13.lec Transverse Space-Charge Effects*

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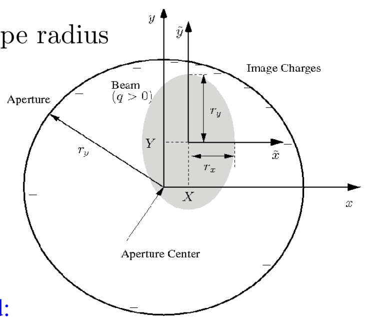
Transverse Space-Charge Effects: Outline

S1: Overview

Analyze transverse centroid and envelope properties of an unbunched $(\partial/\partial z = 0)$ beam

$r_p =$ pipe radius



Expect for linearly focused beam with intense space-charge:
- Beam to look roughly elliptical in shape
- Nearly uniform density within fairly sharp edge

Transverse averages:

$$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \; \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \; f_\perp}$$

Centroid:

$$\boxed{\begin{aligned} X &= \langle x \rangle_\perp \\ Y &= \langle y \rangle_\perp \end{aligned}}$$

x- and *y*-coordinates of beam "center of mass"

Envelope: (edge measure)

$$\boxed{\begin{aligned} r_x &= 2\sqrt{\langle (x-X)^2 \rangle_\perp} \\ r_y &= 2\sqrt{\langle (y-Y)^2 \rangle_\perp} \end{aligned}}$$

x- and *y*-principal axis radii of an elliptical beam envelope
- Apply to general $f_\perp$ but base on uniform density $f_\perp$
- Factor of 2 results from dimensionality (diff 1D and 3D)

Apply the definition of mean-square radius in *x*:

$$\langle (x-X)^2 \rangle_\perp = \frac{\int d^2x' \int d^2x \; (x-X)^2 f_\perp}{\int d^2x' \int d^2x \; f_\perp}$$

Beam distribution function:

$$f_\perp = f_\perp(x, y, x', y'; s)$$

Take norm:

$$n(x, y; s) = \int d^2x' \; f_\perp \;\; = \text{Density}$$

Then:

$$\langle (x-X)^2 \rangle_\perp = \frac{\int d^2x' \int d^2x \; (x-X)^2 f_\perp}{\int d^2x' \int d^2x \; f_\perp} = \frac{\int d^2x \; (x-X)^2 n}{\int d^2x \; n}$$

For a uniform density elliptical beam:

$$n = \begin{cases} \hat{n} = \text{const}, & \text{if } (x-X)^2/r_x^2 + (y-Y)^2/r_y^2 < 1 \\ 0, & \text{if } (x-X)^2/r_x^2 + (y-Y)^2/r_y^2 > 1 \end{cases}$$

Transform the elliptical region within the beam to a unit sphere to more easily carry out the integration in the mean-square radius:

$$x - X = r_x \eta \cos\psi \qquad d^2x = dxdy = r_x r_y \eta d\eta d\psi$$
$$y - Y = r_y \eta \sin\psi \qquad \Longrightarrow \qquad \int_{\text{ellipse}} d^2x \; \cdots = r_x r_y \int_{-\pi}^{\pi} d\psi \int_0^1 d\eta\eta \; \cdots$$

Giving:

$$\langle (x-X)^2 \rangle_\perp = \frac{\int d^2x \; (x-X)^2 n}{\int d^2x \; n}$$
$$= \frac{\hat{n} r_x r_y \int_{-\pi}^{\pi} d\psi \cos^2\psi \; \int_0^1 d\eta\eta \; \eta^2}{\hat{n}\pi r_x r_y} = \frac{r_x^2}{4}$$

and similar in y to show that:

$$\boxed{\begin{aligned} r_x &= 2\sqrt{\langle (x-X)^2 \rangle_\perp} \\ r_y &= 2\sqrt{\langle (y-Y)^2 \rangle_\perp} \end{aligned}}$$

//Aside: Edge Radius Measures and Dimension

The coefficient of rms edge measures of "radii" of a uniform density beam depends on dimension:

1D: Uniform Sheet Beam:

- For accelerator equivalent model details see:
  Lund, Friedman, Bazouin PRSTAB **14,** 054201 (2011)

$$\boxed{x_{\text{width}} \equiv \sqrt{3}\langle \tilde{x}^2 \rangle^{1/2}}$$

2D: Uniform Elliptical Cross-Section:

- See homework problems

$$\boxed{\begin{aligned} r_x &\equiv 2\langle \tilde{x}^2 \rangle_\perp^{1/2} \\ r_y &\equiv 2\langle \tilde{y}^2 \rangle_\perp^{1/2} \end{aligned}}$$

3D: Uniformly Filled Ellipsoid:

- See JJ Barnard Lectures on a mismatched ellipsoidal bunch and and Barnard and Lund, PAC 9VO18 (1997)

Axisymmetric Transverse

$$\boxed{\begin{aligned} r_\perp &\equiv \sqrt{5/2}\langle \tilde{x}^2 + \tilde{y}^2 \rangle^{1/2} \\ r_z &\equiv \sqrt{5}\langle \tilde{z}^2 \rangle^{1/2} \end{aligned}}$$

3D

$$\boxed{\begin{aligned} r_x &\equiv \sqrt{5}\langle \tilde{x}^2 \rangle^{1/2} \\ r_y &\equiv \sqrt{5}\langle \tilde{y}^2 \rangle^{1/2} \\ r_z &\equiv \sqrt{5}\langle \tilde{z}^2 \rangle^{1/2} \end{aligned}}$$

General case uniform density beam:

- For dimension d, the coordinate average along the $j = x, y, z$

$$\boxed{r_j = \sqrt{2+d}\langle \tilde{x}_j^2 \rangle_\perp}$$

///

Oscillations in the statistical beam centroid and envelope radii are the *lowest-order* collective responses of the beam

Centroid Oscillations: Associated with errors and are suppressed to the extent possible:

- Error Sources seeding/driving oscillations:
  - Beam distribution assymetries (even emerging from injector: born offset)
  - Dipole bending terms from imperfect applied field optics
  - Dipole bending terms from imperfect mechanical alignment
- Exception: Large centroid oscillations desired when the beam is kicked (insertion or extraction) into or out of a transport channel as is done in beam insertion/extraction in/out of rings

Envelope Oscillations: Can have two components in periodic focusing lattices

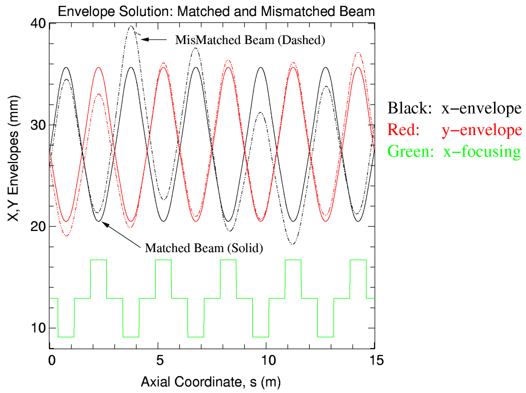
1) Matched Envelope: Periodic "flutter" synchronized to period of focusing lattice to maintain best radial confinement of the beam
   - Properly tuned flutter essential in Alternating Gradient quadrupole lattices

2) Mismatched Envelope: Excursions deviate from matched flutter motion and are seeded/driven by errors

Limiting maximum beam-edge excursions is desired for economical transport
- Reduces cost by Limiting material volume needed to transport an intense beam
- Reduces generation of halo and associated particle loses

Mismatched beams have larger envelope excursions and have more collective stability and beam halo problems since mismatch adds another source of free energy that can drive statistical increases in particle amplitudes

Example: FODO Quadrupole Transport Channel



- Larger machine aperture is needed to confine a mismatched beam
  - Even in absence of beam halo and other mismatch driven "instabilities"

Centroid and Envelope oscillations are the *most important collective modes* of an intense beam

- Force balances based on matched beam envelope equation predict scaling of transportable beam parameters
  - Used to design transport lattices
- Instabilities in beam centroid and/or envelope oscillations can prevent reliable transport
  - Parameter locations of instability regions should be understood and avoided in machine design/operation

Although it is *necessary* to avoid envelope and centroid instabilities in designs, it is not alone *sufficient* for effective machine operation

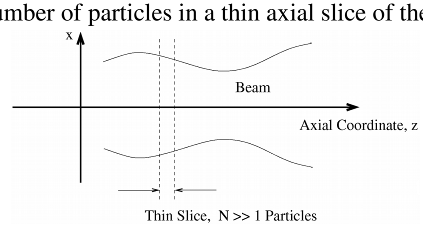
- Higher-order kinetic and fluid instabilities not expressed in the low-order envelope models can can degrade beam quality and control and must also be evaluated
  - see: USPAS lectures on Beam Physics with Intense Space-Charge

S2: Derivation of Transverse Centroid and Envelope Equations of Motion

Analyze centroid and envelope properties of an unbunched $(\partial/\partial z = 0)$ beam

Transverse Statistical Averages:

Let N be the number of particles in a thin axial slice of the beam at axial coordinate s.



