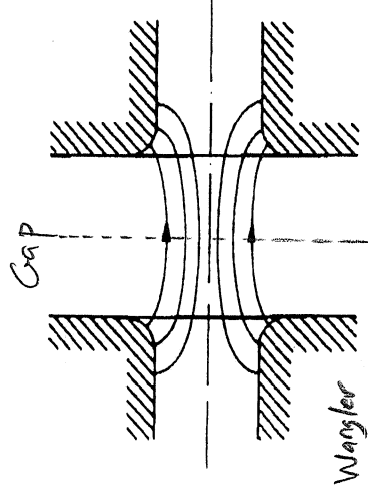
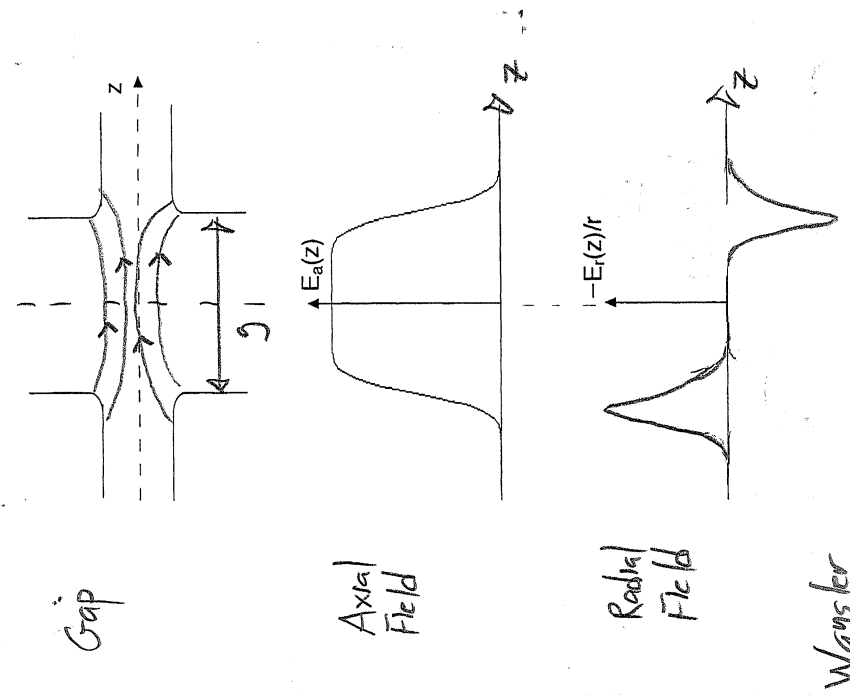
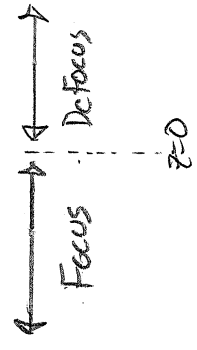


Figure 7.1 Electric field lines in an RF gap.



- * Symmetric if field static and $v \ll c$ const (negligible energy gain) \Rightarrow No optic
- * But: RF field rising in time as particle traverses gap \Rightarrow larger Defocus expected Net defocus.
- * Counter: if larger to right \Rightarrow less dwell time in defocus; Can have RF focusing if energy gain large; Like Einzel lens.
- * BE also present but weaker.

Maxwell's Equations in Cavity:

$$\nabla \cdot \vec{E} = 0$$

$$\Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r E_r) + \frac{\partial E_z}{\partial z} = 0 \quad 1)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\theta}{\partial t} \quad 2)$$

$$\nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

$$-\frac{\partial B_\theta}{\partial z} = \frac{1}{c^2} \frac{\partial E_r}{\partial t} \quad 3)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) = \frac{1}{c^2} \frac{\partial E_z}{\partial t} \quad 4)$$

$$\nabla \cdot \vec{B} = 0$$

satisfied by symmetry

Use these equations to approximate the fields near the axis ($r=0$) where we take E_z to be independent of r

For cavity assume:

$\vec{E} = E_r(r, z, t) \hat{r} + E_z(r, z, t) \hat{z}$ $\vec{B} = B_\theta(r, z, t) \hat{\theta}$	} TM type Mode
--	----------------

Then Lorentz Force Eqn:

$$\frac{d\vec{p}}{dt} = q \vec{E} + q \vec{v} \times \vec{B}$$

gives

$$\frac{dp_r}{dt} = q E_r - q v_z B_\theta$$

$\frac{dp_r}{dt} = q E_r - q \beta c B_\theta$
--

But

$$p_r = m \gamma \frac{dr}{dt} \approx m \gamma \beta c r'$$

$$r' = \frac{dr}{ds}$$

Giving a radial impulse (change in coast measure)

$\Delta(\gamma \beta r') = \frac{q}{mc} \int_{\text{Gap Transit}} [E_r - \beta c B_\theta] dt$
--

Approximate cavity fields near axis where E_z independent of r .
 * Need E_z and B_θ near $r=0$ to calculate impulse.

