RF Cavities

see Conte and Mackay, Chapter 9
Wille, Chapter 5
Wiedemann, § 2.2

11. lec. cavity. pdf
Physics 905 Accelerator Physics

Maxwell's equations in vacuum region:

$$\nabla \cdot \vec{E} = 0$$

Vector Identity; Maxwell Eqn; Maxwell Eqn to eliminate

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \xrightarrow{\nabla\times} \nabla\times(\nabla\times\vec{E}) = \frac{-\partial}{\partial t}\nabla\times\vec{B} \rightarrow \nabla(\underbrace{\nabla\cdot\vec{E}}_{0}) - \nabla^2\vec{E} = \frac{-\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)$$

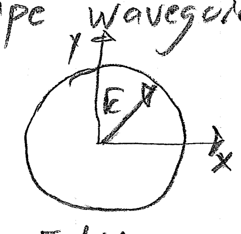
$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \xrightarrow{\nabla\times} \nabla\times(\nabla\times\vec{B}) = \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\times\vec{E} \rightarrow \nabla(\underbrace{\nabla\cdot\vec{B}}_{0}) - \nabla^2\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{-\partial}{\partial t}\vec{B}\right)$$

$$\nabla \cdot \vec{B} = 0$$

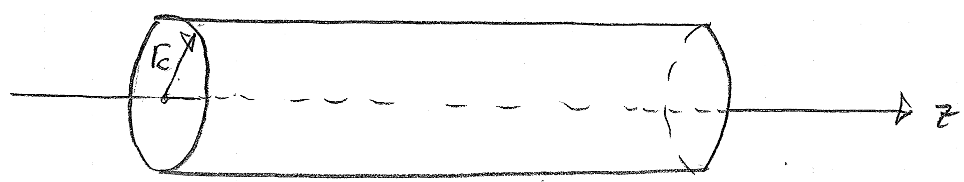
$$\Rightarrow \boxed{\begin{aligned}\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} &= 0\\ \nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} &= 0\end{aligned}}$$

$\vec{E}, \vec{B}$ satisfy Wave equations.

1st step:
We will look for EM wave solutions in a perfectly conducting, cylindrical pipe "waveguide".



End View



Side View

r_c = Radius Cylinder

Maxwell eqns give boundary conds on perfect conductor: $\vec{E}$: Tangential zero; $\vec{B}$: Normal zero

Search for a solution with z-t <u>traveling wave form</u> with harmonic time (t) and z dependance
★ $\sim e^{-i\omega t}$ time variation, $i = \sqrt{-1}$, take Re{ } for physical part.

$$\boxed{\left.\begin{aligned} E_z &= E_z(r,\theta)\cdot e^{i(\omega t - kz)}\\ E_r &= E_r(r,\theta)\cdot e^{i(\omega t - kz)}\\ B_\theta &= B_\theta(r,\theta)\cdot e^{i(\omega t - kz)}\end{aligned}\right\}}$$

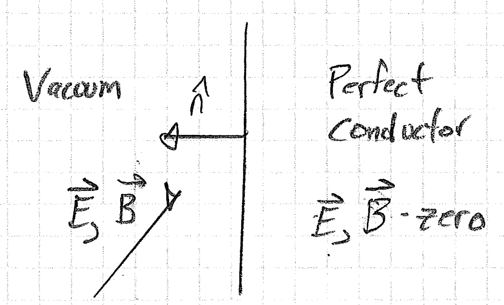
ω = const Angular Frequency
k = const Axial Wavenumber
Transverse Magnetic TM form since want longitudinal E_z for acceleration

Nonzero field components, in cylindrical-polar coordinates.

Later will restrict $E_z(r=r_c) = 0$ to meet boundary conditions.

Field Boundary Conditions: Conducting walls

Apply Maxwell's eqns at boundary of perfect conductor



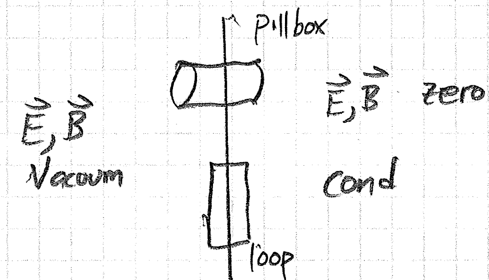
Maxwell Eqns Media

$$\nabla \cdot \vec{D} = \rho$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \cdot \vec{B} = 0$$

Integrate over $\Longrightarrow$ limiting pill box + loop $S, V \to 0$

Boundary Conds

$$\boxed{\begin{aligned} \hat{n} \cdot \vec{D} &= \Sigma \\ \hat{n} \times \vec{E} &= 0 \\ \hat{n} \times \vec{H} &= \vec{K} \\ \hat{n} \cdot \vec{B} &= 0 \end{aligned}}$$



Σ = Surface Charge Density

$\vec{K}$ = Surface Current Density

In vacuum:

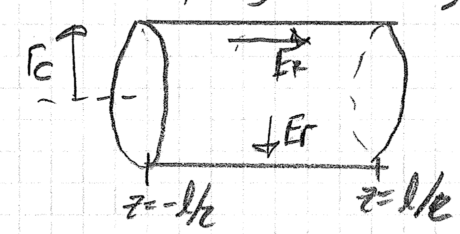
$$\vec{D} = \varepsilon_0 \vec{E}$$
$$\vec{B} = \mu_0 \vec{H}$$

So we have for field boundary conditions in the ideal vacuum / perfect conductor interface:

Exclude $\begin{cases} \vec{E}\,\big|_{tangential} = 0 \\ \vec{B}\,\big|_{normal} = 0 \end{cases}$

Allow $\begin{cases} \vec{E}\,\big|_{normal} \text{ allowed} \\ \vec{B}\,\big|_{tangential} \text{ allowed} \end{cases}$ $\Longrightarrow$ surface charge Σ adjusts to allow; surface current $\vec{K}$ adjusts to allow.

Implications in Pipe Segment: E_z, E_r, B_θ allowed



$E_z \to 0$
$r = r_c$: pipe edge
$z = \pm l/2$: pipe ends

$E_r \to 0$
$z = \pm \frac{l}{2}$: pipe ends

B_θ No restrictions

Examine only E_z:

$$\nabla^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2} \qquad \frac{\partial}{\partial t} = i\omega \qquad \frac{\partial}{\partial z} = -ik$$

$$\nabla^2 E_z - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E_z = 0 \quad \Rightarrow \quad \boxed{\nabla_\perp^2 E_z + \left(\frac{\omega^2}{c^2} - k^2\right) E_z = 0} \qquad \nabla_\perp^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Look for a solution with harmonic azimuthal variation

$$E_z \sim \cos(n\theta) \qquad \text{choose } \theta = 0 \text{ to make true.}$$

$$\Rightarrow \boxed{\begin{gathered}\frac{\partial^2}{\partial r^2} E_z + \frac{1}{r}\frac{\partial}{\partial r} E_z + \left(k_c^2 - \frac{n^2}{r^2}\right) E_z = 0 \\ k_c^2 \equiv \frac{\omega^2}{c^2} - k^2\end{gathered}}$$

Bessel Function Equation.

Recognizing this as Bessel's equation, the general solution is

$$E_z = C_1 J_n(k_c r) + C_2 Y_n(k_c r) \qquad C_1, C_2 \text{ constants}$$

$J_n(x)$ = Ordinary nth order Bessel function of 1st kind
$Y_n(x)$ = Ordinary nth order Bessel function of 2nd kind

$$\lim_{r \to 0} Y_n(k_c r) \to \infty \quad \Rightarrow \quad C_2 = 0 \quad \text{for finite (physical) E-field near } r = 0.$$

Putting back in variation in θ, z, t, we have:

$$\boxed{E_z = E_0 J_n(k_c r) \cos(n\theta)\, e^{i(\omega t - kz)} \qquad E_0 = \text{const.}\ (\text{complex})}$$

We can now substitute this back in the Maxwells' eqns to find the form of B_θ and E_r consistent. But first, simplify by further restricting to $n = 0$ since for accelerating particles we prefer no azimuthal variation.

Then:

$$E_z = E_0 J_0(k_c r)\, e^{i(\omega t - kz)}$$
$$E_r = E_r(r)\, e^{i(\omega t - kz)}$$
$$B_\theta = B_\theta(r)\, e^{i(\omega t - kz)}$$

<u>Maxwell Eqns</u> $\quad \frac{\partial}{\partial z} = -ik, \; \frac{\partial}{\partial t} = i\omega$

$\nabla \cdot \vec{E} = 0$: $\quad \frac{1}{r}\frac{\partial}{\partial r}(r E_r) - ik E_z = 0 \qquad$ 1)

$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \quad \hat{r}$: $\; ik B_\theta = \frac{\omega i}{c^2} E_r \qquad$ 2)

$\hat{z}$: $\; \frac{1}{r}\frac{\partial}{\partial r}(r B_\theta) = \frac{i\omega}{c^2} E_z \qquad$ 3)

$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \quad \frac{\partial E_z}{\partial r} - ik E_r = -i\omega B_\theta \qquad$ 4)

$\nabla \cdot \vec{B} = 0 \quad$ satisfied.

From 2) $\quad B_\theta = \frac{\omega}{c^2 k} E_r$

From 4) $\quad \frac{\partial E_z}{\partial r} = ik E_r - i\omega B_\theta = \left(ik - \frac{i\omega^2}{c^2 k}\right) E_r \quad \rightarrow$

$$\boxed{\begin{aligned} E_r &= \frac{-i/k}{1-\omega^2/(c^2k^2)}\frac{\partial E_z}{\partial r} = \frac{ik}{k_c^2}\frac{\partial E_z}{\partial r} \\ B_\theta &= \frac{-i\omega/(c^2k^2)}{1-\omega^2/(c^2k^2)}\frac{\partial E_z}{\partial r} = \frac{i\omega/c^2}{k_c^2}\frac{\partial E_z}{\partial r}\end{aligned}}$$

$$k_c^2 = \frac{\omega^2}{c^2} - k^2$$

$$k_c^2 = \frac{\omega^2}{c^2} - k^2$$

Using $\quad J_0'(x) = -J_1(x) \;;\; \frac{\partial E_z}{\partial r} = -E_0 k_c J_1(k_c r)\, e^{i(\omega t - kz)}$

$$\boxed{\begin{aligned} E_z &= E_0 J_0(k_c r)\, e^{i(\omega t - kz)} \\ E_r &= -i E_0 \frac{k}{k_c} J_1(k_c r)\, e^{i(\omega t - kz)} \\ B_\theta &= -i \frac{E_0 \omega}{c^2 k_c} J_1(k_c r)\, e^{i(\omega t - kz)}\end{aligned}}$$

$$k_c^2 = \frac{\omega^2}{c^2} - k^2$$

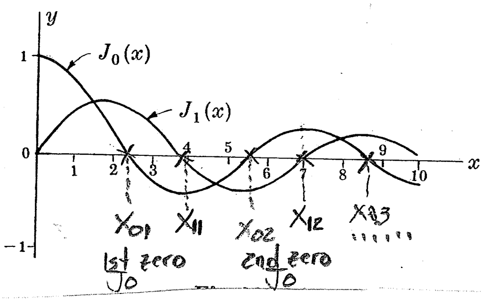
$\Rightarrow \quad E_r(r) = -i E_0 \frac{k}{k_c} J_1(k_c r)$

$\Rightarrow \quad B_\theta(r) = -i \frac{E_0 \omega}{c^2 k_c} J_1(k_c r)$

Finally, need $E_z(r = r_c) = 0$ to satisfy tangential $\vec{E} = 0$ on conducting boundary

$\Rightarrow \quad J_0(k_c r_c) = 0 \quad \Rightarrow \quad k_c r_c = X_{0j}, \quad j = 1, 2, 3, \ldots \quad$ zero of $J_n(X_{nj}) = 0$

Bessel function:



$X_{01} \simeq 2.405$

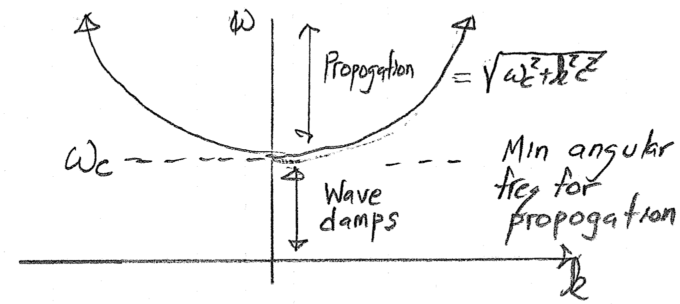
1st zero.