Averages can be equivalently defined in terms of the discreet particles making up the beam or the continuous model transverse Vlasov distribution function:

particles: $$\langle \cdots \rangle_\perp \equiv \frac{1}{N}\sum_{i=1}^{N}\Big|_{\text{slice}} \cdots$$

distribution: $$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \; \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \; f_\perp}$$

- Averages can be generalized to include axial momentum spread

Transverse Particle Equations of Motion

Consistent with earlier analysis [lectures on Transverse Particle Dynamics], take:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

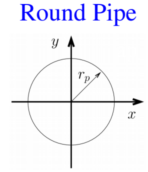
$$\nabla_\perp^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0}$$

$$\rho = q\int d^2x'_\perp \; f_\perp \qquad \phi|_{\text{aperture}} = 0$$

Assume:
- Unbunched beam
- No axial momentum spread
- Linear applied focusing fields described by $\kappa_x, \; \kappa_y$
- Possible acceleration: $\gamma_b\beta_b$ need not be constant

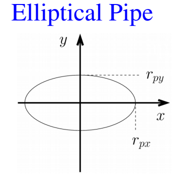
Various apertures are possible influence solution for ϕ. Some simple examples:

Round Pipe



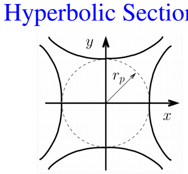
Linac magnetic quadrupoles, acceleration cells, ...

Elliptical Pipe



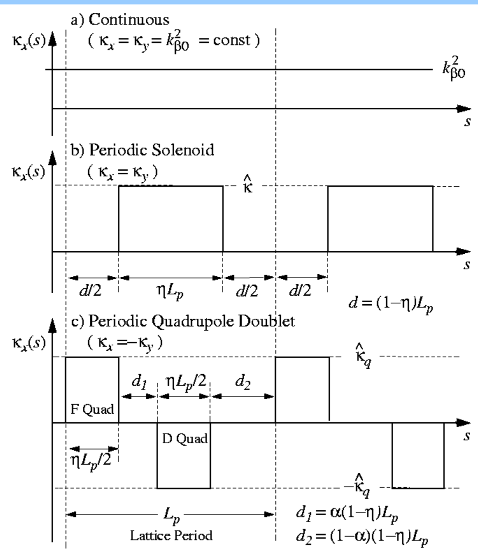
In rings with dispersion: in drifts, magnetic optics, ...

Hyperbolic Sections



Electric quadrupoles

Review: Focusing lattices we will take in examples: Continuous and piecewise constant periodic solenoid and quadrupole doublet



Lattice Period L_p

Occupancy η

$\eta \in [0, 1]$

Solenoid description carried out implicitly in Larmor frame
[see: S.M. Lund lectures on Transverse Particle Dynamics]

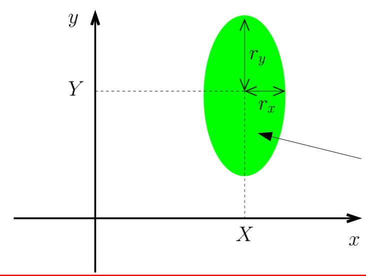
Syncopation Factor α

$$\alpha \in [0, \frac{1}{2}]$$

$$\alpha = \frac{1}{2} \implies FODO$$

Distribution Assumptions

To lowest order, due to Debye screening to applied focusing forces, linearly focused intense beams are expected to be nearly uniform in density within the core of the beam out to an spatial edge where the density falls rapidly to zero



Charge conservation requires:

$\lambda = \text{const}$

Uniform density within beam:

$$\rho = \frac{\lambda}{\pi r_x r_y}$$

$$\rho(x,y) = q \int d^2x'_\perp \, f_\perp \simeq \begin{cases} \frac{\lambda}{\pi r_x r_y}, & (x-X)^2/r_x^2 + (y-Y)^2/r_y^2 < 1 \\ 0, & (x-X)^2/r_x^2 + (y-Y)^2/r_y^2 > 1 \end{cases}$$

$$\lambda = q \int d^2x_\perp \int d^2x'_\perp \, f_\perp \;=\; \int d^2x \, \rho \;= \text{const}$$

Comments:

- Nearly uniform density out to a sharp spatial beam edge expected for near equilibrium structure beam with strong space-charge due to Debye screening
  - See: USPAS course on Beam Physics with Intense Space-Charge
- Simulations support that uniform density model is a good approximation for stable non-equilibrium beams when space-charge is high
  - Variety of initial distributions launched and, where stable, rapidly relax to a fairly uniform charge density core
  - Low order core oscillations may persist with little problem evident
  - See: USPAS course on Beam Physics with Intense Space-Charge
- Assumption of a fixed form of distribution essentially closes the infinite hierarchy of moments that are needed to describe a general beam distribution
  - Need only describe shape/edge and center for uniform density beam to fully specify the distribution
  - Analogous to closures of fluid theories using assumed equations of state etc.
  - Obviously miss much of physics of true collective response where space charge waves are likely to be launched.

Self-Field Calculation

Temporarily, we will consider an *arbitrary* beam charge distribution within an arbitrary aperture to formulate the problem.

Electrostatic field of a line charge in free-space

$$\mathbf{E}_\perp = \frac{\lambda_0}{2\pi\epsilon_0} \frac{(\mathbf{x}_\perp - \tilde{\mathbf{x}})}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}|^2}$$

$\lambda_0 =$ line charge

$\mathbf{x}_\perp = \tilde{\mathbf{x}} =$ coordinate of charge

Resolve the field of the beam into direct (free space) and image terms:

$$\mathbf{E}^s_\perp = -\frac{\partial \phi}{\partial \mathbf{x}_\perp} = \mathbf{E}^d_\perp + \mathbf{E}^i_\perp$$

and superimpose free-space solutions for direct and image contributions

Direct Field

$$\mathbf{E}^d_\perp(\mathbf{x}_\perp) = \frac{1}{2\pi\epsilon_0} \int d^2\tilde{x}_\perp \, \frac{\rho(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp)}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2}$$

$\rho(\mathbf{x}_\perp) =$ beam charge density

Image Field

$$\mathbf{E}^i_\perp(\mathbf{x}_\perp) = \frac{1}{2\pi\epsilon_0} \int d^2\tilde{x}_\perp \, \frac{\rho^i(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp)}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2}$$

$\rho^i(\mathbf{x}_\perp) =$ beam image charge density induced on aperture

// Aside: 2D Field of Line-Charges in Free-Space

$$\nabla_\perp \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \rho(r) = \lambda \frac{\delta(r)}{2\pi r} \qquad \lambda = \int d^2x\, \rho$$

Line charge at origin, apply Gauss' Law to obtain the field as a function of the radial coordinate r :

$$E_r = \frac{\lambda}{2\pi\epsilon_0 r} \qquad \mathbf{E}_\perp = \hat{\mathbf{r}} E_r$$

For a line charge at $\mathbf{x}_\perp = \tilde{\mathbf{x}}_\perp$, shift coordinates and employ vector notation:

$$\boxed{\mathbf{E}_\perp = \frac{\lambda}{2\pi\epsilon_0} \frac{\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2}}$$

Use this and linear superposition for the field due to direct and image charges

- Metallic aperture replaced by collection of images external to the aperture in free-space to calculate consistent fields interior to the aperture

$$\boxed{\mathbf{E}_\perp = \frac{1}{2\pi\epsilon_0} \int d^2x_\perp\, \rho(\tilde{\mathbf{x}}_\perp) \frac{\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2}}$$

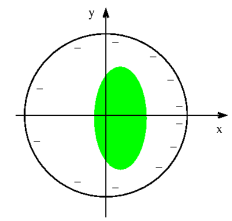
//

Comment on Image Fields

Actual charges on the conducting aperture are induced on a thin (surface charge density) layer on the inner aperture surface. In the method of images, these are replaced by a distribution of charges outside the aperture in vacuum that meet the conducting aperture boundary conditions

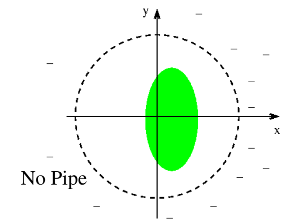
- Field within aperture can be calculated using the images in vacuum
- Induced charges on the inner aperture often called "image charges"
- Magnitude of induced charge on aperture is equal to beam charge and the total charge of the images

Physical



Images

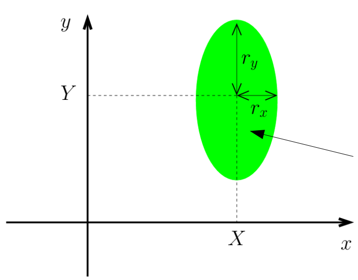
- No pipe
- Schematic only (really continuous image dist)



Direct Field:

The direct field solution for an umbunched uniform density beam in free-space can can be solved analytically