Using 1) with $\partial E_r / \partial r \approx 0$!

$$\frac{1}{r} \frac{\partial}{\partial r} (r E_r) + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \frac{\partial}{\partial r} (r E_r) = -\frac{\partial E_z}{\partial z} r$$

$$\textcircled{1} \quad E_r = -\frac{1}{2} \frac{\partial E_z}{\partial z} r$$

Using 3): $\sqrt{\text{use 1)}$

$$\frac{\partial B_\theta}{\partial z} = -\frac{1}{c^2} \frac{\partial E_r}{\partial t} = \frac{1}{c^2} \frac{\partial}{\partial z} \frac{\partial E_z}{\partial z} \frac{\partial r}{\partial t} \Rightarrow \int \text{integrate } B_\theta(r=0)=0$$

$$\textcircled{2} \quad B_\theta = \int \frac{\partial E_z}{\partial z} \frac{r^2}{c^2} dz$$

We take for the gap:

$$\textcircled{3} \quad E_z = E_0(z) \cos(\omega t + \phi)$$

harmonic accel. field, $E_z = E_0(z) \cos \phi$
 $t=0 \Rightarrow z=0$

Using 1) and 2) in the radial impulse formula:

$$\Delta(\gamma \beta r') = \frac{q}{mc} \int_{\text{Gap TransH}} [E_r - \beta c B_\theta] dt = \frac{q}{2mc} \int_{\text{Gap TransH}} \left[-\frac{\partial E_z}{\partial z} \cdot \frac{r}{c} - \beta \frac{\partial E_z}{\partial z} r \right] dt$$

$$= \frac{q}{2mc} \int_{-L/2}^{L/2} r \left[\frac{\partial E_z}{\partial z} + \beta \frac{\partial E_z}{\partial z} \right] dz \cdot dt = \frac{q}{2mc} \frac{dz}{\beta c}$$

Approximate further in single gap

- $r \approx \text{const}$
- $\beta \approx \text{const}$
- Impulse approx.
- Accel weak

These may break down at very low energies. Then more detailed analysis needed.

Then we can pull γ and β through the integral

$$\Delta(\chi \beta r') = \frac{-q \gamma}{2 \beta m c} \int_{-L/2}^{L/2} \left[\frac{\partial E_z}{\partial z} + \beta \frac{\partial E_z}{\partial t} \right] dz$$

But

$$\frac{\partial E_z}{\partial z} = \frac{\partial E_z}{\partial z} + \frac{\partial E_z}{\partial z} + \frac{\partial E_z}{\partial z} + \frac{\partial E_z}{\partial z} + \frac{\partial E_z}{\partial z} + \frac{\partial E_z}{\partial z} + \frac{\partial E_z}{\partial z} + \frac{\partial E_z}{\partial z}$$

Using this result to eliminate $\partial E_z / \partial z$:

$$\Delta(\chi \beta r') = \frac{-q \gamma}{2 \beta m c} \int_{-L/2}^{L/2} \left[\frac{\partial E_z}{\partial z} + (\beta - 1) \frac{\partial E_z}{\partial t} \right] dz$$

L contains field so no contribution

$$\left(\beta - \frac{1}{\beta} \right) = \frac{\beta^2 - 1}{\beta} = -\frac{(1 - \beta^2)}{\beta} = \frac{1}{\gamma \beta}$$

$$\Delta(\chi \beta r') = \frac{q \gamma}{2 (\gamma \beta)^2 m c^2} \int_{-L/2}^{L/2} \frac{\partial E_z}{\partial t} dz$$

Now use the harmonic accel field

$$E_z = E_0(z) \cos(\omega t + \phi) \Rightarrow \frac{\partial E_z}{\partial t} = -\omega E_0(z) \sin(\omega t + \phi)$$

For gap

$$\omega t = \frac{2\pi}{\beta \lambda} z$$

$$\frac{\partial E_z}{\partial t} = -\omega E_0(z) \sin\left(\frac{2\pi}{\beta \lambda} z + \phi\right)$$

Insert this full expression in impulse formula!

$$\Delta(x\beta r') = -\frac{q\omega}{2(\gamma\beta)^2 mc^2} \int_{-L/2}^{L/2} E_0(z) \sin\left(\frac{2\pi z}{\beta\lambda T} + \phi\right) dz$$

$$= -\frac{q\omega}{2(\gamma\beta)^2 mc^2} \int_{-L/2}^{L/2} E_0(z) \left\{ \sin\left(\frac{2\pi z}{\beta\lambda T}\right) \cos\phi + \cos\left(\frac{2\pi z}{\beta\lambda T}\right) \sin\phi \right\} dz$$

if $E_0(z)$ even function; usual for symmetric gap

$$\Delta(x\beta r') = -\frac{q\omega}{2(\gamma\beta)^2 mc^2} \int_{-L/2}^{L/2} E_0(z) \cos\left(\frac{2\pi z}{\beta\lambda T}\right) dz$$