Choose 1st zero to try and get flat field near $r \simeq 0$.

$$\Rightarrow \quad k_c r_c = \sqrt{\frac{\omega^2}{c^2} - k^2}\; r_c = X_{01}$$

Dispersion Relation

$$\boxed{\omega^2 = \omega_c^2 + k^2 c^2 \; ; \quad \omega_c \equiv \frac{X_{01}\, c}{r_c}} \quad \text{cutoff freq}$$



Wave phase velocity

$$\psi = \omega t - kz = \text{const}$$

$$\dot{\psi} = \omega - k\dot{z} = 0 \quad \Rightarrow \quad \dot{z} \equiv v_{phase} = \frac{\omega}{k} = \frac{\omega c}{\sqrt{\omega^2 - \omega_c^2}} = \frac{c}{\sqrt{1 - \omega_c^2/\omega^2}} > c$$

$$\boxed{v_{phase} = \frac{c}{\sqrt{1 - \omega_c^2/c^2}} > c \quad \text{Cannot maintain resonance with particle}}$$

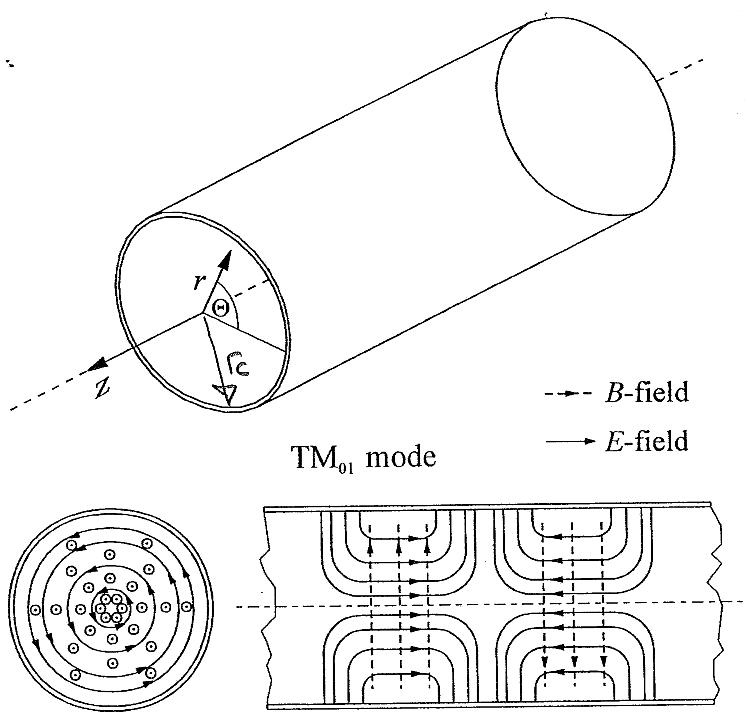
Note energy propogation speed at group velocity $\quad \omega = (\omega_c^2 + k^2 c^2)^{1/2}$

$$\boxed{v_{group} = \frac{d\omega}{dk} = \frac{kc}{(\omega_c^2 + k^2 c^2)^{1/2}} = \frac{c^2}{\omega/k} = \frac{c^2}{v_{phase}} < c}$$

★ $v_{group} < c$ as must be case for physical energy transmission.

Note ! $v_{group} \cdot v_{phase} = c^2 = \text{const.}$

Cylindrical Waveguide TM_{01} Modes



Wille **Fig. 5.2** Cylindrical waveguide with TM_{01} wave.

Nonzero Fields:

$$E_z = E_0 \cdot J_0(k_c r)\, e^{i(\omega t - kz)}$$

$$E_r = -i E_0 \frac{k}{k_c} J_1(k_c r)\, e^{i(\omega t - kz)}$$

$$B_\theta = -i \frac{E_0 \omega}{c^2 k_c} J_1(k_c r)\, e^{i(\omega t - kz)}$$

Nomenclature:

TM = Transverse Magnetic (Longitudinal E_z)

$TM_{n_\theta n_r}$

n_θ = azimuthal θ-harmonic E_z
$= 0 \Rightarrow$ None

n_r = number radial zeros E_z
$= 1 \Rightarrow$ One at $r = r_c$
(min needed for BC with nonzero sol.)

$\Rightarrow$ TM_{01} Mode

// Side Point: Traveling wave accelerator works by adding disks to waveguide to slow down EM wave phase velocity to maintain particle resonance:

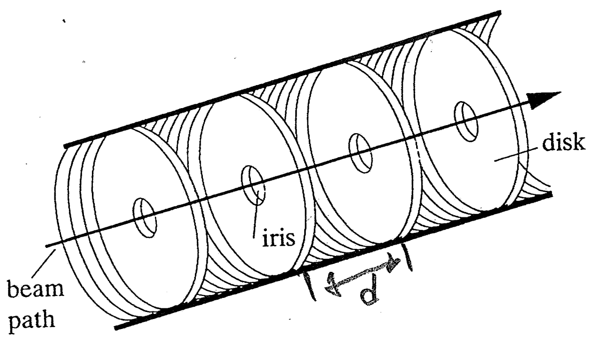
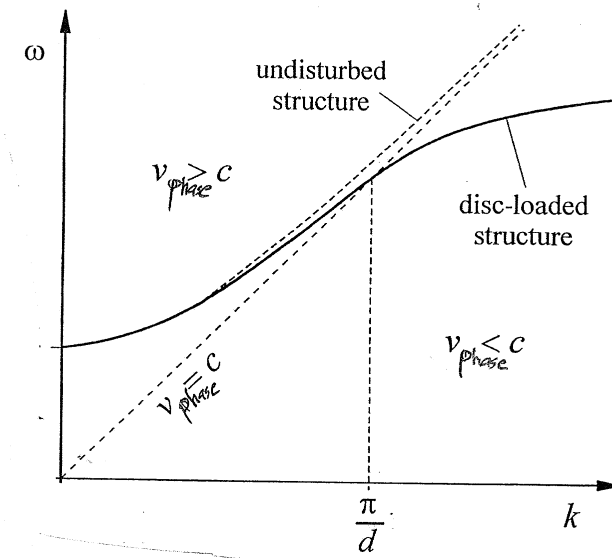


Fig. 2.8. Disk loaded accelerating structure for an electron linear accelerator (schematic)
Wiedemann



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Irises give partial reflections allowing loss free propogation only at RF wavelengths with integer multiples of the iris seperation d.

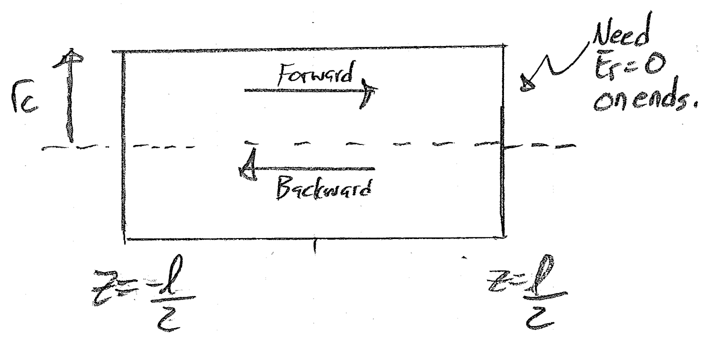
This method is commonly used in e^- accelerators. See Wangler for details.

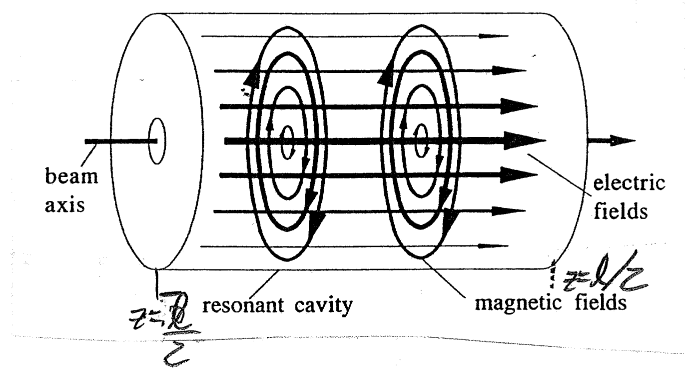
☆ washer loaded waveguide behaves like (weakly) coupled cavities.

//

So what do we do in our case? <u>Make resonant cavity.</u>

- ★ Add conducting walls at $z=0$, $z=\ell$
- ★ Superimpose <u>forward</u> and <u>backward</u> waves in cavity to meet boundary conditions and setup standing wave.
- ★ Time phasing of particles traversing cavity to gain energy and focus
  - Use formulation developed in earlier notes.





Wiedemann

For cavity E_z: Superimpose Waves:

$$E_z = \frac{\tilde{E}_0}{2} J_0(k_c r)\, e^{i(\omega t - kz)} + \frac{\tilde{E}_0}{2} J_0(k_c r)\, e^{i(\omega t + kz)}$$

Forward Wave (½ Amp) — Reflected Backward Wave (½ Amp) $(k \to -k)$

$$e^{ikz} + e^{-ikz} = 2\cos(kz)$$

$$\Rightarrow \quad E_z = \tilde{E}_0\, J_0(k_c r) \cos(kz)\, e^{i\omega t}$$

$\tilde{E}_0$ Amplitude (Complex)

No issue with end-plate boundary conditions

<u>E_r</u>

$$E_r = -i\frac{\tilde{E}_0}{2}\frac{k}{k_c}J_1(k_c r)\,e^{i(\omega t - kz)} + i\frac{\tilde{E}_0}{2}\frac{k}{k_c}J_1(k_c r)\,e^{i(\omega t + kz)}$$

Forward Wave
(1/2 Amp)

Reflected Backward Wave $(k \to -k)$
(1/2 Amp)

$$i\left[e^{ikz} - e^{-ikz}\right] = 2i\,i\,\sin(kz) = -2\sin(kz)$$

$$E_r = -\tilde{E}_0\frac{k}{k_c}J_1(k_c r)\sin(kz)\,e^{i\omega t}$$

To meet end-plate boundary conditions $E_r\big|_{z=\pm l/2} = 0$

$$\sin(kz)\Big|_{z=\pm l/2} = 0 \quad \Rightarrow \quad \frac{kl}{2} = n_z\pi \qquad n_z = 0, 1, 2, \ldots$$

<u>B_θ</u>

$$B_\theta = -\frac{i\tilde{E}_0\omega}{2c^2k_c}J_1(k_c r)\,e^{i(\omega t - kz)} - \frac{i\tilde{E}_0\omega}{2c^2k_c}J_1(k_c r)\,e^{i(\omega t + kz)}$$

Forward Wave
(1/2 Amp)

Reflected Backward Wave $(k \to -k)$
(1/2 Amp)

$$e^{ikz} + e^{-ikz} = 2\cos(kz)$$

No issues meeting boundary conditions at end-plates

$$B_\theta = -\frac{i\tilde{E}_0\omega}{c^2k_c}J_1(k_c r)\cos(kz)\,e^{i\omega t}$$

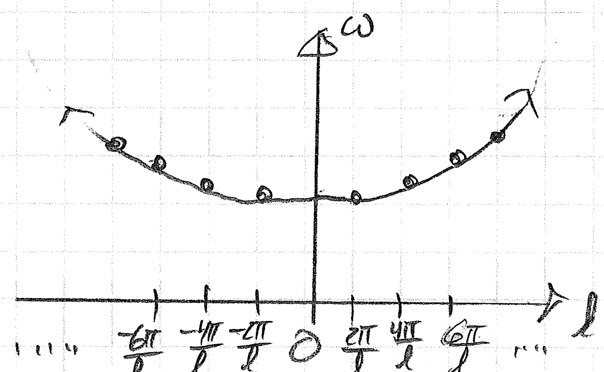
For the Pill-box cavity, due to E_r boundary condition

$$k = \frac{2 n_z \pi}{l} \qquad n_z = 0, 1, 2, 3$$

Inserting in the previous dispersion relation

$$\boxed{\omega^2 = \omega_c^2 + k^2 c^2 = \omega_c^2 + \left(\frac{2 n_z \pi c}{l}\right)^2}$$

$$\omega_c = \frac{x_{01} c}{r_c}$$



Only discrete values k now allowed, for standing wave.

Choose the <u>simplest possible solution</u>

$n_z = 0 \quad \Rightarrow k = 0$

Also gives no z-variation in E_z.

Label $TM_{n_\theta n_r n_z} = TM_{010}$ mode

$$\Rightarrow \boxed{\begin{aligned} E_z &= \tilde{E}_0 J_0(k_c r) e^{i\omega t} \\ E_r &= 0 \\ B_\theta &= -i \frac{\tilde{E}_0 \omega}{c^2 k_c} J_1(k_c r) e^{i\omega t} \end{aligned}}$$

$\omega = \omega_c$

$\frac{\omega}{c k_c} = 1$

$k_c = \frac{\omega_c}{c} = \frac{\omega}{c}$

$= \frac{x_{01}}{r_c}$

$$\boxed{\begin{aligned} E_z &= \tilde{E}_0 J_0\left(\frac{x_{01}}{r_c} r\right) e^{i\omega t} \\ E_r &= 0 \\ B_\theta &= -\frac{i\tilde{E}_0}{c} J_1\left(\frac{x_{01} r}{r_c}\right) e^{i\omega t} \end{aligned}}$$

$$\tilde{E}_0 \equiv E_0 e^{i\phi}$$

E_0 = Amp. (Real)
ϕ = Phase (Real)

and take the fields to be given by the Real part of the complex expression

$$\mathrm{Re}\left[\tilde{E}_0 e^{i\omega t}\right] = \mathrm{Re}\left[E_0 e^{i(\omega t+\phi)}\right] = E_0 \cos(\omega t+\phi)$$

$$\mathrm{Re}\left[i\tilde{E}_0 e^{i\omega t}\right] = \mathrm{Re}\left[iE_0 e^{i(\omega t+\phi)}\right] = -E_0 \sin(\omega t+\phi)$$

Giving

TM010 cavity fields

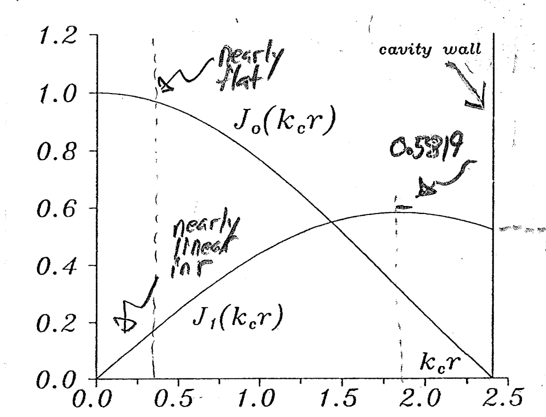
$$\boxed{\begin{aligned} E_z &= E_0 J_0\left(\frac{x_{01} r}{r_c}\right) \cos(\omega t+\phi) \\ E_r &= 0 \\ B_\theta &= -\frac{E_0}{c} J_1\left(\frac{x_{01} r}{r_c}\right) \sin(\omega t+\phi) \end{aligned}}$$

← Used phase choices for previous convention ($t=0$ at $z=0$ center of cavity)

← Used $k_c = \frac{\omega}{c}$ for $k=0$ in B_θ + reflection terms with consistent phase choices

Comments:

* All other field terms zero. $E_r = 0$ due to $k=0$
* Finite beam apeture at ends will allow $E_r \neq 0$ for this mode.