- See: USPAS lectures on Beam Physics with Intense Space-Charge



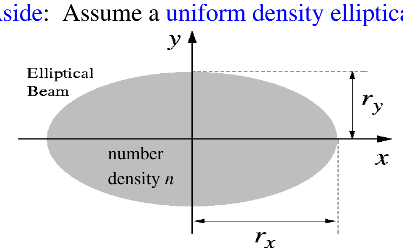
Uniform density in beam:

$$\rho = \frac{\lambda}{\pi r_x r_y} = \text{const}$$

$$\boxed{\begin{aligned} E_x^d &= \frac{\lambda}{\pi\epsilon_0} \frac{x - X}{(r_x + r_y) r_x} \\ E_y^d &= \frac{\lambda}{\pi\epsilon_0} \frac{y - Y}{(r_x + r_y) r_y} \end{aligned}}$$

Expressions are valid only *within the elliptical density beam* -- where they will be applied in taking averages

// Aside: Assume a uniform density elliptical beam in a periodic focusing lattice



Line-Charge:

$$\lambda = qn(s)\pi r_x(s) r_y(s) = \text{const} \quad \text{(charge conservation)}$$

Beam Edge:

$$\frac{x^2}{r_x^2(s)} + \frac{y^2}{r_y^2(s)} = 1 \quad \text{(ellipse)}$$

Free-space self-field solution within the beam (see USPAS: Beam Physics with Intense Space Charge) is:

- This is a non-trivial solution: originally derived in Astrophysics in Classical gravitational models of stars with ellipsoidal density profiles

$$\phi = -\frac{\lambda}{2\pi\epsilon_0}\left[\frac{x^2}{(r_x+r_y)r_x} + \frac{y^2}{(r_x+r_y)r_y}\right] + \text{const}$$

$$\Longrightarrow \boxed{\begin{aligned} -\frac{\partial\phi}{\partial x} &= \frac{\lambda}{\pi\epsilon_0} \frac{x}{(r_x+r_y)r_x} \\ -\frac{\partial\phi}{\partial y} &= \frac{\lambda}{\pi\epsilon_0} \frac{y}{(r_x+r_y)r_y} \end{aligned}}$$

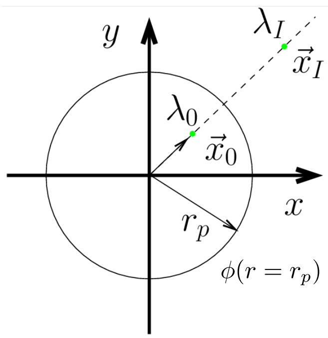
valid only within the beam!

- Nonlinear outside beam

//

Image Field:

Image structure depends on the aperture. Assume a round pipe (most common case) for simplicity.



$\lambda_I = -\lambda_0$ image charge

$\mathbf{x}_I = \frac{r_p^2}{|\mathbf{x}_0|^2}\mathbf{x}_0$ image location

Will be derived in the the problem sets.

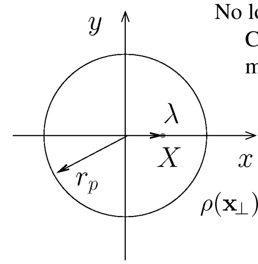
Superimpose all images of beam to obtain the image contribution in aperture:

$$\boxed{\mathbf{E}^i_\perp(\mathbf{x}_\perp) = -\frac{1}{2\pi\epsilon_0}\int_{\text{pipe}} d^2\tilde{x}_\perp \frac{\rho(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - r_p^2\tilde{\mathbf{x}}_\perp/|\tilde{\mathbf{x}}_\perp|^2)}{|\mathbf{x}_\perp - r_p^2\tilde{\mathbf{x}}_\perp/|\tilde{\mathbf{x}}_\perp|^2|^2}}$$

- Difficult to calculate even for ρ corresponding to a uniform density beam

Examine limits of the image field to build intuition on the range of properties:

1) Line charge along x-axis:



No loss in generality:
Can always choose coordinates to make charge lie on axis

$\rho(\mathbf{x}_\perp) = \lambda\delta(\mathbf{x}_\perp - X\hat{\mathbf{x}})$

$$\mathbf{E}^i_\perp = \frac{\lambda^i}{2\pi\epsilon_0}\frac{\mathbf{x}_\perp - \mathbf{x}^i_\perp}{|\mathbf{x}_\perp - \mathbf{x}^i_\perp|^2}$$

$$\lambda^i = -\lambda$$

$$\mathbf{x}^i_\perp = \frac{r_p^2}{X}\hat{\mathbf{x}}$$

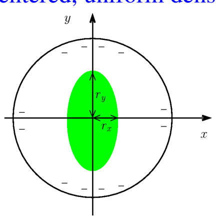
Plug this density in the image charge expression for a round-pipe aperture:

- Need only evaluate at $\mathbf{x}_\perp = X\hat{\mathbf{x}}$ since beam is at that location

$$\boxed{\mathbf{E}^i_\perp(\mathbf{x}_\perp = X\hat{\mathbf{x}}) = \frac{\lambda}{2\pi\epsilon_0(r_p^2/X - X)}\hat{\mathbf{x}}}$$

- Generates nonlinear field at position of direct charge
- Field creates attractive force between direct and image charge
  - Therefore image charge should be expected to "drag" centroid further off
  - Amplitude of centroid oscillations expected to increase if not corrected (steering)

2) Centered, uniform density elliptical beam:



$$\rho(\mathbf{x}_\perp) = \begin{cases} \frac{\lambda}{\pi r_x r_y}, & x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}$$

Expand using complex coordinates starting from the general image expression:

- Image field is in vacuum aperture so complex methods help calculation
- Follow procedures in Multipole Models of applied focusing fields

$$\underline{E}^{i*} = E^i_x - iE^i_y = \sum_{n=2,4,\cdots}^{\infty} \underline{c}_n \underline{z}^{n-1} \qquad \underline{c}_n = \frac{1}{2\pi\epsilon_0}\int_{\text{pipe}} d^2x_\perp\, \rho(\mathbf{x}_\perp)\frac{(x-iy)^n}{r_p^{2n}}$$

$$\underline{z} = x + iy \qquad i = \sqrt{-1} \qquad = \frac{\lambda n!}{2\pi\epsilon_0 2^n (n/2+1)!(n/2)!}\left(\frac{r_x^2 - r_y^2}{r_p^4}\right)^{n/2}$$

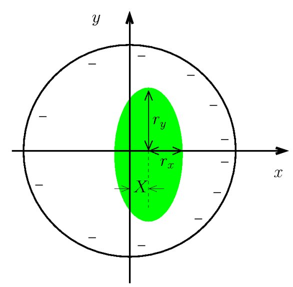
The linear (n = 2) components of this expansion give:

$$\boxed{E^i_x = \frac{\lambda}{8\pi\epsilon_0}\frac{r_x^2 - r_y^2}{r_p^4}x, \qquad E^i_y = -\frac{\lambda}{8\pi\epsilon_0}\frac{r_x^2 - r_y^2}{r_p^4}y}$$

- Rapidly vanish (higher order n terms more rapidly) as beam becomes more round
- Case will be analyzed further in the problem sets

3) Uniform density elliptical beam with a small displacement along the x-axis:

$Y = 0 \qquad |X|/r_p \ll 1$



Expand using complex coordinates starting from the general image expression:

- Complex coordinates help simplify very messy calculation

E.P. Lee, E. Close, and L. Smith, Nuclear Instruments and Methods, 1126 (1987)

Leading order terms expanded in $|X|/r_p$ without assuming small ellipticity obtain:

$$E^i_x = \frac{\lambda}{2\pi\epsilon_0 r_p^2}\left[f\cdot(x-X)+g\cdot X\right]+\Theta\left(\frac{X}{r_p}\right)^3$$

$$E^i_y = -\frac{\lambda}{2\pi\epsilon_0 r_p^2} f\cdot y+\Theta\left(\frac{X}{r_p}\right)^3$$

Where *f* and *g* are focusing and bending coefficients that can be calculated in terms of $X,\ r_x,\ r_y$ (which all may vary in *s*) as:

FocusingTerm:

$$f=\frac{r_x^2-r_y^2}{4r_p^2}+\frac{X^2}{r_p^2}\left[1+\frac{3}{2}\left(\frac{r_x^2-r_y^2}{r_p^2}\right)+\frac{3}{8}\left(\frac{r_x^2-r_y^2}{r_p^2}\right)^2\right]$$

BendingTerm:

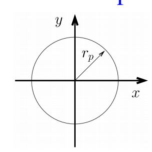
$$g=1+\frac{r_x^2-r_y^2}{4r_p^2}+\frac{X^2}{r_p^2}\left[1+\frac{3}{4}\left(\frac{r_x^2-r_y^2}{r_p^2}\right)+\frac{1}{8}\left(\frac{r_x^2-r_y^2}{r_p^2}\right)^2\right]$$