This can be further simplified using our formula for the transit time factor of a symmetric gap:

$$T = \int_{-L/2}^{L/2} E_0(z) \cos\left(\frac{2\pi z}{\beta\lambda T}\right) dz$$

Transit Time

$$E_0 L = \int_{-L/2}^{L/2} E_0(z) dz$$

Avg Field over Cell

(L large enough to contain $E_0(z)$, usually take to be cell length $\Rightarrow E_0$ is cell avg field.)

$$\frac{\omega}{c} = \frac{2\pi}{\beta\lambda c} = \frac{2\pi}{\lambda c}$$

Then we have

Radial
Impulse
from
RF Gap

$$\Delta(\chi\beta r') = \frac{\pi (g_{EOLT}) \sin(-\phi) \times r}{\chi\beta (mc^2 (\chi\beta)^2)}$$

Comments:

* Linear optic: Impulse $\propto r$

* For $\phi < 0$ (RF stability) is defocusing

* $\sim 1/(\chi\beta)^3 \Rightarrow$ quickly becomes weak for relativistic particles
 \Rightarrow will be stronger for NR heavy ions (FRIB).

* More detailed analysis by Gluckstern (see Wangler of 7.4) shows that impulse can become focusing or significantly weakened when

β varies strongly in gap (low energy ions). In this context the

Enercol lens electrostatic focus impulse part compensates or offsets the effect of the rising RF field during transit.

Quasistatic Modeling of RF Gap Field

Wangler, p 5.14

In the previous treatment, we took E_z to be independent of r to calculate the approximate cavity detuning impulse. It one needs a better approx:

- 1) Import cavity fields from a cavity design code into a particle simulation.
- 2) Carry out more advanced analysis to better approx. fields and acceleration effects within gap.

Within the context of 2), the so-called quasistatic approx. can be useful.

Cavity fields satisfy the wave eqn:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} = 0$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

For harmonically varying fields: $\sim \cos(\omega t + \phi)$

$$\left(\nabla^2 - \frac{\omega^2}{c^2} \right) \vec{E} = 0$$

But

$$\frac{\omega}{c} = \frac{2\pi}{\lambda_{RF}} = \frac{2\pi}{\lambda_{RF}} \Rightarrow \left[\nabla^2 - \left(\frac{2\pi}{\lambda_{RF}} \right)^2 \right] \vec{E} = 0$$

If the gap has characteristic length scales $l_{gap} \ll \lambda_{RF}$, expect

$$\nabla^2 \sim \frac{1}{l_{gap}^2} \gg \left(\frac{2\pi}{\lambda_{RF}} \right)^2 \Rightarrow \boxed{\nabla^2 \vec{E} \approx 0}$$

But \vec{E} satisfies (vector calculus, any field):

$$\nabla \times (\nabla \times \vec{E}) = \nabla \cdot (\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

in cavity

$$\nabla \cdot \vec{E} = 0$$

$$\nabla^2 \vec{E} = 0$$

Giving

$$\nabla \times (\nabla \times \vec{E}) = 0 \Rightarrow \nabla \times \vec{E} = 0 \text{ solution.}$$

Satisfied if we take

$$\vec{E} = -\nabla \phi_e$$

since $\nabla \times \nabla \phi_e = 0$ for any ϕ_e .

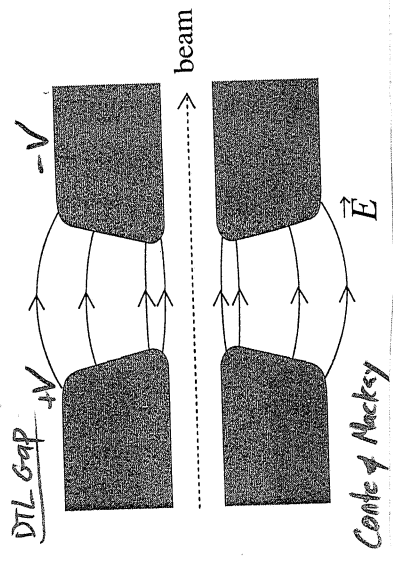
The Potential also must satisfy:

$$\nabla \cdot \vec{E} = -\nabla^2 \phi_e = 0 \Rightarrow \nabla^2 \phi_e = 0$$

ϕ_e satisfies Electrostatic Laplace eqn.

* can only apply locally (say near short gap) with $\lambda \ll \lambda_{RF}$

* Approx decouples electric and magnetic fields since $-\frac{\partial \vec{B}}{\partial t}$ has been neglected in Faraday's Law.



* Used to model gap field regions.

* Also applied in RFQ analysis (poles ripples small relative to RF wavelength) and analysis of induction accelerator gaps.