Wiedemann

$k_c = \frac{x_{01}}{r_c} \simeq \frac{2.405}{r_c}$

$J_1(x_{01}) \simeq 0.519$

* Beam will only fill a small fraction of r_c ⇒ $k_c r/r_c \ll 1$

$J_0(k_c r) \approx 1$ Nearly uniform E_z

$J_1(k_c r) \approx \frac{k_c r}{2}$; $B_\theta \propto r$ ⇒ Linear focus optic. (usually limited impact)

Now:

Max B_θ at $k_c r = 1.891$ where $J_1(k_c r) = J_1(1.891) \simeq 0.5819$

Max $E_z = E_0$ at $r = 0$ where $J_0(0) = 1$

Therefore: $\dfrac{cB_{Max}}{E_{Max}} = \dfrac{J_1(1.891)}{J_0(0)} = \dfrac{0.5819}{1} = 0.5819$

This number can have implications for the cavity field stress / breakdown.

$E_{Max} = E_0$ as large as possible for strong acceleration.

However, larger E_{Max} can trigger breakdown issues and larger $E_{Max} \Rightarrow$ larger B_{Max} (on cavity ends) which can also induce a quench for superconducting cavities. Realistic cavities shaped to try to limit these issues.

Pillbox cavity resonant frequency:

$$\omega = 2\pi f = \omega_c = \frac{X_{01} c}{r_c} \qquad X_{01} \cong 2.405$$

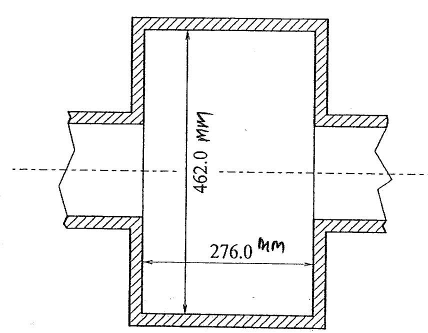
$$\boxed{f = \frac{2.405\, c}{2\pi r_c}}$$ Cavity Frequency

Some numbers:

$$2r_c = \frac{2.405 c}{\pi f}$$

Cavity freq f	Cavity Diameter $2r_c$
1 MHz	240 m
10 MHz	24 m
50 MHz	5 m
100 MHz	2.5 m
500 MHz	45.9 cm
1 GHz	25 cm
3 GHz	8 cm

Higher frequencies desired to limit size of cavities and control cost.



DORIS Storage Ring Cavity
German Electron Synchrotron Lab DESY

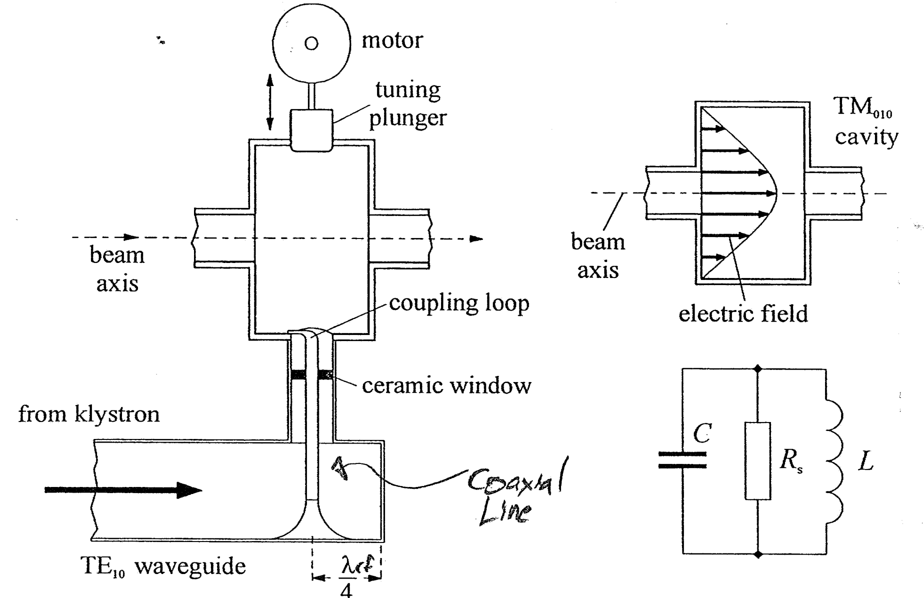
Wille

Fig. 5.3 Example of a single-cell cavity. It is chosen to have the dimensions $D = 462$ mm and $l = 276$ mm used in the accelerating structure developed for the storage ring DORIS, designed for a resonant frequency of 500 MHz.

Cavities must be connected to an RF source such as a klystron. Typical connection sketched below.

- Waveguide carries TE_{10} mode from klystron.
- Waveguide terminated near RF cavity
- Coaxial cable pickup $\sim \lambda_{rf}/4$ from waveguide termination ($\sim E$ max location)
- Connections shaped to inhibit reflections/losses.
- Ceramic window separates waveguide / coaxial cable (normal pressure) from cavity (high vacuum) without impeding RF wave.
  - Cavity window technology demanding for high power/voltages!
- RF wave coupled to TM_{010} symmetry of cavity by a loop.

Many details to do optimally; just a brief outline here:



Wille

Fig. 5.4 Design of a single-cell accelerating structure using the TM_{010} mode. The exact resonant frequency is adjusted using a tuning plunger. The resonator is excited by an inductive coupling loop.

A stable standing wave will exist in cavity only if the resonance condition of the TM_{010} mode is precisely satisfied.

Following an identification of cavity equivalent circuit parameters, will show that

$$Q = \frac{\omega_{res}}{\Delta\omega} = \frac{R_s}{|Z|} >> 1$$

$\Rightarrow \Delta\omega$ small

ω_{res} = resonant cavity ω
$\Delta\omega$ = Frequency bandwidth for ½ power.

Cavity Stored Energy: Pillbox Cavity TM_{010} mode

Field Energy Densities

$$\rho_E = \frac{\varepsilon_0}{2}\vec{E}^2$$

$$\rho_M = \frac{1}{2\mu_0}\vec{B}^2$$

At any given instant in time t the energy stored in an RF cavity is:

$$\boxed{U = \frac{\varepsilon_0}{2}\int_{\text{cavity}} \vec{E}^2 d^3x + \frac{1}{2\mu_0}\int_{\text{cavity}} \vec{B}^2 d^3x = \text{Stored EM Energy}}$$

Use Pillbox cavity fields and take $\omega t + \phi = 0$ ∵ $U = \text{const}$ so can take any time.
☆ This choice ⇒ all energy in E-field.

$$E_z = E_0 J_0(k_c r)\cos(\omega t + \phi) = E_0 J_0(k_c r)$$

$$B_\theta = \frac{-E_0}{c} J_1(k_c r)\sin(\omega t + \phi) = 0$$

$$k_c = \frac{x_{01}}{r_c}$$

and:

$$U = \frac{\varepsilon_0}{2}\int_{\text{cavity}} E_z^2 d^3x = \frac{\varepsilon_0}{2}(2\pi)E_0^2 \ell \int_0^{r_c} [J_0(k_c r)]^2 r\,dr$$

$$\int_{\text{cavity}} dz = \ell = \text{length cavity}$$

$$\int_{\text{cavity}} d\theta = 2\pi$$

Using

$$\int_0^1 t J_n(x_{nj}t) J_n(x_{nk}t)\,dt = \frac{1}{2}[J_n'(x_{nj})]^2 \delta_{jk}$$

$$\int_0^1 dt\, t J_0(x_{01}t)^2 = \frac{1}{2}[J_0'(x_{01})]^2 = \frac{1}{2}[J_1(x_{01})]^2$$

$$J_0'(t) = -J_1(t)$$

We have

$$\int_0^{r_c} \left[J_0\left(\frac{x_{01}}{r_c}r\right)\right]^2 r\,dr = r_c^2 \int_0^1 [J_0(x_{01}t)]^2 t\,dt = \frac{r_c^2}{2}[J_1(x_{01})]^2.$$

$$\Rightarrow \boxed{U = \frac{\varepsilon_0}{2} E_0^2 \pi r_c^2 \ell \cdot [J_1(x_{01})]^2}$$

$$J_1(x_{01}) \triangleq J_1(2.405) \simeq 0.51911$$

$$U \simeq 0.423\varepsilon_0 E_0^2 r_c^2 \ell$$

Cavity Dissipation: Pillbox Cavity ; Ref: Pick favorite EM Book.

No perfect conductors exist, but conductivity can be high:

Copper $\frac{1}{\sigma} \simeq 1.7 \times 10^{-8}\ \Omega \cdot m$

For a good but imperfect conductor, the fields penetrate the conductor in a thin surface layer where they fall off rapidly beyond a "skin depth" δ for fields varying at harmonic frequency ω:

$$\boxed{\text{Skin Depth} \quad \delta = \sqrt{\frac{2}{\sigma \mu_0 \omega}}}$$

Copper @ 100 MHz
$\Rightarrow \delta \simeq 10^{-6} m \sim 1\ \mu m$

Because of skin depth AC and DC resistances are not equal.

$$\boxed{\text{RF Surface Resistance} \quad R_{surf} = \frac{1}{\sigma \delta} = \sqrt{\frac{\mu_0 \omega}{2\sigma}}}$$

$\propto [\text{RF frequency}]^{1/2}$

AC and DC resistance varies.

Copper @ 100 MHz
$R_{surf} \sim$ milli-Ohm.

Electromagnetic theory texts show that the time averaged power loss to the walls over the RF cycle is given by:

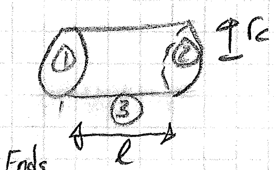
$$\boxed{\begin{aligned} \langle P_{loss} \rangle_{rf} &= \frac{1}{\tilde{T}_{rf}} \int_0^{T_{rf}} P_{loss}\, dt = \frac{R_{surf}}{2} \int |\vec{H}_t|^2 ds \\ \vec{H}_t &= \hat{n} \times \vec{H} = \text{Tangential component } \vec{H} \text{ at conductor.} \\ &\sim e^{-i\omega t} \text{ vary} \end{aligned}}$$

Interpretation: $H_t \rightarrow$ surface current.
Integrate loss over cavity surface.

Apply this loss formula to RF pillbox cavity:

$$\langle P_{loss}\rangle_{rf} = \frac{R_{surf}}{2}\int |\vec{H}_t|^2 ds \qquad B_\theta = \mu_0 H_\theta = -\frac{E_0}{c} J_1(k_c r)\, e^{-i\omega t} \qquad k_c = \frac{x_{01}}{r_c}$$

Will be contributions from



① + ② Ends
③ Side

$$\langle P_{loss}\rangle_{rf} = \frac{R_{surf}}{2}\Big\{ \underbrace{2 \times 2\pi\left(\frac{E_0}{\mu_0 c}\right)^2 \int_0^{r_c} J_1\!\left(\frac{x_{01}}{r_c} r\right) r\,dr}_{\text{① + ② Ends (same each end)}} + \underbrace{\pi r_c l\left(\frac{E_0}{\mu_0 c}\right)^2 [J_1(x_{01})]^2}_{\text{③ Side}} \Big\}$$

$$\tfrac{1}{2} \times 2\pi r_c \times l \times |H_t|^2\Big|_{r=r_c}$$

Algebra + Integral Tables

$$\boxed{\langle P_{loss}\rangle_{rf} = \pi\, r_c(r_c + l)\, R_{surf}\left(\frac{E_0}{\mu_0 c}\right)^2 [J_1(x_{01})]^2} \simeq 0.847\, r_c(r_c+l) R_{surf}\left(\frac{E_0}{\mu_0 c}\right)^2$$

Numerically $\pi [J_1(x_{01})]^2 \simeq 0.847$

Typical result:

* Loss depends on surface resistance (R_{surf}), peak field (E_0) and geometric parameters. (Cavity geom specific)
* Need Low R_{surf} for low losses.

Scaling of R_{surf}:

Normal Conducting:

$$R_{surf} = \sqrt{\frac{\mu_0 \omega}{2\sigma}} \propto f_{rf}^{1/2}$$

Copper at $f_{rf} \sim 100$ MHz

$R_{surf} \sim$ milli-Ohm

Superconducting Niobium Ref. Wangler

$$R_{surf} = 9\times 10^{-5}\, \frac{f_{rf}^2(\text{GHz})}{T(^\circ K)} \exp\left(-\eta \frac{T_c}{T}\right) \text{Ohms} + R_{residual}$$

(AC BCS Theory; $R_{residual}$ from imperfections)

$\propto f_{rf}^2$

$\eta = 1.92$

$T_c = 9.2^\circ K$ Critical Temp.

$R_{residual}$ = Residual Resistance $\sim 10^{-9} - 10^{-8}\ \Omega$ typical

$\Rightarrow \quad R_{surf} \sim 10^{-5} \times (R_{surf}$ of Copper$)$

Quality Factor :

Define in full generality (any cavity):

$$\boxed{\text{Quality Factor} = Q = 2\pi \frac{U}{\langle P_{loss}\rangle_{rf} \cdot T_{rf}} = 2\pi \times \frac{\text{Energy Stored}}{\text{Energy Dissipated in RF Cycle}}}$$

$$Q = \frac{2\pi}{T_{rf}} \frac{U}{\langle P_{loss}\rangle_{rf}} = \frac{\omega U}{\langle P_{loss}\rangle_{rf}} \Rightarrow \boxed{Q = \frac{\omega U}{\langle P_{loss}\rangle_{rf}}}$$

Pillbox Cavity Q

Using previous results for pillbox cavity

$$Q = \omega \frac{\overbrace{\left[\frac{\varepsilon_0}{2} E_0^2 \pi r_c^2 l \left[J_1(x_{01})\right]^2\right]}^{\text{" } U}}{\underbrace{\left[\pi r_c (r_c + l) R_{surf} \left(\frac{E_0}{\mu_0 c}\right)^2 \left[J_1(x_{01})\right]^2\right]}_{\text{" } \langle P_{loss}\rangle_{rf}}}$$

$$= \frac{\omega \overbrace{(\varepsilon_0 \mu_0 c^2)}^{\text{" } 1} \mu_0}{2 R_{surf}} \frac{r_c^2 l}{r_c (r_c + l)} \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$= \frac{\omega}{c} \frac{c \mu_0}{2 R_{surf}} \frac{r_c l}{r_c + l}$$

But at resonant frequency

$$\frac{\omega}{c} = \frac{x_{01}}{r_c} \qquad x_{01} \simeq 2.405$$

Pillbox Cavity

$$\boxed{Q = \frac{x_{01}\sqrt{\mu_0/\varepsilon_0}}{2 R_{surf}} \frac{1}{1 + r_c/l}} \qquad x_{01} \simeq 2.405$$

Want very high Q for cavity

$\Rightarrow R_s$ low : good conductor or superconductor

NC Example: DESY DORIS pillbox CU cavity $Q \simeq 38{,}000$ @ 500 MHz

SC Example: FRIB Quarter Wave SRF Cavity $Q \sim 10^9 - 10^{10}$ range.