- Expressions become even more complicated with simultaneous *x*- and *y*-displacements and more complicated aperture geometries !
- *f* quickly become weaker as the beam becomes more round and/or for a larger pipe
- Similar comments apply to *g* other than it has a term that remains for a round beam

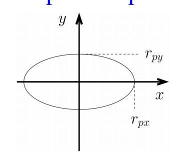
Comments on images:

- Sign is generally such that it will tend to increase beam centroid displacements
  - Also (usually) weak linear focusing corrections for an elliptical beam
- Can be very difficult to calculate explicitly
  - Even for simple case of circular pipe
  - Special cases of simple geometry and case formulas help clarify scaling
  - Generally suppress by making the beam small relative to characteristic aperture dimensions and keeping the beam steered near-axis
  - Simulations typically applied
- Depend strongly on the aperture geometry
  - Generally varies as a function of *s* in the machine aperture due to changes in accelerator lattice elements and/or as beam symmetries evolve

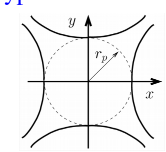
Round Pipe



Elliptical Pipe



Hyperbolic Sections



Coupled centroid and envelope equations of motion for a uniform density elliptical beam

Consistent with the assumed structure of the distribution (uniform density elliptical beam), denote:

Beam Centroid: (phase-space)

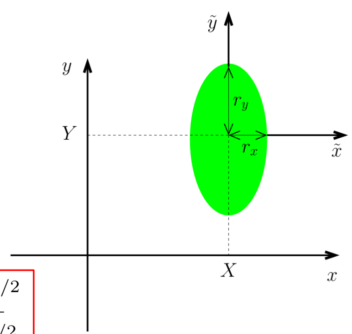
$$X\equiv\langle x\rangle_\perp \qquad X'=\langle x'\rangle_\perp$$
$$Y\equiv\langle y\rangle_\perp \qquad Y'=\langle y'\rangle_\perp$$

Coordinates with respect to centroid:

$$\tilde{x}\equiv x-X \qquad \tilde{x}'=x'-X'$$
$$\tilde{y}\equiv y-Y \qquad \tilde{y}'=y'-Y'$$

Envelope Edge Radii: (phase-space)

$$r_x\equiv 2\sqrt{\langle\tilde{x}^2\rangle_\perp} \qquad r_x'=2\langle\tilde{x}\tilde{x}'\rangle_\perp/\langle\tilde{x}^2\rangle_\perp^{1/2}$$
$$r_y\equiv 2\sqrt{\langle\tilde{y}^2\rangle_\perp} \qquad r_y'=2\langle\tilde{y}\tilde{y}'\rangle_\perp/\langle\tilde{y}^2\rangle_\perp^{1/2}$$



With the *assumed* uniform elliptical beam, all moments can be calculated in terms of: $X, X'\ \ Y, Y' \qquad r_x, r_x'\ \ r_y, r_y'$

- Such truncations follow when the form of the distribution is "frozen"

Derive 2nd order equations of motion to describe the evolution of the beam centroid and envelope.

- Derive by taking averages over the equations of motion while applying the assumed (uniform density) form of the beam distribution
- Cast equations of motion in a form that allows easy interpretation

Derive centroid equations: First use the self-field resolution for a uniform density beam, then the equations of motion for a particle within the beam are:

Perveance: $Q \equiv \dfrac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2}$ (not necessarily constant if beam accelerates)

average equations using: $\langle x' \rangle_\perp = \langle x \rangle_\perp' = X'$ etc., to obtain:

Centroid Equations: (see derivation steps next slide)

$$\boxed{\begin{aligned} X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \kappa_x X &= Q\left[\frac{2\pi\epsilon_0}{\lambda}\langle E_x^i\rangle_\perp\right] \\ Y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}Y' + \kappa_y Y &= Q\left[\frac{2\pi\epsilon_0}{\lambda}\langle E_y^i\rangle_\perp\right] \end{aligned}}$$

Note: the electric image field will cancel the coefficient $2\pi\epsilon_0/\lambda$

$$\mathbf{E}_\perp^i = \frac{1}{2\pi\epsilon_0}\int d^2\tilde{x}_\perp \frac{\rho^i(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp)}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2}$$

- $\langle E_x^i\rangle_\perp$ will generally depend on: X, Y and r_x, r_y

1) Derivation of centroid equations of motion

Start with particle equation of motion:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

Use (valid within beam):

$$-\frac{\partial\phi}{\partial x} = \underbrace{\frac{\lambda}{\pi\epsilon_0}\frac{x-X}{(r_x+r_y)r_x}}_{\text{Direct}} + \underbrace{E_x^i}_{\text{Image}}$$

$$\implies \quad x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x - \frac{2Q}{(r_x+r_y)r_x}(x-X) = \frac{q}{m\gamma_b^3\beta_b^2c^2}E_x^i$$

Perveance:

$$Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2}$$

Image Field:

$$\mathbf{E}_\perp^i = \frac{1}{2\pi\epsilon_0}\int d^2\tilde{x}_\perp \frac{\rho^i(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp)}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2}$$

Giving (valid within beam):

$$\boxed{\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x - \frac{2Q}{(r_x+r_y)r_x}(x-X) &= \frac{q}{m\gamma_b^3\beta_b^2c^2}E_x^i \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y y - \frac{2Q}{(r_x+r_y)r_y}(y-Y) &= \frac{q}{m\gamma_b^3\beta_b^2c^2}E_y^i \end{aligned}}$$

Direct Terms Image Terms

Equation of motion:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x - \frac{2Q}{(r_x+r_y)r_x}(x-X) = \frac{q}{m\gamma_b^3\beta_b^2c^2}E_x^i$$

Take average of equation of motion pulling through terms that depend on on *s*:

$$\langle\cdots\rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \ \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \ f_\perp}$$

$$\langle x''\rangle_\perp + \langle\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x'\rangle_\perp + \langle\kappa_x x\rangle_\perp - \langle\frac{2Q}{(r_x+r_y)r_x}(x-X)\rangle_\perp = \langle\frac{q}{m\gamma_b^3\beta_b^2c^2}E_x^i\rangle_\perp$$

$$\begin{aligned} \langle x\rangle_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\langle x'\rangle_\perp + \kappa_x\langle x\rangle_\perp - \frac{2Q}{(r_x+r_y)r_x}\langle x-X\rangle_\perp \\ = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2}\left[\frac{2\pi\epsilon_0}{\lambda}\right]\langle E_x^i\rangle_\perp \\ = Q\left[\frac{2\pi\epsilon_0}{\lambda}\right]\langle E_x^i\rangle_\perp \end{aligned}$$

Perveance:

$$Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2}$$

$$\begin{aligned} \langle x\rangle_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\langle x'\rangle_\perp + \kappa_x\langle x\rangle_\perp - \frac{2Q}{(r_x+r_y)r_x}\langle x-X\rangle_\perp \\ = Q\left[\frac{2\pi\epsilon_0}{\lambda}\right]\langle E_x^i\rangle_\perp \end{aligned}$$

Use:

$$X = \langle x\rangle_\perp \qquad X' = \langle x'\rangle_\perp$$

$$\langle x - X\rangle_\perp = X - X = 0$$

$$\implies \quad X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \kappa_x X = Q\left[\frac{2\pi\epsilon_0}{\lambda}\langle E_x^i\rangle_\perp\right]$$

+ Analogous equation obtained in *y*-plane

$$\boxed{\begin{aligned} X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \kappa_x X &= Q\left[\frac{2\pi\epsilon_0}{\lambda}\langle E_x^i\rangle_\perp\right] \\ Y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}Y' + \kappa_y Y &= Q\left[\frac{2\pi\epsilon_0}{\lambda}\langle E_y^i\rangle_\perp\right] \end{aligned}}$$

2) Derivation of envelope equation of motion

To derive equations of motion for the envelope radii, r_x, r_y 1st subtract the centroid equations from the particle equations of motion:

Particle equation:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x - \frac{2Q}{(r_x+r_y)r_x}(x-X) = \frac{q}{m\gamma_b^3\beta_b^2c^2}E_x^i$$

Subtract centroid equation:

$$X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \kappa_x X = Q\left[\frac{2\pi\epsilon_0}{\lambda}\langle E_x^i\rangle_\perp\right]$$

Giving:

$$\tilde{x} \equiv x - X$$
$$\tilde{x}' \equiv x' - X'$$

$$\boxed{\begin{aligned}\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa_x\tilde{x} - \frac{2Q\tilde{x}}{(r_x+r_y)r_x} &= \frac{q}{m\gamma_b^3\beta_b^2c^2}\left[E_x^i - \langle E_x^i\rangle_\perp\right]\\ \tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa_y\tilde{y} - \frac{2Q\tilde{y}}{(r_x+r_y)r_x} &= \frac{q}{m\gamma_b^3\beta_b^2c^2}\left[E_y^i - \langle E_y^i\rangle_\perp\right]\end{aligned}}$$

Next, differentiate the equation for the envelope radius twice:

$$r_x = 2\langle\tilde{x}^2\rangle_\perp^{1/2}$$

$\Longrightarrow$ 1st derivative: $$r_x' = \frac{2\langle\tilde{x}\tilde{x}'\rangle_\perp}{\langle\tilde{x}^2\rangle_\perp^{1/2}} = \frac{4\langle\tilde{x}\tilde{x}'\rangle_\perp}{r_x}$$

2nd derivative:
$$\begin{aligned} r_x'' &= \frac{2\langle\tilde{x}\tilde{x}''\rangle_\perp}{\langle\tilde{x}^2\rangle_\perp^{1/2}} + \frac{2\langle\tilde{x}'^2\rangle_\perp}{\langle\tilde{x}^2\rangle_\perp^{1/2}} - \frac{2\langle\tilde{x}\tilde{x}'\rangle_\perp^2}{\langle\tilde{x}^2\rangle_\perp^{3/2}}\\ &= 4\frac{\langle\tilde{x}\tilde{x}''\rangle_\perp}{[2\langle\tilde{x}^2\rangle_\perp^{1/2}]} + \frac{16\left[\langle\tilde{x}^2\rangle_\perp\langle\tilde{x}'^2\rangle_\perp - \langle\tilde{x}\tilde{x}'\rangle_\perp^2\right]}{[2\langle\tilde{x}^2\rangle_\perp^{1/2}]^3}\\ &= 4\frac{\langle\tilde{x}\tilde{x}''\rangle_\perp}{r_x} - \frac{16\left[\langle\tilde{x}^2\rangle_\perp\langle\tilde{x}'^2\rangle_\perp - \langle\tilde{x}\tilde{x}'\rangle_\perp^2\right]}{r_x^3}\end{aligned}$$

Define a statistical rms edge emittance:

$$\boxed{\varepsilon_x \equiv 4\varepsilon_{x,\text{rms}} \equiv 4\left[\langle\tilde{x}^2\rangle_\perp\langle\tilde{x}'^2\rangle_\perp - \langle\tilde{x}\tilde{x}'\rangle_\perp^2\right]^{1/2}}$$

Then we have:

$$\boxed{\begin{aligned} r_x'' &= 4\frac{\langle\tilde{x}\tilde{x}''\rangle_\perp}{r_x} + \frac{16[\langle\tilde{x}^2\rangle_\perp\langle\tilde{x}'^2\rangle_\perp - \langle\tilde{x}\tilde{x}'\rangle_\perp^2]}{r_x^3}\\ &= 4\frac{\langle\tilde{x}\tilde{x}''\rangle_\perp}{r_x} + \frac{\varepsilon_x^2}{r_x^3}\end{aligned}}$$

and employ the equations of motion to eliminate $\tilde{x}''$ in $\langle\tilde{x}\tilde{x}''\rangle_\perp$ with steps below

Using the equation of motion:

$$\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa_x\tilde{x} - \frac{2Q\tilde{x}}{(r_x+r_y)r_x} = \frac{q}{m\gamma_b^3\beta_b^2c^2}\left[E_x^i - \langle E_x^i\rangle_\perp\right]$$

Multiply the equation by $\tilde{x}$, average, and pull s-varying coefficients and constants through the average terms to obtain

$$\begin{aligned}\langle\tilde{x}\tilde{x}''\rangle_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\langle\tilde{x}\tilde{x}'\rangle_\perp + \kappa_x\langle\tilde{x}^2\rangle_\perp - \frac{2Q\langle\tilde{x}^2\rangle_\perp}{(r_x+r_y)r_x}\\ = \frac{q}{m\gamma_b^3\beta_b^2c^2}\left[\langle\tilde{x}E_x^i\rangle_\perp - \langle\tilde{x}\langle E_x^i\rangle_\perp\rangle_\perp\right]\end{aligned}$$

But:

$$\langle\tilde{x}\langle E_x^i\rangle_\perp\rangle_\perp = \cancelto{0}{\langle\tilde{x}\rangle_\perp}\langle E_x^i\rangle_\perp = 0$$

Giving:

$$\langle\tilde{x}\tilde{x}''\rangle_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\langle\tilde{x}\tilde{x}'\rangle_\perp + \kappa_x\langle\tilde{x}^2\rangle_\perp - \frac{2Q\langle\tilde{x}^2\rangle_\perp}{(r_x+r_y)r_x} = \frac{q}{m\gamma_b^3\beta_b^2c^2}\langle\tilde{x}E_x^i\rangle_\perp$$

$$\langle\tilde{x}\tilde{x}''\rangle_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\frac{r_x r_x'}{4} + \kappa_x\frac{r_x^2}{4} - \frac{Qr_x/2}{r_x+r_y} = \frac{q}{m\gamma_b^3\beta_b^2c^2}\langle\tilde{x}E_x^i\rangle_\perp$$

Using this moment in the equation for r_x''

$$r_x'' = 4\frac{\langle\tilde{x}\tilde{x}''\rangle_\perp}{r_x} + \frac{\varepsilon_x^2}{r_x^3}$$

then gives the envelope equation with the image charge couplings as:

Envelope Equations:

$$\boxed{\begin{aligned} r_x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}r_x' + \kappa_x r_x - \frac{2Q}{r_x+r_y} - \frac{\varepsilon_x^2}{r_x^3} = 8Q\left[\frac{\pi\epsilon_0}{\lambda}\langle\tilde{x}E_x^i\rangle_\perp\right] \\ r_y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}r_x' + \kappa_y r_y - \frac{2Q}{r_x+r_y} - \frac{\varepsilon_y^2}{r_y^3} = 8Q\left[\frac{\pi\epsilon_0}{\lambda}\langle\tilde{y}E_y^i\rangle_\perp\right] \end{aligned}}$$

- $\langle\tilde{x}E_x^i\rangle_\perp$ will generally depend on: $X,\ Y$ and $r_x,\ r_y$

Comments on Centroid/Envelope equations:

- Centroid and envelope equations are *coupled* and must be solved simultaneously when image terms on the RHS cannot be neglected
- Image terms contain nonlinear terms that can be difficult to evaluate explicitly
  - Aperture geometry changes image correction
- The formulation is not self-consistent because a frozen form (uniform density) charge profile is assumed
  - Uniform density choice motivated by KV results and Debye screening see: USPAS, lectures on Beam Physics with Intense Space-Charge
  - The assumed distribution form not evolving represents a fluid model closure
  - Typically find with simulations that uniform density frozen form distribution models can provide reasonably accurate approximate models for centroid and envelope evolution

Comments on Centroid/Envelope equations (Continued):

- Constant (normalized when accelerating) emittances are generally assumed
  - For strong space charge emittance terms small and limited emittance evolution does not strongly influence evolution outside of final focus

$\beta_b,\ \gamma_b,\ \lambda$ s-variation set by acceleration schedule

$$\begin{aligned} \varepsilon_{nx} &= \gamma_b\beta_b\varepsilon_x = \text{const} \\ \varepsilon_{ny} &= \gamma_b\beta_b\varepsilon_y = \text{const} \end{aligned} \quad\longrightarrow\quad \text{used to calculate } \varepsilon_x,\ \varepsilon_y$$

$$Q = \frac{q\lambda}{2\pi m\epsilon_0\gamma_b^3\beta_b^2c^2}$$

Interpretation of the dimensionless perveance Q

The dimensionless perveance:

$$\boxed{Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2}} \qquad \begin{aligned} \lambda &= q\hat{n}\pi r_x r_y = \text{line-charge} = \text{const} \\ \hat{n} &= \text{beam density} \end{aligned}$$

- Scales with size of beam (λ), but typically has small characteristic values even for beams with high space charge intensity (~ 10^{-4} to 10^{-8} common)
- Even small values of Q can matter depending on the relative strength of other effects from applied focusing forces, thermal defocusing, etc.