High Q corresponds to:

- ★ Low heat generation
- ★ High efficiency
- ★ High stability : to variations in RF drive and beam loading

To understand the stability point, suppose an isolated cavity has stored energy U_0 in oscillatory mode with angular frequency ω: If the drive is removed, the energy U will chang as:

$$\frac{dU}{dt} = -\langle P_{loss}\rangle_{rf} = -\frac{\omega U}{Q} \qquad \text{since} \quad Q \equiv \frac{\omega U}{\langle P_{loss}\rangle_{rf}}$$

This has solution:

$$U(t) = U_0\, e^{-\omega t/Q} \qquad \Rightarrow \text{slow decay for } Q \text{ large, giving good stability}$$

A commonly used Figure of merit of an RF acceleration system is the so-called <u>shunt impedance</u> See Wangler, Sec. 2.5

$$V_0 = E_0 L \quad = \text{Effective cavity voltage}$$

$$\boxed{\text{Shunt Impedance:} \quad R_s \equiv \frac{V_0^2}{\langle P_{loss}\rangle_{rf}}}$$

Note Ohms Law: $V = IR$, $P = VI = \frac{V^2}{R}$

Caution: (Beware of 2 factor) Sometimes defined as $R_s = \frac{V_0^2}{2\langle P_{loss}\rangle_{rf}}$ due to interpretation of harmonic averaging factors.

Large shunt impedance $\Rightarrow$ Large accelerating potential relative to cavity dissipation. for economical acceleration.

But due to transit time factor, the accel potential V_0 is not fully imparted to particles. Therefore, define an "effective shunt impedance" to take this into account using synchronous phase $\phi_s = 0$ (Max accel.)

$$\Delta W = q(E_0 L)T\cos\phi_s$$

$$\Rightarrow \Delta W_{Max} = q V_0 T \qquad T = \text{Transit Time}$$

$\Rightarrow$ $E_0 L \Rightarrow E_0 L T$, $V_0 \Rightarrow V_0 T$ in previous formulas for "effective" measures.

Effective Shunt Impedance $$\boxed{R_{s,eff} = \frac{(V_0 T)^2}{\langle P_{loss}\rangle_{rf}} = \left(\frac{V_0^2}{\langle P_{loss}\rangle_{rf}}\right) T^2 = R_s T^2}$$

Sometimes these are analyzed per axial length L for long systems:

$$\boxed{\frac{R_{s,eff}}{L} = \frac{E_0^2 L^2 T^2}{L\langle P_{loss}\rangle_{rf}} = \frac{(E_0 T)^2}{\langle P_{loss}\rangle_{rf}/L}}$$

Typically given in $M\Omega$/meter

Another figure of merit is "R over Q":

$$\boxed{R \text{ over } Q: \quad \frac{R}{Q} \equiv \frac{R_{s,eff}}{Q} = \frac{(V_0 T)^2}{\cancel{\langle P_{loss}\rangle_{rf}}} \frac{\cancel{\langle P_{loss}\rangle_{rf}}}{\omega U} = \frac{(V_0 T)^2}{\omega U}}$$

☆ Measures efficiency acceleration per unit stored energy at specific frequency RF

☆ Function only of cavity geometry. – Independent of surface properties of power loss.

Energy imparted to beam particles must also come from RF cavity fields.

Instantaneous Power Delivered by Beam

$$\boxed{P_B = (\#\ \text{Particles}) \cdot \Delta W = \frac{I_{beam}}{q} \Delta W}$$

I_{beam} = beam electrical current.

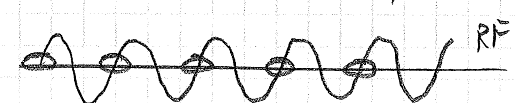
The total average power delivered will be

$$\langle P_{Total} \rangle_{rf} = \langle P_{loss} \rangle_{rf} + \langle P_B \rangle_{rf}$$

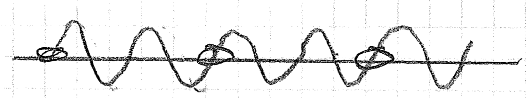
Take $\langle P_B \rangle_{rf} = \eta_{fill} \dfrac{\langle I_{beam} \rangle_{rf}}{q} \Delta W$ η_{fill} = Bucket Fill-Factor

$\langle I_{beam} \rangle_{rf} = \dfrac{Q_{bunch}}{\tau_{rf}}$ $Q_{bunch}/q = N_{bunch}$ = # particles in bunch

η_{fill} = Bucket fill fraction in machine pulse



$\eta_{fill} = 1$ All buckets filled



$\eta_{fill} = \frac{1}{2}$ Half buckets filled.

$$\boxed{\langle P_{Total} \rangle_{rf} = \langle P_{loss} \rangle_{rf} + \eta_{fill} \frac{N_{bunch} \cdot}{\tau_{rf}} \Delta W}$$

The efficiency of the accelerating structure can be

$$\boxed{\mathcal{E} \equiv \frac{\langle P_B \rangle_{rf}}{\langle P_{Total} \rangle_{rf}}} \qquad \text{Efficiency}$$

For "wall-plug" efficiency must account for other losses:

- ☆ RF Generation
- ★ Focusing + Bending magnet dissipation
- ★ Front end
- ☆ Cryo-Plant efficiencies for any superconducting systems
  ⋮

More efficient accelerators opens the door for more applications:

- ☆ Material processing
- ★ Energy Production : Subcritical reactors, Actinide Burning, Fusion drivers .....

  ⋮

Generally want more beam current for high efficiency and this can make accelerator physics much more difficult due to beam space-charge effects.

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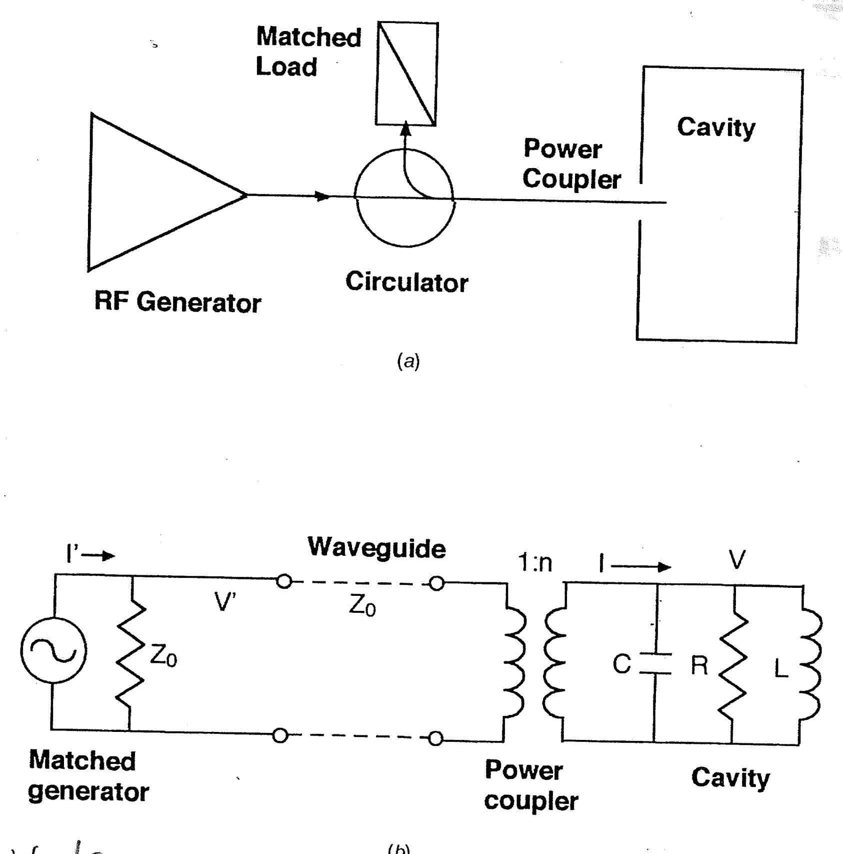
Equivalent Circuit for RF Cavity

Motivated by the qualitative correspondance to circuit parameters for the RF cavity the response of the system is idealized in terms of an equivalent circuit:

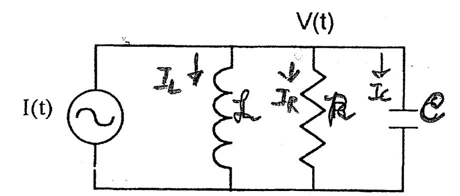
Equivalent Circuit



Wangler

Figure 5.3 (a) Block diagram of RF system components and (b) the equivalent circuit.

Cavity Component (idealized)



Wangler

$V(t) \Rightarrow$ Cavity Voltage $\sim E_0 L$

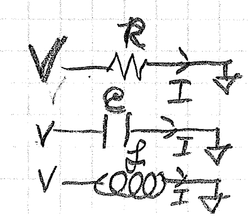
$$I(t) = I_L + I_R + I_c$$

$$= \frac{\int V dt}{\mathcal{L}} + \frac{V}{\mathcal{R}} + \mathcal{C}\frac{dV}{dt}$$

$$\dot{I} = \frac{V}{\mathcal{L}} + \frac{\dot{V}}{\mathcal{R}} + \mathcal{C}\ddot{V}$$

$V(t) =$ RF Cavity Voltage $\sim E_0 \cdot L$

Recall:



Resistor: $V = I\mathcal{R}$

Capacitor: $I = \mathcal{C}\, dV/dt$

Inductor: $V = \mathcal{L}\, dI/dt$

Driving current $I(t)$ produces voltage $V(t)$

$V(t) \sim V_0 e^{i\omega t}$ $\Leftrightarrow$ Axial accelerating voltage $V = E_0 L$ of cavity. with harmonic variation.

$\frac{1}{2} \mathcal{C} V_0^2 = U$ $\Leftrightarrow$ Energy U stored in the cavity. Sets capacitance $\mathcal{C}$

$\langle P_{loss} \rangle_{rf} = \frac{1}{2} \frac{V_0^2}{\mathcal{R}}$ $\Leftrightarrow$ Power lost in cavity. Sets restance $\mathcal{R}$

Express equation $\dot{I} = \frac{V}{\mathcal{L}} + \frac{\dot{V}}{\mathcal{R}} + \mathcal{C}\ddot{V}$ as:

$$\ddot{V} + \underbrace{\frac{1}{\mathcal{R}\mathcal{C}}}_{\text{Damping}} \dot{V} + \underbrace{\frac{1}{\mathcal{L}\mathcal{C}}}_{\text{Restore}} V = \underbrace{\frac{\dot{I}}{\mathcal{C}}}_{\text{Drive}}$$

$$\boxed{\ddot{V} + \frac{\omega_{res}}{Q} \dot{V} + \omega_{res}^2 V = \frac{\dot{I}}{\mathcal{C}}}$$

$$\boxed{\omega_{res} = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} = \text{Resonant Freq} \quad \Leftrightarrow \quad \text{Set } \mathcal{L} \text{ to get correct angular freq.}}$$

Motivated From prev damping analysis:

$$\frac{1}{\mathcal{R}\mathcal{C}} = \frac{\omega_{res}}{Q} = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}\,Q} \Rightarrow Q = \mathcal{R}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}} = \omega_{res} \frac{U}{\langle P_{loss} \rangle_{rf}} = \omega_{res} \mathcal{R}\mathcal{C}$$

$$\boxed{Q = \omega_{res} \frac{U}{\langle P_{loss} \rangle_{rf}} = \omega_{res} \mathcal{R}\mathcal{C} \quad \Leftrightarrow \quad \text{Set } \mathcal{R} \text{ to get correct damping}}$$

Search for a harmonic <u>steady-state</u> solution $(t \to \infty)$

$$I(t) = I_0 e^{i\omega t} \qquad \begin{array}{l}\omega = \text{const angular freq.} \\ I_0 = \text{const}\end{array} \quad (\text{need not satisfy } \omega = \omega_{res})$$

Analysis shows that

$$\boxed{V(t) = \frac{R I_0 e^{i(\omega t + \phi)}}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega}\right)}} \qquad \phi = -\tan^{-1}\left[Q\left(\frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega}\right)\right]}$$

Denote

$$\boxed{\Delta\omega = \omega - \omega_{res}}$$

Then

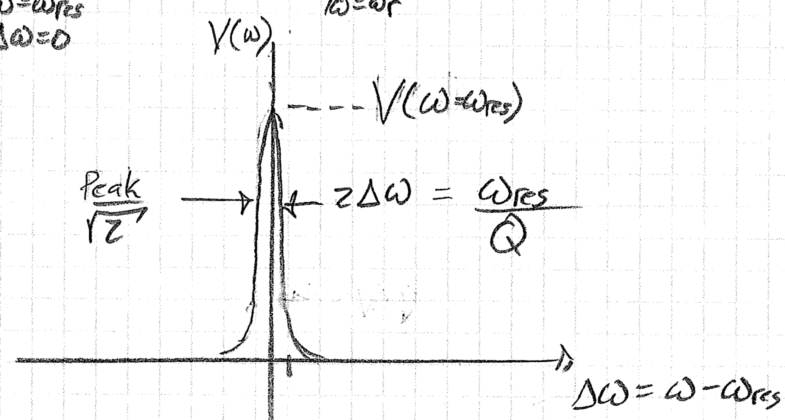
$$Q\left[\frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega}\right] = Q\left[1 + \frac{\Delta\omega}{\omega_{res}} - \frac{1}{1 + \frac{\Delta\omega}{\omega_{res}}}\right] \simeq 2Q\,\frac{\Delta\omega}{\omega_{res}}$$