Can be expressed equivalently in several ways:

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \frac{qI_b}{2\pi\epsilon_0 m\gamma_b^3\beta_b^3c^3} = \frac{2}{(\gamma_b\beta_b)^3}\frac{I_b}{I_A}$$

$$= \frac{q^2\pi r_x r_y\hat{n}}{2\pi\epsilon_0 m\gamma_b^3\beta_b^3c^3} = \frac{\hat{\omega}_p^2 r_x r_y}{2\gamma_b^3\beta_b^2c^2}$$

$$\begin{aligned} I_b &= \lambda\beta_b c = \text{beam current} \\ I_A &= 4\pi\epsilon_0 mc^3/q = \text{Alfven current} \\ \hat{\omega}_p &= \sqrt{q^2\hat{n}/(m\epsilon_0)} = \text{plasma freq.} \end{aligned}$$

- Forms based on $\lambda,\ I_b$ generalize to *nonuniform density beams*

To better understand the perveance Q, consider a round, uniform density beam with

$$r_x = r_y = r_b$$

then the solution for the potential within the beam reduces:

$$\begin{aligned} \phi &= -\frac{\lambda}{2\pi\epsilon_0}\left[\frac{x^2}{(r_x+r_y)r_x} + \frac{y^2}{(r_x+r_y)r_y}\right] + \text{const} \\ &= -\frac{\lambda}{4\pi\epsilon_0}\frac{r^2}{r_b^2} + \text{const} \end{aligned}$$

$$\Longrightarrow\quad \Delta\phi = \phi(r=0) - \phi(r=r_b) = \frac{\lambda}{4\pi\epsilon_0} \qquad \text{for potential drop across the beam}$$

If the beam is also nonrelativistic, then the axial kinetic energy $\mathcal{E}_b$ is

$$\mathcal{E}_b = (\gamma_b - 1)mc^2 \simeq \frac{1}{2}m\beta_b^2c^2$$

and the perveance can be alternatively expressed as

$$\boxed{Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} \simeq \frac{q\Delta\phi}{\mathcal{E}_b}}$$

- Perveance can be interpreted as space-charge potential energy difference across beam relative to the axial kinetic energy

S3: Centroid Equations of Motion
Single Particle Limit: Oscillation and Stability Properties

Neglect image charge terms, then the centroid equation of motion becomes:

$$\boxed{\begin{aligned} X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \kappa_x X = 0 \\ Y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}Y' + \kappa_y Y = 0 \end{aligned}}$$

- Usual Hill's equation with acceleration term
- Single particle form. Apply results from S.M. Lund lectures on Transverse Particle Dynamics: phase amplitude methods, Courant-Snyder invariants, and stability bounds, ...

Assume that applied lattice focusing is tuned for constant phase advances with normalized coordinates (effective κ_x, κ_y) and/or that acceleration is weak and can be neglected. Then single particle stability results give immediately:

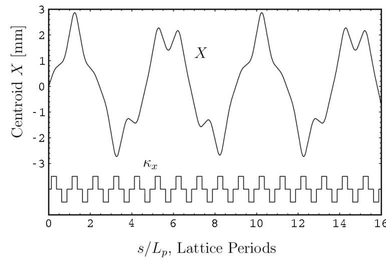
$$\begin{aligned} \frac{1}{2}|\mathrm{Tr}\ \mathbf{M}_x(s_i+L_p|s_i)| \leq 1 \\ \frac{1}{2}|\mathrm{Tr}\ \mathbf{M}_y(s_i+L_p|s_i)| \leq 1 \end{aligned} \quad \Longleftrightarrow \quad \boxed{\begin{aligned} \sigma_{0x} < 180^\circ \\ \sigma_{0y} < 180^\circ \end{aligned}}$$

centroid stability
1st stability condition

/// Example: FODO channel centroid evolution for a coasting beam

Mid-drift launch:

$X(0) = 0$ mm
$X'(0) = 1$ mrad



lattice/beam parameters:

$\gamma_b\beta_b = \text{const}$
$\sigma_{0x} = 80^\circ$
$L_p = 0.5$ m
$\eta = 0.5$

- Centroid exhibits expected characteristic stable betatron oscillations
  - Stable so oscillation amplitude does not grow
  - Courant-Snyder invariant (i.e, initial centroid phase-space area set by initial conditions) and betatron function can be used to bound oscillation
- Motion in *y*-plane analogous

///

Designing a lattice for single particle stability by limiting undepressed phases advances to less that 180 degrees per period means that the centroid will be stable

- Situation could be modified in very extreme cases due to image couplings

Effect of Driving Errors

The reference orbit is ideally tuned for zero centroid excursions. But there will *always* be driving errors that can cause the centroid oscillations to accumulate with beam propagation distance:

$$\boxed{X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \sum_n \frac{G_n}{G_0}\kappa_n(s)X = \sum_n \frac{G_n}{G_0}\kappa_n(s)\Delta_{xn}}$$

$\kappa_q(s) = \sum_n \kappa_n(s)$ $\quad \kappa_n(s)$ nominal gradient function, *n*th quadrupole

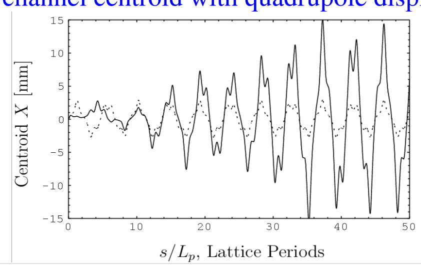
$\frac{G_n}{G_0} =$ *n*th quadrupole gradient error (unity for no error; *s*-varying)

$\Delta_{xn} =$ *n*th quadrupole transverse displacement error (*s*-varying)

/// Example: FODO channel centroid with quadrupole displacement errors

$\frac{G_n}{G_0} = 1$
$\Delta_{xn} = [-0.5, 0.5]$ mm
(uniform dist)

same lattice and initial condition as previous



solid – with errors
dashed – no errors

///

Errors will result in a characteristic random walk increase in oscillation amplitude due to the (generally random) driving terms

- Can also be systematic errors with different (not random walk) characteristics depending on the nature of the errors

Control by:

- Synthesize small applied dipole fields to regularly steer the centroid back on-axis to the reference trajectory: X = 0 = Y, X' = 0 = Y'
- Fabricate and align focusing elements with higher precision
- Employ a sufficiently large aperture to contain the oscillations and limit detrimental nonlinear image charge effects (analysis to come)

Economics dictates the optimal strategy
- Usually sufficient control achieved by a combination of methods

Effects of Image Charges

Model the beam as a displaced line-charge in a circular aperture. Then using the previously derived image charge field, the equations of motion reduce to:

$$\boxed{X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \kappa_x X = \frac{QX}{r_p^2 - X^2}}$$

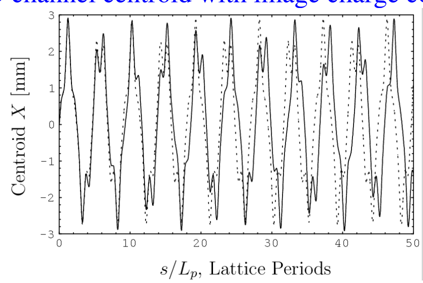
examine oscillation along *x*-axis

$$\frac{QX}{r_p^2 - X^2} \simeq \underbrace{\frac{Q}{r_p^2}X}_{\text{linear correction}} + \underbrace{\frac{Q}{r_p^4}X^3}_{\text{Nonlinear correction (smaller)}}$$

Example: FODO channel centroid with image charge corrections

$r_p = 30$ mm
$Q = 2 \times 10^{-4}$

same lattice as previous



solid – with images
dashed – no images

Main effect of images is typically an accumulated phase error of the centroid orbit
- This will complicate extrapolations of errors over many lattice periods

Control by:
- Keeping centroid displacements X, Y small by correcting
- Make aperture (pipe radius r_p) larger

Comments:
- Images contributions to centroid excursions typically less problematic than misalignment errors in focusing elements
- More detailed analysis show that the coupling of the envelope radii $r_x, \ r_y$ to the centroid evolution in X, Y is often weak
- Fringe fields are more important for accurate calculation of centroid orbits since orbits are not part of a matched lattice
  - Single orbit vs a bundle of orbits, so more sensitive to the timing of focusing impulses imparted by the lattice
- Over long path lengths many nonlinear terms can also influence oscillation phase
- Lattice errors are not typically known a priori so one must often analyze characteristic error distributions to see if centroids measured are consistent with expectations
  - Often model a uniform distribution of errors or Gaussian with cutoff tails since quality checks should render the tails of the Gaussian inconceivable to realize

S4: Envelope Equations of Motion

Overview: Reduce equations of motion for $r_x, \ r_y$
- Find that couplings to centroid coordinates $X \ \ Y$ are weak
  - Centroid ideally zero in a well tuned system
- Envelope eqns are most important in designing transverse focusing systems
  - Expresses average radial force balance (see following discussion)
  - Can be difficult to analyze analytically for scaling properties
  - "Systems" or design scoping codes often written using envelope equations, stability criteria, and practical engineering constraints
- Instabilities of the envelope equations in periodic focusing lattices must be avoided in machine operation
  - Instabilities are strong and real: not washed out with realistic distributions without frozen form
  - Represent lowest order "KV" modes of a full kinetic theory
- Previous derivation of envelope equations relied on Courant-Snyder invariants in linear applied and self-fields. Analysis shows that the same force balances result for a uniform elliptical beam with no image couplings.
  - Debye screening arguments suggest assumed uniform density model taken should be a good approximation for intense space-charge

KV/rms Envelope Equations: Properties of Terms

The envelope equation reflects low-order force balances:

$$r_x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}r_x' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}r_y' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

Terms:	Streaming	Applied Acceleration	Applied Focusing	Space-Charge Defocusing	Thermal Defocusing
	Inertial	Lattice	Lattice	Perveance	Emittance

The "acceleration schedule" specifies both $\gamma_b\beta_b$ and λ then the equations are integrated with:

$$\boxed{\begin{aligned}\gamma_b\beta_b\varepsilon_x &= \text{const} \\ \gamma_b\beta_b\varepsilon_y &= \text{const}\end{aligned}}$$

normalized emittance conservation (set by initial value)

$$\boxed{Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2}}$$

specified perveance

As the beam expands, perveance term will eventually dominate emittance term:
[see: Lund and Bukh, PRSTAB **7**, 024801 (2004)]

Consider a free expansion $(\kappa_x = \kappa_y = 0)$ for a coasting beam with $\gamma_b\beta_b = \text{const}$

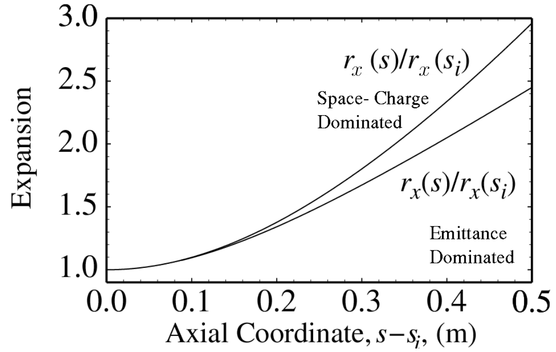
Initial conditions:

$$r_x(s_i) = r_y(s_i) \qquad \frac{Q}{r_x(s_i)} = \frac{\varepsilon_x^2}{r_x^3(s_i)}$$
$$r'_x(s_i) = r'_y(s_i) = 0$$
$$Q = \frac{\varepsilon_x^2}{r_x^2(s_i)} = 10^{-3}$$

Cases:

Space-Charge Dominated: $\varepsilon_x = 0$

Emittance Dominated: $Q = 0$



See next page: solution is analytical in bounding limits shown

Parameters are chosen such that *initial* defocusing forces in two limits are equal to compare case

For an emittance dominated beam in free-space, the envelope equation becomes:

$$\frac{Q}{r_x + r_y} \ll \frac{\varepsilon_{x,y}^2}{r_{x,y}^3} \quad \Longrightarrow \quad r''_j - \frac{\varepsilon_j^2}{r_j^3} = 0 \qquad j = x, y$$

The envelope Hamiltonian gives:

$$\frac{1}{2}r'^2_j + \frac{\varepsilon_j^2}{2r_j^2} = \text{const}$$

which can be integrated from the initial envelope at $s = s_i$ to show that:

Emittance Dominated Free-Expansion $(Q = 0)$

$$r_j(s) = r_j(s_i)\sqrt{1 + \frac{2r'_j(s_i)}{r_j(s_i)}(s - s_i) + \left[1 + \frac{r_j^2(s_i)r'^2_j(s_i)}{\varepsilon_j^2}\right]\frac{\varepsilon_j^2}{r_j^4(s_i)}(s-s_i)^2}$$

$j = x, y$

Conversely, for a space-charge dominated beam in free-space, the envelope equation becomes:

$$\frac{Q}{r_x + r_y} \gg \frac{\varepsilon_{x,y}^2}{r_{x,y}^3} \quad \Longrightarrow \quad \begin{aligned} r''_+ - \frac{Q}{r_+} &= 0 \\ r''_- &= 0 \end{aligned} \qquad r_\pm \equiv \frac{1}{2}(r_x \pm r_y)$$

The equations of motion

$$r''_+ - \frac{Q}{r_+} = 0$$
$$r''_- = 0$$

can be integrated from the initial envelope at $s = s_i$ to show that:

- r_- equation solution trivial
- r_+ equation solution exploits Hamiltonian $\frac{1}{2}r'^2_+ - Q\ln r_+ = \text{const}$

Space-Charge Dominated Free-Expansion $(\varepsilon_x = \varepsilon_y = 0)$

$$r_+(s) = r_+(s_i)\exp\left(-\frac{r'^2_+(s_i)}{2Q} + \left[\text{erfi}^{-1}\left\{\text{erfi}\left[\frac{r'_+(s_i)}{\sqrt{2Q}}\right] + \sqrt{\frac{2Q}{\pi}}e^{\frac{r'^2_+(s_i)}{2Q}}\frac{(s-s_i)}{r_+(s_i)}\right\}\right]^2\right)$$

$$r_-(s) = r_-(s_i) + r'_-(s_i)(s - s_i)$$

$$r_\pm = \frac{1}{2}(r_x \pm r_y)$$

Imaginary Error Function

$$\text{erfi}(z) \equiv \frac{\text{erf}(iz)}{i} \equiv \frac{2}{\sqrt{\pi}}\int_0^z dt\ \exp(t^2)$$
$$i \equiv \sqrt{-1}$$

The free-space expansion solutions for emittance and space-charge dominated beams will be explored more in the problems

S5: Matched Envelope Solution: Lund and Bukh, PRSTAB **7**, 024801 (2004)

Neglect acceleration $(\gamma_b\beta_b = \text{const})$ or use transformed variables:

$$r''_x(s) + \kappa_x(s)r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0$$
$$r''_y(s) + \kappa_y(s)r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0$$
$$r_x(s + L_p) = r_x(s) \qquad r_x(s) > 0$$
$$r_y(s + L_p) = r_y(s) \qquad r_y(s) > 0$$

Matching involves finding specific initial conditions for the envelope to have the periodicity of the lattice:

Find Values of:

$$\begin{matrix} r_x(s_i) & r'_x(s_i) \\ r_y(s_i) & r'_y(s_i) \end{matrix}$$

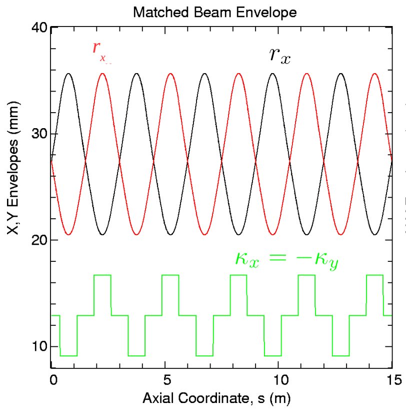
Such That: (periodic)

$$\begin{matrix} r_x(s_i + L_p) = r_x(s_i) & r'_x(s_i + L_p) = r'_x(s_i) \\ r_y(s_i + L_p) = r_y(s_i) & r'_y(s_i + L_p) = r'_y(s_i) \end{matrix}$$

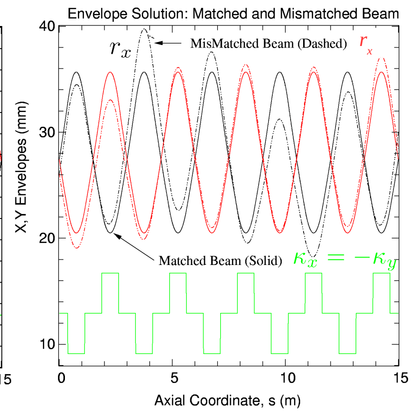
- Typically constructed with numerical root finding from estimated/guessed values
  - Can be surprisingly difficult for complicated lattices (high σ_0) with strong space-charge
- Iterative technique developed to numerically calculate without root finding; Lund, Chilton and Lee, PRSTAB **9**, 064201 (2006)
  - Method exploits Courant-Snyder invariants of depressed orbits within the beam

Typical Matched vs Mismatched solution for FODO channel:

Matched



Mismatched



The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport

- Matching uses optics most efficiently to maintain radial beam confinement

The matched solution to the KV envelope equations reflects the symmetry of the focusing lattice and must, in general, be calculated numerically

Envelope equation very nonlinear

$$\boxed{\begin{aligned} r_x(s+L_p) &= r_x(s) \\ r_y(s+L_p) &= r_y(s) \\ \varepsilon_x &= \varepsilon_y \end{aligned}}$$

Parameters

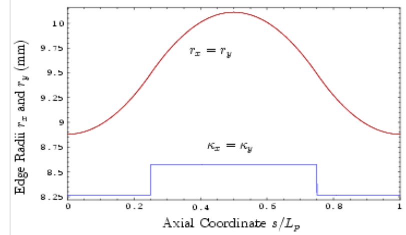
$L_p = 0.5$ m, $\sigma_0 = 80°$, $\eta = 0.5$

$\varepsilon_x = 50$ mm-mrad

$\sigma/\sigma_0 = 0.2$ Perveance Q iterated to obtain matched solution with this tune depression