The frequency shift $\Delta\omega$ to reduce the voltage amplitude to $1/\sqrt{2}$ the value (i.e., the ½ power value) relative to on resonance is:

$$V_{res}(t) = V(t)\Big|_{\substack{\omega = \omega_{res} \\ \Delta\omega = 0}} = R I_0 e^{i\omega_r t}_{\substack{\phi = 0 \\ \omega = \omega_r}} \qquad V(t)\Big|_{\omega = \omega_{res} + \Delta\omega} = \frac{\left(R I_0 e^{i\omega_r t}\right) e^{i(\Delta\omega t + \phi)}}{\sqrt{1 + 4Q^2 \frac{\Delta\omega^2}{\omega_{res}^2}}} \equiv V(\omega) e^{i\omega t}$$

$$= \frac{\left(V(t)\big|_{\Delta\omega = 0}\right)}{\sqrt{2}} \bullet \text{phase}$$

$\sim \frac{1}{2}$ Power $\rightsquigarrow$



for $2Q\frac{\Delta\omega}{\omega_{res}} = 1 \Rightarrow \Delta\omega = \frac{\omega_{res}}{2Q}$

$\boxed{\therefore \text{ High } Q \text{ means very sharply tuned resonant frequency.}}$

Frequency Scaling in RF Cavity Figures of Merit   Wangler 2.7

One of the most important parameters to choose in design is the cavity frequency.

$$\boxed{\omega = \frac{2\pi}{T_{rf}} = 2\pi f_{rf}}$$

Take:

$$\left.\begin{aligned} E_0 &= \text{const} \\ \Delta W &= \text{const} \end{aligned}\right\} \quad \text{fixed independent of } f_{rf} \text{ and fix length } L$$

Scale all other cavity dimensions with RF wavelength $\lambda_{rf} = c\,T_{rf} = \dfrac{c}{f_{rf}}$

Then

Transit Time T independent of f

Cavity surface Area S $\sim r_c \sim \dfrac{1}{f_{rf}}$

Cavity Volume V $\sim r_c^2 \sim \dfrac{1}{f_{rf}^2}$ $\Rightarrow$ Cavity Stored Energy U $\sim \dfrac{1}{f_{rf}^2}$

Surface Resistance $R_{surf} \sim \begin{cases} f_{rf}^{1/2} & \text{Normal Cond (NC)} \\ f_{rf}^{2} & \text{Superconducting (SC)} \end{cases}$ ← Skin depth scaling; ← Neglect residual resistance (good approx large f_{rf})

Avg. Power Loss $\langle P_{loss} \rangle_{rf} \sim \dfrac{R_{surf}|B|^2}{\mu_0} S \sim S\,R_{surf} \sim \dfrac{1}{f_{rf}} \begin{cases} f_{rf}^{1/2} & NC \\ f_{rf}^{2} & SC \end{cases} \sim \begin{cases} f_{rf}^{-1/2} & NC \\ f_{rf} & SC \end{cases}$

Quality Factor $Q = \dfrac{\omega U}{\langle P_{loss} \rangle_{rf}} \sim (f_{rf})(1/f_{rf}^2) \begin{cases} f_{rf}^{1/2} & NC \\ f_{rf}^{-1} & SC \end{cases} \sim \begin{cases} f_{rf}^{-1/2} & NC \\ f_{rf}^{-2} & SC \end{cases}$

Effective Shunt "Impedance"

$$R_{s,\mathrm{eff}} = \frac{(E_0 T)^2}{\langle P_{loss} \rangle_{rf}} \sim \frac{1}{\langle P_{loss} \rangle_{rf}} \sim \begin{cases} f_{rf}^{1/2} & NC \\ f_{rf}^{-1} & SC \end{cases}$$

★ Effective shunt impedance per unit axial length scales same.

R over Q

$$\frac{R}{Q} \equiv \frac{R_{s,\mathrm{eff}}}{Q} \sim \frac{(V_0 T)^2}{\cancel{\langle P_{loss} \rangle_{rf}}} \frac{\cancel{\langle P_{loss} \rangle_{rf}}}{\omega U} \sim \frac{1}{\omega U} \sim f_{rf} \sim \begin{cases} f_{rf} & NC \\ f_{rf} & SC \end{cases}$$

★ R over Q scales same for NC and SC since it should be independent of surface properties.

Phase-space Area Bucket that can be accelerated

$$\simeq \frac{3\pi}{2} \tan(-\phi_s) \sqrt{\frac{Z_q E_0 T (\gamma_s \beta_s)^3 \lambda_{rf}}{\pi m c^2} (\sin(-\phi_s) - \phi_s \cos\phi_s)} \overset{f_{rf}\lambda_{rf} = c}{\sim} f_{rf}^{-1/2}$$

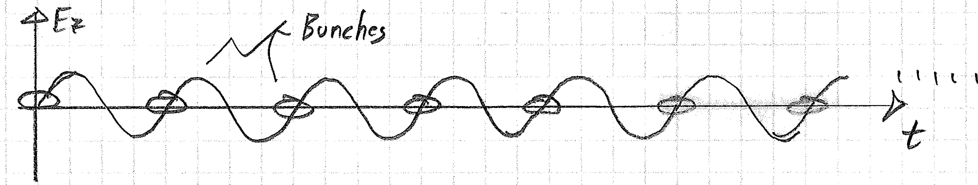
★ Higher frequency will lead to lower longitudinal "acceptance" for phase space area that can be accelerated by bucket.

- "Matching" important too if frequency transitions.

Comment: If linac has frequency transitions only harmonics and sub-harmonics are possible for a wave train of RF buckets. In certain cases only a limited fraction of buckets will be filled.

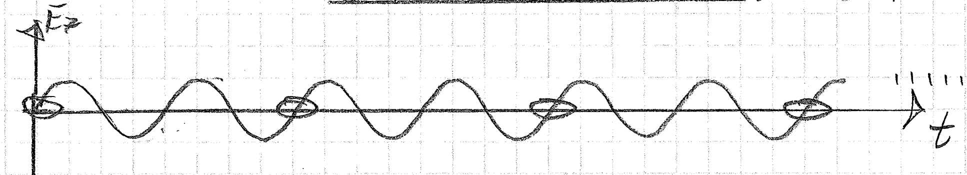
RF Bunch Structures

All Buckets Filled Continuous Wave



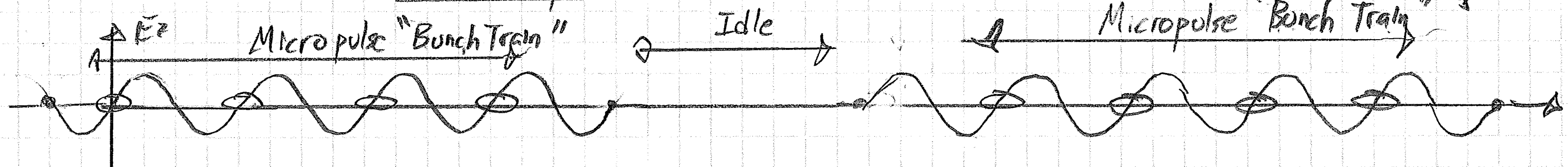
* Highest intensity on target
☆ Max use of RF

50% Buckets Filled. Continuous Wave



☆ Skip any # buckets to reduce intensity on target.

2 Micropulses With all Buckets Filled



"Bunch Train" also called "Batch" in Synchrotrons

☆ Trains of bunches for consistency with sources etc.

+ Many Variants.

Many reasons for various micro-pulse structures.

* ★ RF Structure limits in power (more idle time)
* ☆ Source limitations of particles
* ☆ Frequency changes: transitions to higher frequencies for more compact structures.
* ☆ Target limitations

More on Cavities

RF Cavities very diverse topic. Can teach whole courses on just aspects of technology.

Beam tube on pillbox cavity adds complication:

- ☆ Want field concentrated on gap for larger transit-time factor.
- ☆ Opening large enough to get beam in and out of cavity $\Rightarrow$ E_r generated.
- ☆ Peak E may no longer be on-axis.

E_{acc} = Accelerating E-field

$E_{peak} \sim 2-3 \times E_{acc}$

Figure of Merit $= \dfrac{E_{peak}}{E_{acc}}$

SC Cavities

High E_{peak} $\Rightarrow$ Field emission e^-'s, decreased efficiency + damage possible

NC Cavities

High E_{peak} $\Rightarrow$ Electric Breakdown, Cavity damage + loss of E_{acc}

- ☆ Resonant cavity angular freq ω more sensitive to cavity dimensions.
- ☆ Large B_θ on outer walls of cavity can quench if superconducting critical magnetic field exceed. The critical field depends on temperature.

$B_{critical} \sim 0.2$ Tesla for $2-4.2\,°K$ Niobium

Impurities reduce:

$B_{Max} \sim 0.1$ Tesla typical

For pill box cavity $\dfrac{cB_{Max}}{E_{Max}} = \dfrac{cB_{Max}}{E_{acc}} = 0.5819$

but this value can increase on drift-tubes, nose cones, etc.

<u>Electron Field Emission</u> Limits SC Cavity E_{Max} ; Wangler 5.10

e^- emitted from surface in strong $\vec{E}$ field. $\Rightarrow$ strike cavity after gaining energy and generate heat + X-rays when stopping.

Lowers Q

Fowler-Nordheim Law:

$$\boxed{\underset{\text{Current Density}}{J} \propto \frac{E_{peak}}{\Phi} \exp\left(-\frac{a\Phi^{3/2}}{E_{peak}}\right)}$$

Φ = Work function $\approx 4.3\,eV$ for Niobium

E_{peak} = peak electric field on surface.

a = const.

$$E_{peak} \sim 250 \times \left(\begin{array}{c}E_{Max} \text{ of Cavity} \\ \text{on Surface}\end{array}\right)$$

Due to surface roughness.

Very important for superconducting surfaces to be clean and smooth!

<u>RF Electric Breakdown</u> Limits NC Cavity E_{Max} ; Wangler 5.11

It is found empherically by Kilpatrick (Rev. Sci Inst., 28, 824 (1957)).

* for a given freq f_{rf} the peak E field on the surface before breakdown given by

$$\boxed{f_{rf}(MHz) = 1.64\, E_{Max}^2\, e^{-8.5/E_{Max}} \quad E_{Max} \text{ in } MV/m} \quad (*) \quad \Rightarrow \text{Plot}$$

* Somewhat conservative, often take

$E_{Max} = B\,(E_{Max}$ from Kilpatrick)

B = "bravery factor" 1-2 typical

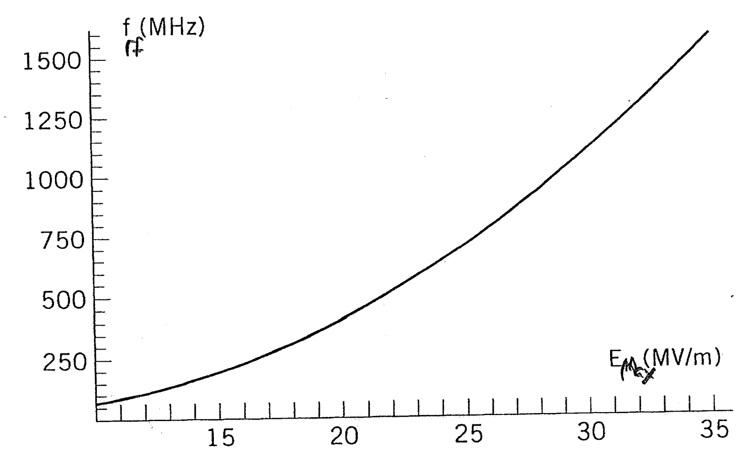


Figure 5.14 Kilpatrick formula from Eq. ~~(5.80)~~. *

Idealized Pillbox Cavity is distorted to better optimize.

Translate circuit model to a cavity model: Directly driven, re-entrant RF cavity

Outer region: Large, single turn Inductor

$$\mathcal{L} = \frac{\mu_o \pi a^2}{2\pi(R+a)}$$

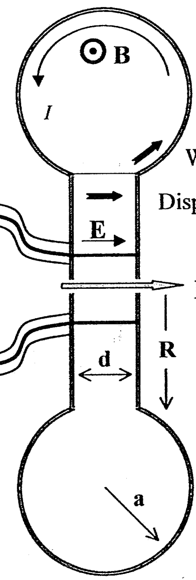
Central region: Large plate Capacitor

$$\mathcal{C} = \varepsilon_o \frac{\pi R^2}{d}$$

$$\omega_o = 1/\sqrt{\mathcal{LC}} = c\left[\frac{2((R+a)d}{\pi R^2 a^2}\right]^{1/2}$$

Q – set by resistance in outer region

$$Q = \sqrt{L/C}/R \; ?$$



Expanding outer region raises Q

Wall current

Displacement current

Beam (Load) current

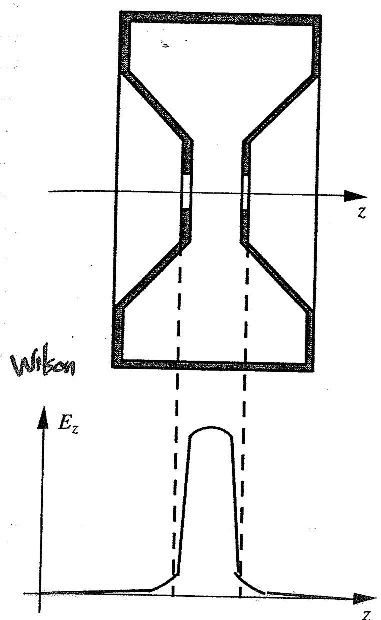
Narrowing gap raises shunt impedance

Source: Humphries, Charged Particle Accelerators

Barletta

US PARTICLE ACCELERATOR SCHOOL

Wilson



$E_{acc} \sim V_0$

$U = \frac{1}{2}\mathcal{C} V_0^2$

$\langle P_{loss} \rangle_{rf} = \frac{1}{2} V_0 / R$

$\omega_{res} = \frac{1}{\sqrt{\mathcal{LC}}}$

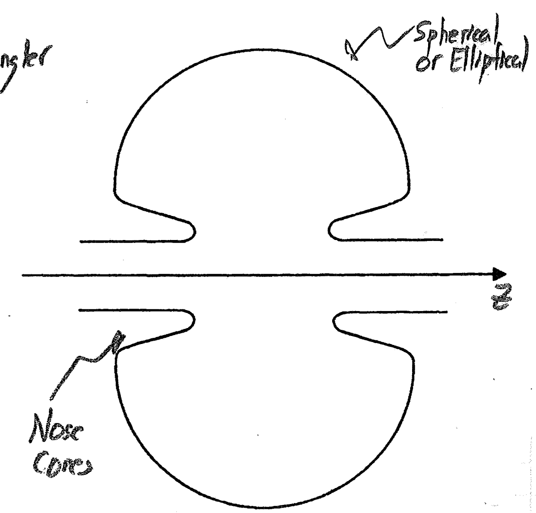
$Q = \omega_{res} \frac{U}{\langle P_{loss} \rangle_{rf}}$

Elliptical Cavity

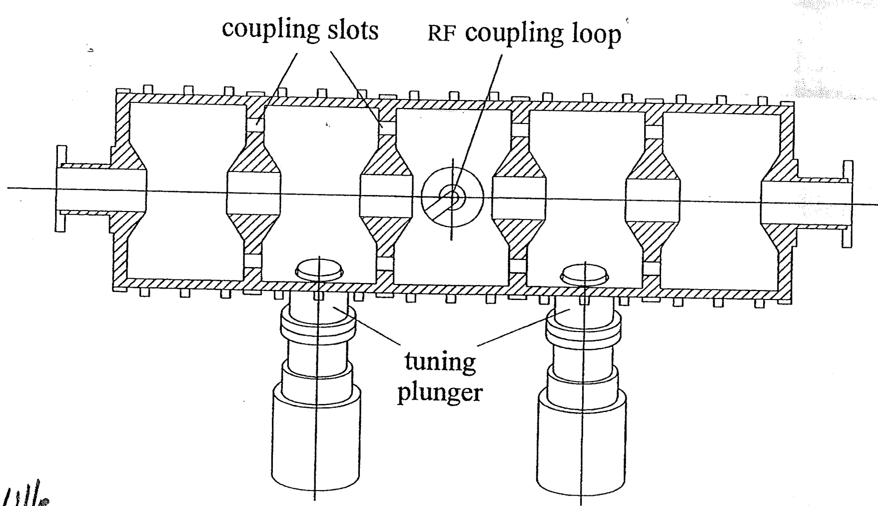
want:

- Small gap d for efficient accel.
  - Transit time Factor T large
  - Raise effective shunt impedance $R_{s,eff}$
- Expand outer region, raises Q

Wangler



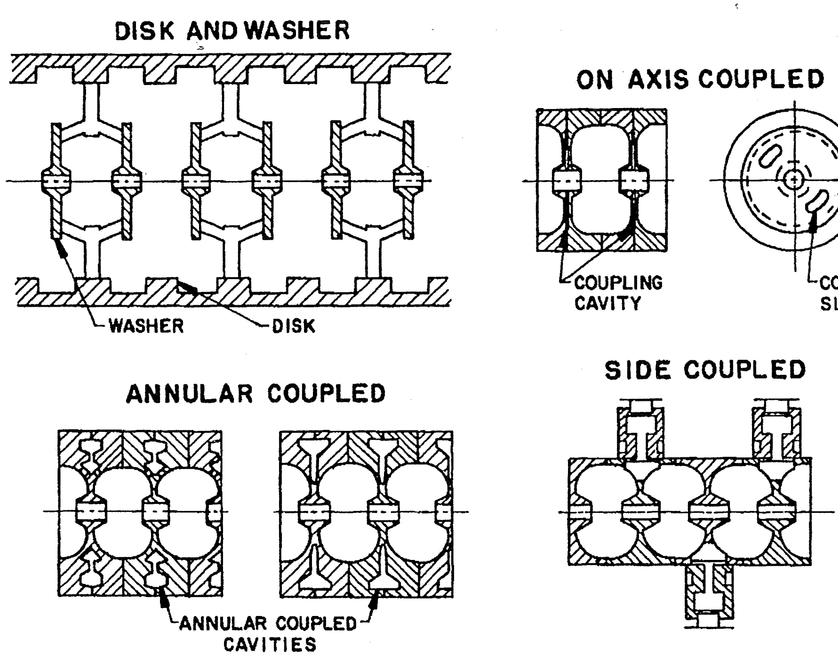
Coupled Cavities Groups of adjacent RF cavities are coupled together to maintain relative RF Phase control



Wille

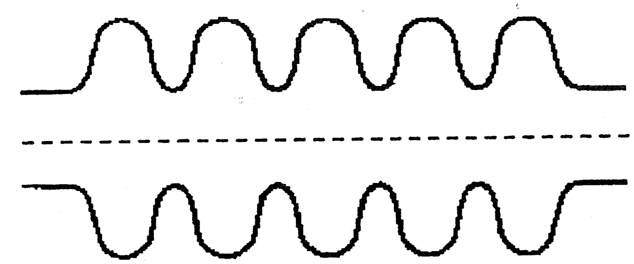
Fig. 5.5 Layout of a five-cell accelerating structure. The power feed is coupled to the middle cell and two tuning plungers are sufficient for the entire structure.

- ☆ Common for high β particle acceleration
  - Simplifies RF drive
  - Saves cost
  - Many possible geometries
- ☆ Coupling can be through beam apetures or slots, or sometimes special coupling cavities
  - Coupling cavities sometimes off axis or minimal length to save space.
- ☆ Usually transverse focusing placed between banks of cavities.



Wangler **Figure 4.17** Four examples of coupled-cavity linacs.

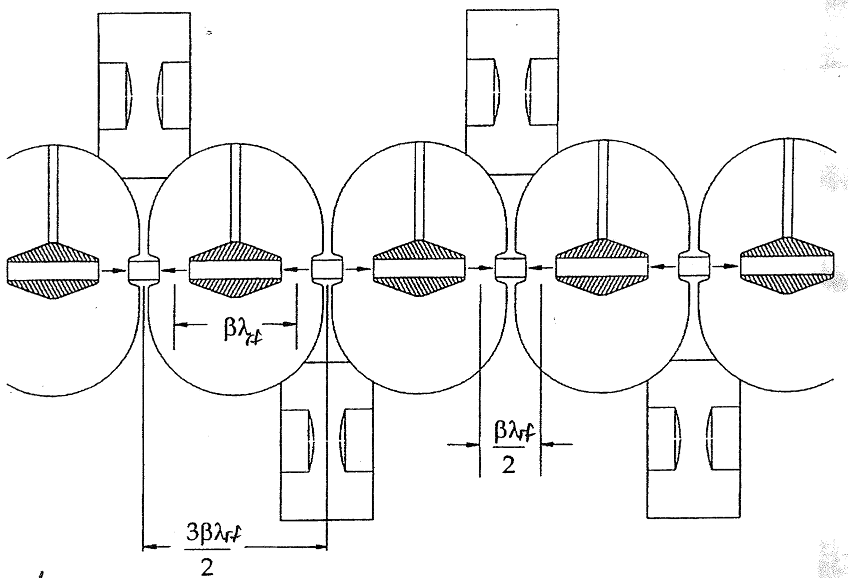
Elliptical : 5-Cell Bank



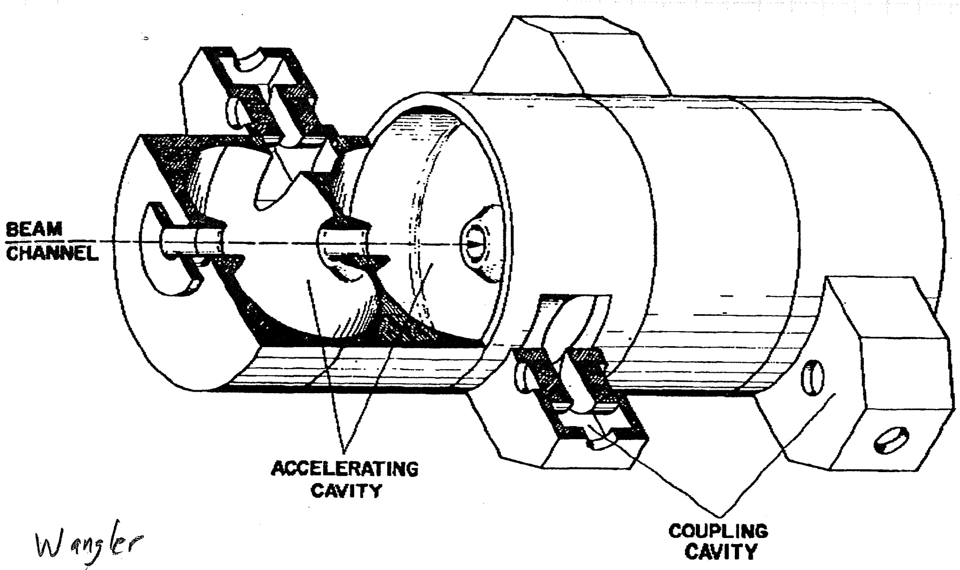
Wangler

Figure 2.5 Cross section of a $\beta = 0.82$ elliptical cavity designed for a superconducting proton linac. The cross section for each cell consists of an outer circular arc, an ellipse at the iris, and a connecting straight line.

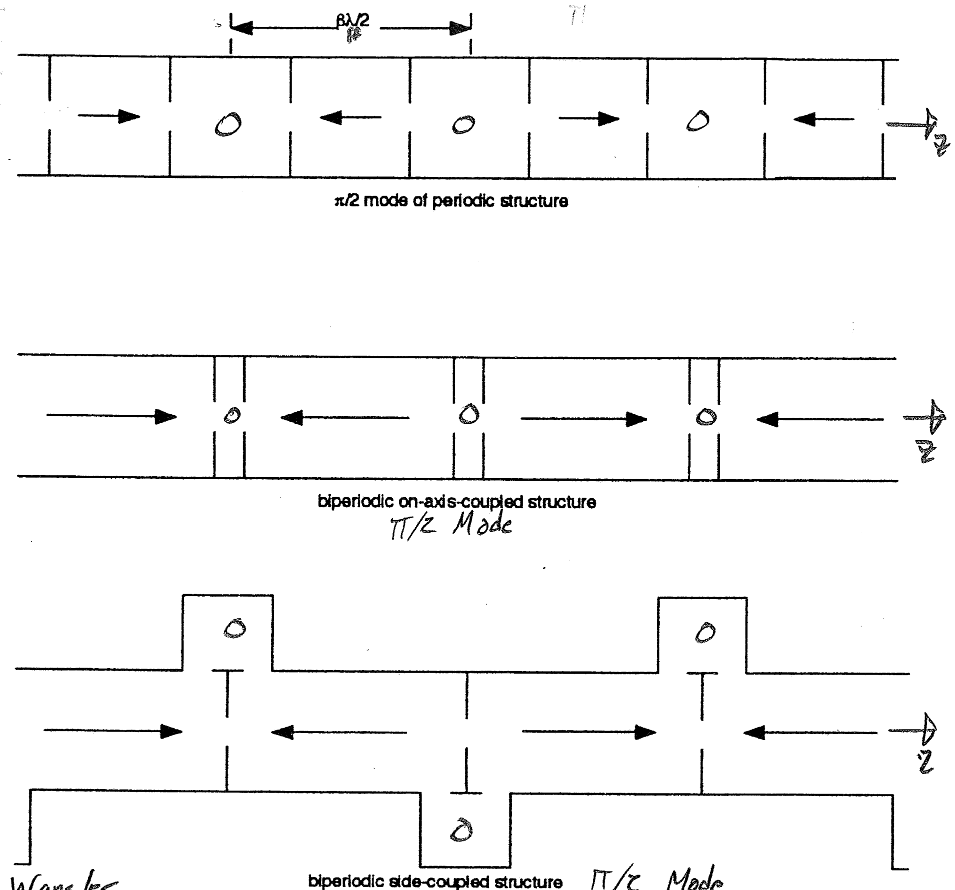
Phase relations between E-fields in cavities can vary.



Wangler
Figure 12.2 Coupled-cavity drift-tube linac (CCDTL) structure with a single drift tube in each accelerating cavity.



Wangler
Figure 4.11 Side-coupled linac structure as an example of a coupled-cavity linac structure. The cavities on the beam axis are the accelerating cavities. The cavities on the side are nominally unexcited and stabilize the accelerating-cavity fields against perturbations from fabrication errors and beam loading.



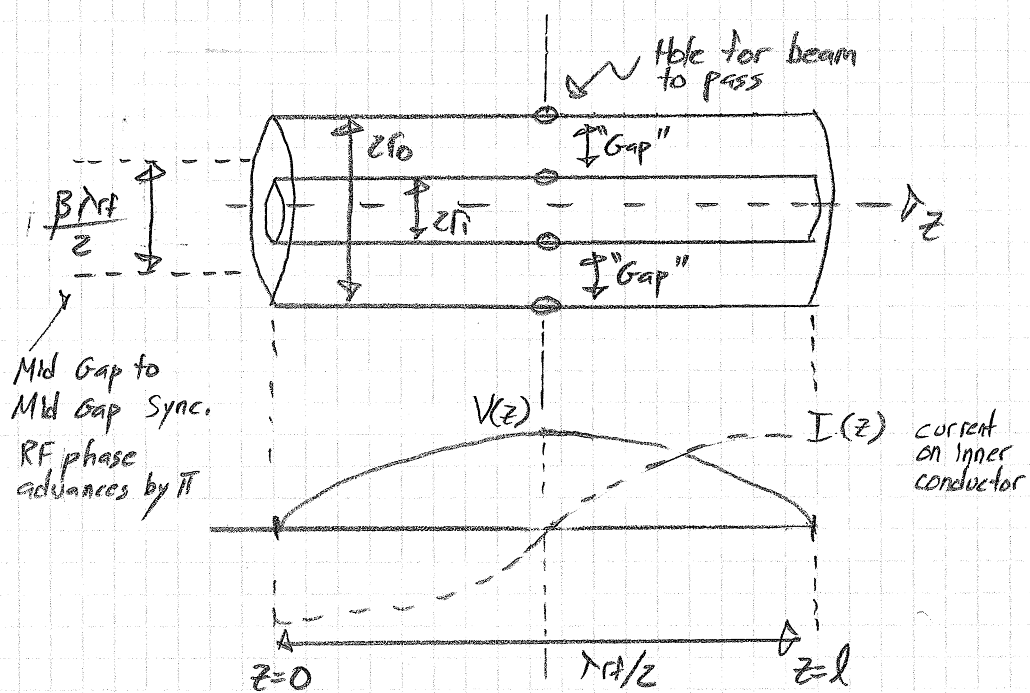
Wangler
Figure 4.15 $\pi/2$-like-mode operation of a cavity resonator chain. From top to bottom are shown a periodic structure in $\pi/2$ mode, a biperiodic on-axis coupled-cavity structure in $\pi/2$ mode, and a biperiodic side-coupled cavity in $\pi/2$ mode.

Low Frequency Half and Quarter Wave RF Structures

For low freq. ion acceleration with cavities operating with $f_{rf} \lesssim 100$ MHz, cavities based on coaxial resonators are employed.

★ Used in FRIB. 1/4 and 1/2 wave SRF Cavities.

Basic Idea : Half Wave Structure



Will show on a homework problem that an EM standing wave solution exists with

$$E_r = -2\sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{I_0}{2\pi r} \sin\left(\frac{p\pi z}{\ell}\right) \sin(\omega t + \phi)$$

$$B_\theta = \frac{\mu_0 I}{\pi r} \cos\left(\frac{p\pi z}{\ell}\right) \cos(\omega t + \phi)$$

$p = 1, 2, 3, \ldots$

$p = 1 \Rightarrow$ Half-Wave

$$\omega = \frac{p\pi c}{\ell}$$

I_0 = Amplitude of traveling wave current components on inner conductor.

$$V = \int_{r_i}^{r_o} E_r \, dr = \text{Accel. Voltage.}$$

★ Beam holes at $z = \ell/2$ where voltage is maximum

★ Beam moves on radial path sees no field when inside inner conductor (like drift tube).

★ RF phase advances by π when traversing the inner conductor so that the particle can be accelerated on both entrance and exit sides.

– Conductor radii chosen for max energy gain on each side

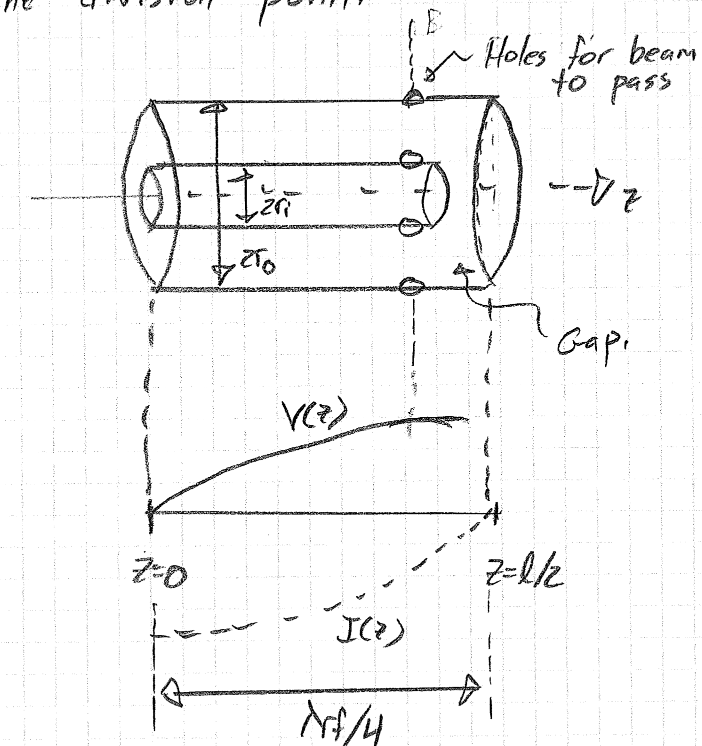
★ Effectively forms 2 gap cavity. 2 Gap Transit Time factor of HW problems applies.

Will also show in homework problems for the ½-wave coaxial resonator:

$$\boxed{\begin{aligned} U &= \frac{\mu_0 \ell I_0^2}{2\pi} \ln(r_0/r_i) && \text{RF Energy Stored} \\ Q &= \frac{p\pi}{R_{surf}} \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{\ln(r_0/r_i)}{\ell[1/r_i + 1/r_0] + 4\ln(r_0/r_i)} && \text{Quality Factor} \end{aligned}}$$

Quarter Wave Structure

Essentially split the half-wave structure divided in two with a capacitive termination at the division point.



★ Has a lesser degree of symmetry and fields will be distorted more than in the half-wave resonator.

Design formulas including the contribution to the fields from the capacitive gap termination can be found in

Moreno, Microwave Transmission Design Data, Dover, NY 1948, pp. 227-230.

Both Quarter and Half-Wave structures produce more compact low freq. cavities:

★ Save RF power
★ Cheaper superconducting (less material, less losses to cool, ...)

Coupling to RF Cavities

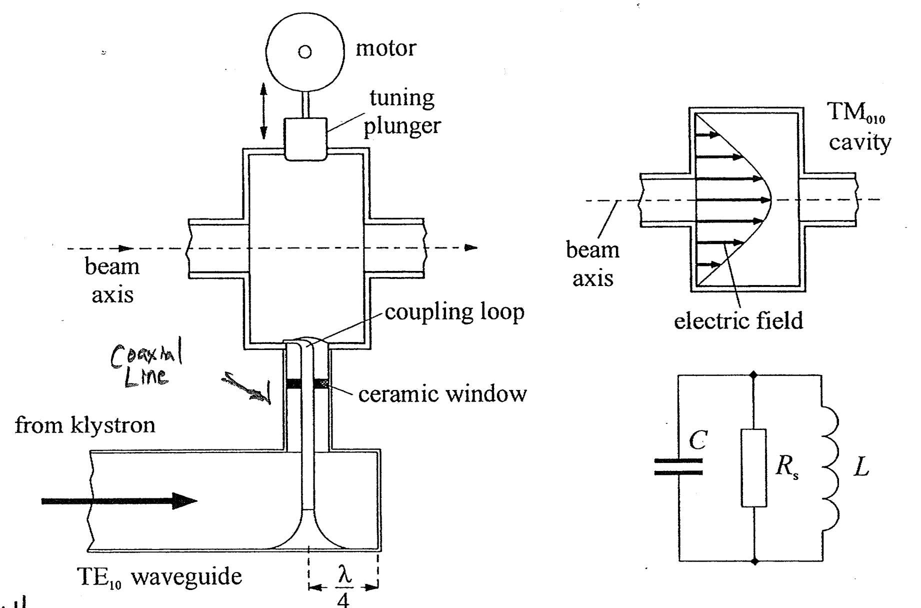
See Wille, "The Physics of Particle Accelerators", Chapter 5
Wilson, "An Introduction to Particle Accelerators", Chapter 5
Wangler, "RF Linear Accelerators" Chapter 5

Beyond scope to discuss in this class.
Many ways to couple RF power to resonant cavities.
Most common may be with a loop to couple with magnetic field of EM TM010 type standing wave.

- ☆ Place where magnetic field high in outer radial extent of cavity
- ☆ Field created by loop should have component in common with B_θ of TM010 type mode (or whatever mode) desired to excite.

Coupling of klystron to waveguide + coaxial cable also an issue. Much to consider.

Magnetic Coupling Loop at end of Coaxial Transmission Cable



Wille

Fig. 5.4 Design of a single-cell accelerating structure using the TM_{010} mode. The exact resonant frequency is adjusted using a tuning plunger. The resonator is excited by an inductive coupling loop.

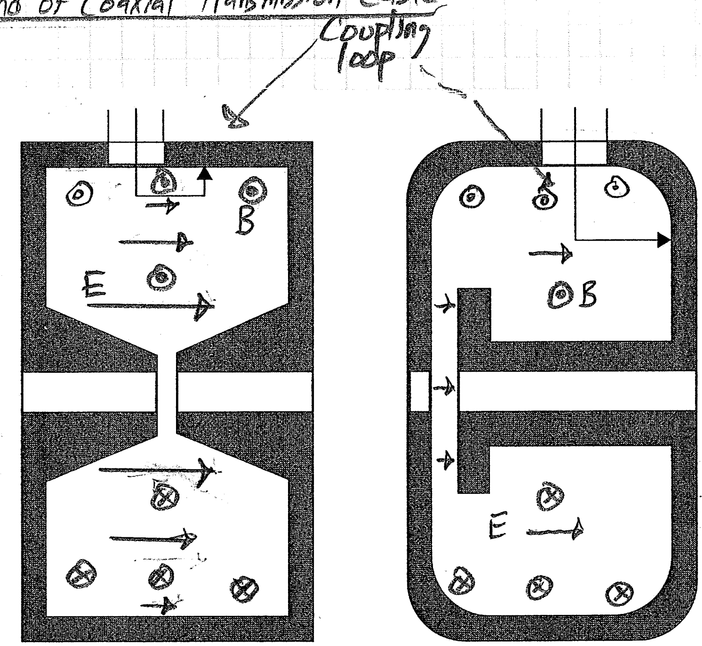


Fig. 10.15 Two examples of loop coupling.
Wilson

TM010 type mode

Common Methods Coupling.

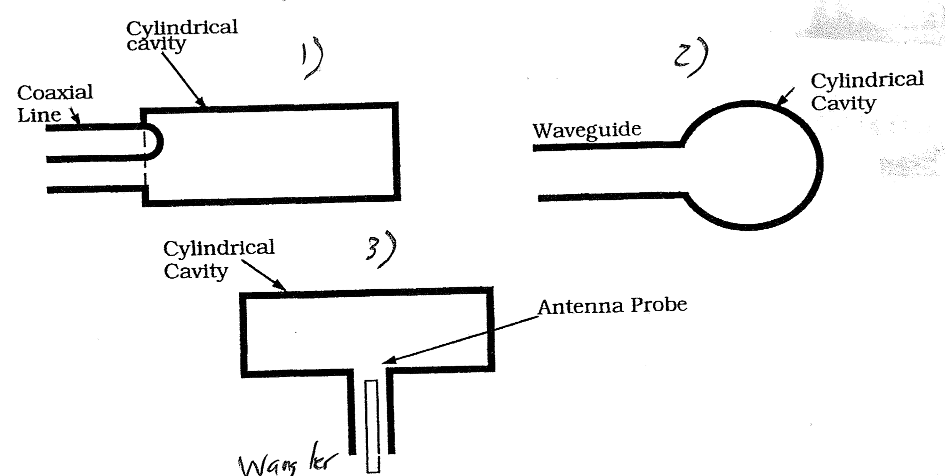


Figure 5.2 Methods of coupling to cavities.

1) Magnetic Loop at end of coaxial transmission line connected to cavity

2) Hole or Apeture in cavity wall connected to a wave guide

3) Electric Coupling Probe or Antenna using the central conductor of a coaxial transmission line.

Comments:

* Want structure using low order mode to make easy to excite and avoid coupling to higher order modes.
  - Preclude coupling to Higher order modes by frequency choice.

* Couplers have much difficult engineering
  - Heat leak for SRF structures.

RF Sources

See Wille, "The Physics of Particle Accelerators," Chapter 5
Wilson, "An Introduction to Particle Accelerators," Chapter 5

Harmonically varying RF power needed for accelerating structures ranging from a few kW to MW power levels. Pulses may be short, long, or continuous wave (CW).

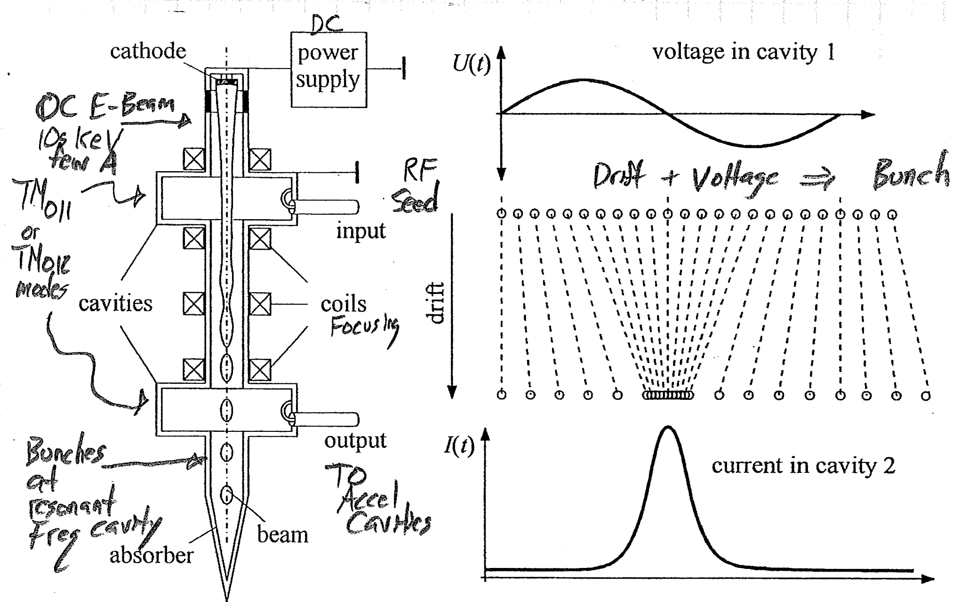
1) Triode / Tetrode: few MHz → few 100 MHz ; high power broad band
2) Klystron: few 100 MHz + — Most common in accelerator systems. Narrow band, tuned, very high power

Most common for Accel. Applications →

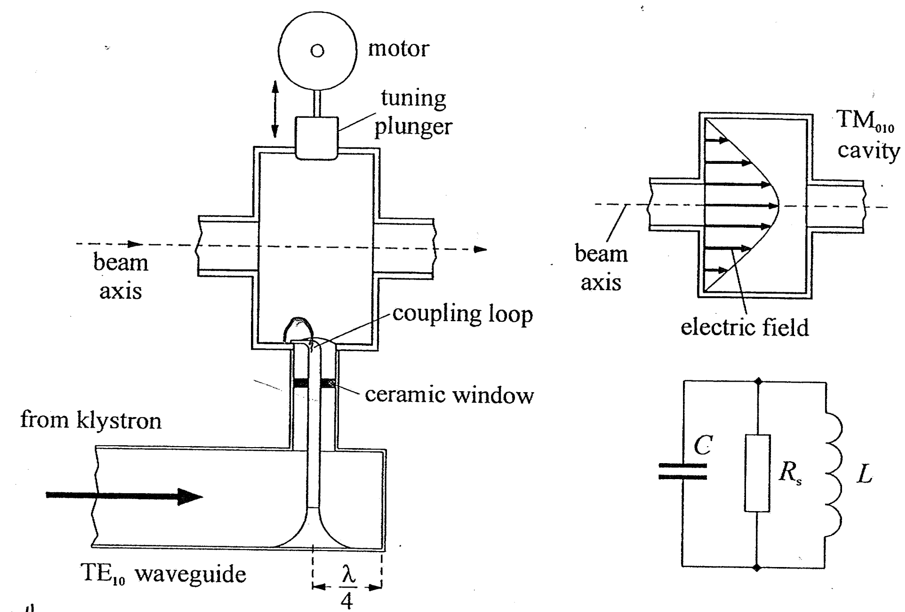
3) Also: Traveling Wave Tubes, Magnetrons, Cross-Field Amplifiers, Gyrotrons, ...

Klystron

Drift long enough to bunch.
Using TM_{011} or TM_{012}



Wille
Fig. 5.11 The classical microwave klystron, operating in the ten centimetre region.



Wille
Fig. 5.4 Design of a single-cell accelerating structure using the TM_{010} mode. The exact resonant frequency is adjusted using a tuning plunger. The resonator is excited by an inductive coupling loop.

Power delivered by klystron

e^- beam source large: $I_{beam} \sim 10\,A$ typical
$V \sim 10$'s kV Source Voltage typical

$P_{klystron} = \eta V I_{beam}$ η = Efficiency 45% → 65% typical

~ 1.2 MW per tube now achieved in CW operation.
@ 350 – 500 MHz

~ 250 kW typical CW values.

Real klystrons may use several resonators to extract more energy.

Many variants including relativistic klystrons using hiher (MeV) energy e-beams.

Numerous topics on RF sources, Coupling, measurement, engineering.
Many texts exist on topic. Often older books on Engineering.
Additional important topics:

* Microwave coupling to cavities/waveguides
* Slater perturbation theorem - bead pull of small metal structure used to measure cavity frequencies.
* Cavity tuning : usually via deformation

⋮

The US Particle Accelerator School regularly offers courses on

Microwave Sources
Microwave Measurements and Beam Instrumentation
Microwave Linear Accelerators

as part of the core curriculum. These courses overview the topic from an accelerator perspective.

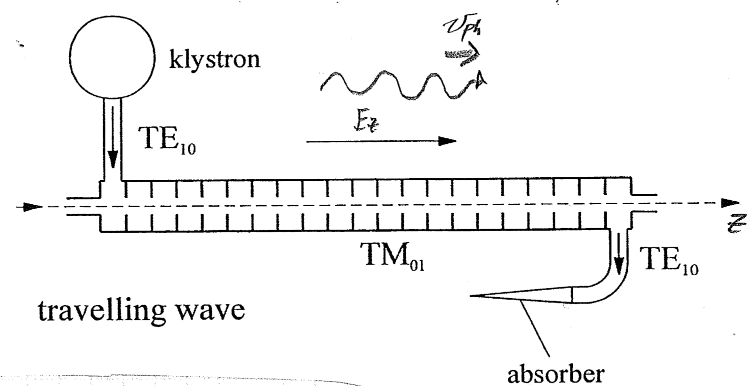
Traveling Wave RF Acceleration

See for more info:
Wangler, RF Linear Accelerators, Chapter 3
Wille, The Physics of Particle Accelerators, Chapter 5.

Rather than exchanging energy with an RF standing wave in a fixed cavity, a traveling RF wave can be setup to resonate with a particle to accelerate it.

Consider an EM wave propogating along $\hat{z}$ in a waveguide structure:

Wille



$$\boxed{\begin{aligned} E_z(z,t) &= E(z)\cos\left[\omega t - \int_0^z k(z)\,dz + \phi\right] \\ k(z) &= \frac{\omega}{v_p(z)} \qquad v_p = \text{phase velocity of wave.} = \frac{\omega}{k} \end{aligned}}$$

For efficient acceleration of a particle with axial velocity v_z, want

$$\boxed{v_z \simeq v_p}$$

* deviations will result in wave "slipping" in phase particle

Consider a particle of charge q with $v_z = v_{ph}$ at each instant in time. Then the particle arrives at position z at time

$$\boxed{t(z) = \int_0^z \frac{dz}{v_z(z)}}$$

So the E_z at the particle is:

$$E_z = E(z)\cos\left[\omega t - \int_0^z k(z)\,dz + \phi\right] \qquad k(z) = \frac{\omega}{v_p(z)}$$

$$= E(z)\cos\left[\cancel{\omega\int_0^z \frac{dz}{v_z(z)}} - \cancel{\omega\int_0^z \frac{dz}{v_p(z)}} + \phi\right] = E(z)\cos\phi$$

In the traveling wave approach, this particle is called the synchronous particle and $\phi = \phi_s$. The synchronous particle will gain kinetic energy.

$$\boxed{\Delta W_s = q\int_0^z E(z)\cos\phi_s\,dz}$$

As previously discussed, $v_p > c$ for a simple cylindrical pipe waveguide. But periodic structures can be placed in the waveguide to produce partial reflections to reduce the phase velocity to satisfy $v_p \leq c$. The structure can be analyzed as a periodic array of coupled resonant cavities. Various techniques exist to analyze this situation and gain intuition; See Wangler Chapter 3.

Iris Loaded Waveguide for $v_p < c$

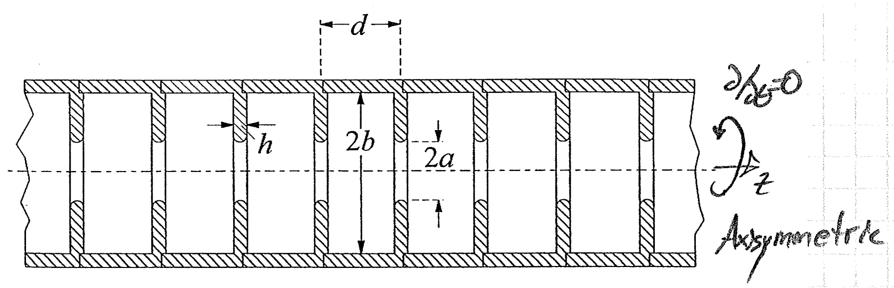
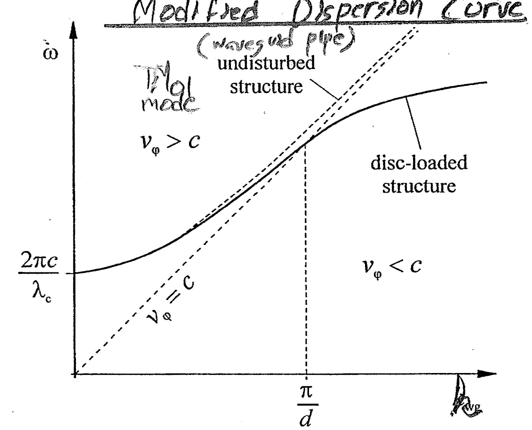


Fig. 5.6 Cross-section through a typical linac structure. The phase velocity of the RF wave is reduced to the particle velocity by the insertion of irises.

Wille

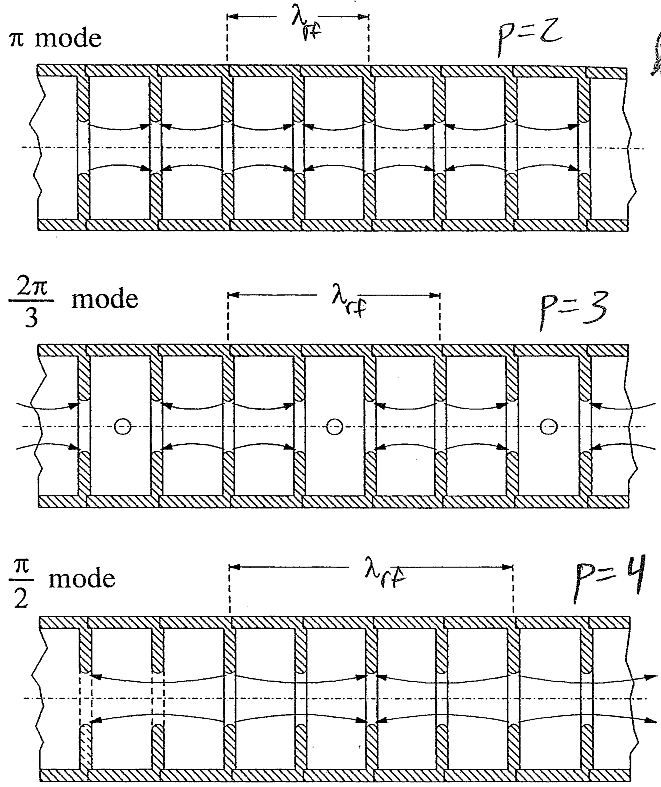
Modified Dispersion Curve



Wille

Fig. 5.7 Dispersion curve for a cylindrical waveguide, with and without irises. The frequency ω is plotted as a function of the wavenumber k_z of the waveguide.

Analogously to cavities in a linac, various wave symmetries are compatible with acceleration!



$k_z d = \pi$ ← Requires long settle time so not good for short pulse

$k_z d = \frac{2\pi}{3}$ ← Short settle time and high shunt impedance. Most common operation

$k_z d = \frac{\pi}{2}$ ← Low shunt impedance so energy gain modest relative to RF power

Wille

Fig. 5.10 The field configurations of the three most important modes in linac structures.

In smooth waveguides all k above cutoff allowed.

For disc loaded waveguide only

$$\lambda_{rf} = p\,d \qquad p = 1, 2, 3, \ldots$$

d = Iris seperation

allowed.

$$\Rightarrow \quad k_z = \frac{2\pi}{\lambda_{rf}} = \frac{2\pi}{pd}$$

$p = 1, 2, 3, \ldots$

Analysis shows the <u>max</u> energy gain of a particle : see Wille

☆ Max $\Longleftrightarrow$ Synchronous condition : $v_z = v_p$

☆ Note: Traveling wave structures often used for <u>relativistic electrons</u> with $\beta \simeq 1$, So there is <u>no need for phase focusing and operation is for max acceleration.</u> The only need is for the bunch to fit into the bucket.

$$\boxed{\Delta W = K \cdot \sqrt{P_{RF}\, l \left(\frac{R_s}{L}\right)} \quad \text{Max Kinetic Energy Gain}}$$

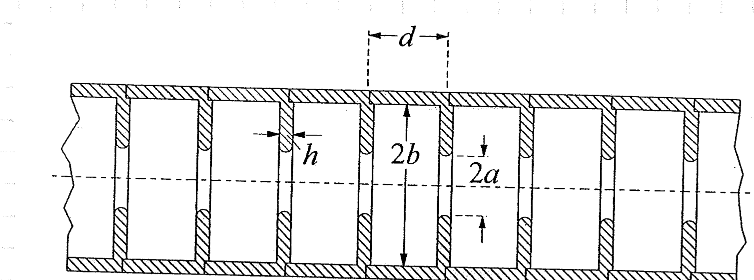
K = correction factor $\simeq 0.8$ typical
P_{RF} = Supplied RF Power
l = Interaction length
$\left(\frac{R_s}{L}\right)$ = Shunt Impedance per meter of structure.

Emperical formula applied that works well.

$$\left(\frac{R_s}{L}\right) = 5.12 \times 10^{8} \cdot \frac{\beta(1-\eta)^2}{p + 2.61\,\beta(1-\eta)} \left(\frac{\sin D/2}{D/2}\right)^2$$

$\beta = v_p/c$
$\eta = \frac{h}{d}$
p = # irises per wavelength
$D = \frac{2\pi}{p}(1-\eta)$

SLAC Structure



$2b = 82.5$ mm
$2a = 22.6$ mm
$h = 5.8$ mm
$d = 35.0$ mm

For the SLAC traveling wave linacs:

$\beta \simeq 1$ Relativistic e^-

$p = 3$ $\frac{2\pi}{3}$ Mode

$$\left(\frac{R_s}{L}\right) = .53\times10^6 \frac{\Omega}{\text{meter}}$$

$$P_{RF} = 35\,MW$$

$$\ell = 3\,\text{meter}$$

Gives

$$\Delta W = K\sqrt{P_{RF}\,\ell\left(\frac{R_s}{L}\right)} = 59.7\,MV$$

$$\Rightarrow \frac{\Delta W}{\ell} = \frac{59.7\,MV}{3\,\text{meter}} = 19.9\,\frac{MV}{\text{meter}}$$

Typical Values achieved for traveling wave structures $15\frac{MV}{m} \to 20\frac{MV}{m}$

Values $\sim 100\frac{MV}{m}$ possible for short structures.

Typically only short pulses possible to avoid heat damage to normal conducting structure.

Want very high gradients for any future linear e^+e^- collider?

- Otherwise linac too long
- Synchrotron radiation precludes ring.