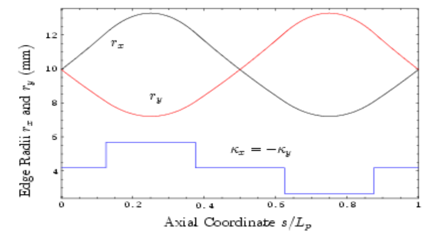
Solenoidal Focusing

$(Q = 6.6986 \times 10^{-4})$



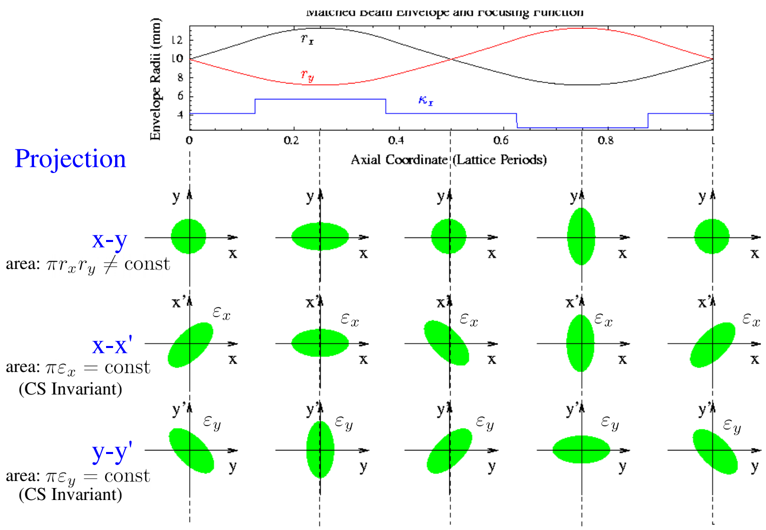
FODO Quadrupole Focusing

$(Q = 6.5614 \times 10^{-4})$



Symmetries of a matched beam are interpreted in terms of a local rms equivalent KV beam and moments/projections of the KV distribution

[see: S.M. Lund, lectures on Transverse Equilibrium Distributions]



S6: Particle Orbits with Space-Charge

The envelope equation reflects low-order force balances

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

Terms:	Applied Focusing	Space-Charge Defocusing	Thermal Defocusing
	Lattice	Perveance	Emittance

Matched Solution:

$r_x(s+L_p) = r_x(s)$

$r_y(s+L_p) = r_y(s)$

$\kappa_x(s+L_p) = \kappa_x(s)$

$\kappa_y(s+L_p) = \kappa_y(s)$

Comments:

- Envelope equation is a projection of a 4D (linear field) invariant distribution
  - Envelope evolution equivalently given by moments of the 4D equilibrium distribution
- Most important basic design equation for transport lattices with high space-charge intensity
  - Simplest consistent model incorporating applied focusing, space-charge defocusing, and thermal defocusing forces
  - Starting point of almost all practical machine design!

The matched solution to the envelope equations reflects the symmetry of the focusing lattice and must in general be calculated numerically

Matching Condition

$$r_x(s+L_p) = r_x(s)$$
$$r_y(s+L_p) = r_y(s)$$

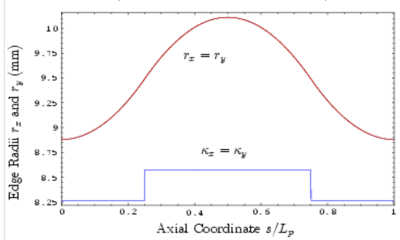
Example Parameters

$L_p = 0.5\ \mathrm{m}, \quad \sigma_0 = 80^\circ, \quad \eta = 0.5$

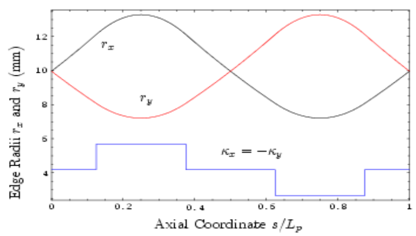
$\varepsilon_x = \varepsilon_y = 50\ \text{mm-mrad}$

$\sigma/\sigma_0 = 0.2$

Solenoidal Focusing $(Q = 6.6986 \times 10^{-4})$



FODO Quadrupole Focusing $(Q = 6.5614 \times 10^{-4})$



The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport

Particle orbits in the presence of uniform space-charge can be strongly modified – space charge slows the orbit response:

The particle equations of motion:

$$x'' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \qquad -\frac{\partial\phi}{\partial x} = \frac{\lambda}{\pi\epsilon_0}\frac{x}{(r_x+r_y)r_x}$$

$$y'' + \kappa_y y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y} \qquad -\frac{\partial\phi}{\partial y} = \frac{\lambda}{\pi\epsilon_0}\frac{y}{(r_x+r_y)r_y}$$

become within the beam:

$$x''(s) + \left\{\kappa_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)}\right\}x(s) = 0$$

$$y''(s) + \left\{\kappa_y(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)}\right\}y(s) = 0$$

Here, Q is the dimensionless perveance defined by:

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \text{const}$$

If we regard the envelope radii r_x, r_y as specified functions of s, then these equations of motion are Hill's equations familiar from elementary accelerator physics:

$$x''(s) + \kappa_x^{\text{eff}}(s)x(s) = 0$$
$$y''(s) + \kappa_y^{\text{eff}}(s)y(s) = 0$$
$$\kappa_x^{\text{eff}}(s) = \kappa_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)}$$
$$\kappa_y^{\text{eff}}(s) = \kappa_y(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)}$$

Review (1): The Courant-Snyder invariant of Hill's equation

[Courant and Snyder, Annl. Phys. **3**, 1 (1958)]

Hill's equation describes a zero space-charge particle orbit in linear applied focusing fields:

$$x''(s) + \kappa(s)x(s) = 0$$

As a consequence of Floquet's theorem, the solution can be cast in phase-amplitude form:

$$x(s) = A_i w(s)\cos\psi(s) \qquad \psi'(s) \equiv \frac{1}{w^2(s)}$$

where $w(s)$ is the periodic amplitude function satisfying

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s+L_p) = w(s) \qquad w(s) > 0$$

$\psi(s)$ is a phase function given by

$$\psi(s) = \psi_i + \int_{s_i}^{s}\frac{d\tilde{s}}{w^2(\tilde{s})}$$

A_i and ψ_i are constants set by initial conditions at $s = s_i$

Review (2): The Courant-Snyder invariant of Hill's equation

From this formulation, it follows that

$$x(s) = A_i w(s) \cos\psi(s) \qquad \psi'(s) \equiv \frac{1}{w^2(s)}$$

$$x'(s) = A_i w'(s)\cos\psi(s) - \frac{A_i}{w(s)}\sin\psi(s)$$

or

$$\frac{x}{w} = A_i \cos\psi$$

$$wx' - w'x = A_i \sin\psi$$

square and add equations to obtain the Courant-Snyder invariant

$$\boxed{\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2 = \text{const}}$$

- Simplifies interpretation of dynamics
- Extensively used in accelerator physics

Phase-amplitude description of particles evolving within a uniform density beam:

Phase-amplitude form of x-orbit equations:

$$x(s) = A_{xi} w_x(s) \cos\psi_x(s)$$

$$x'(s) = A_{xi} w_x'(s)\cos\psi_x(s) - \frac{A_{xi}}{w_x(s)}\sin\psi_x(s)$$

initial conditions yield: $(s = s_i)$

$$A_{xi} = \text{const}$$

$$\psi_{xi} = \psi_x(s = s_i) = \text{const}$$

where

$$\boxed{\begin{gathered} w_x''(s) + \kappa_x(s) w_x(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_x(s)} w_x(s) - \frac{1}{w_x^3(s)} = 0 \\ w_x(s + L_p) = w_x(s) \qquad w_x(s) > 0 \\ \psi_x(s) = \psi_{xi} + \int_{s_i}^{s} \frac{d\tilde{s}}{w_x^2(\tilde{s})} \end{gathered}}$$

identifies the Courant-Snyder invariant

$$\boxed{\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_{xi}^2 = \text{const}}$$

Analogous equations hold for the y-plane

The KV envelope equations:

Define *maximum* Courant-Snyder invariants:

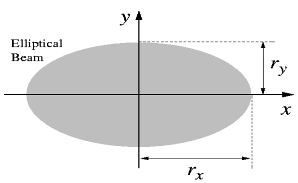
$$\boxed{\begin{aligned} \varepsilon_x &\equiv \text{Max}(A_{xi}^2) \\ \varepsilon_y &\equiv \text{Max}(A_{yi}^2) \end{aligned}} \qquad x = A_{xi} w_x \cos\psi_x \xrightarrow{\cos\psi_x = 1} r_x = A_{x,\max} w_x$$

Values must correspond to the beam-edge radii:

$$\boxed{\begin{aligned} r_x(s) &= \sqrt{\varepsilon_x} w_x(s) \\ r_y(s) &= \sqrt{\varepsilon_y} w_y(s) \end{aligned}}$$

Edge Ellipse:

$$\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} = 1$$



The equations for w_x and w_y can then be rescaled to obtain the familiar KV envelope equations for the matched beam envelope

$$\boxed{\begin{gathered} r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0 \\ r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0 \\ r_x(s + L_p) = r_x(s) \qquad r_x(s) > 0 \\ r_y(s + L_p) = r_y(s) \qquad r_y(s) > 0 \end{gathered}}$$

Contrast: Review, the undepressed particle phase advance calculated in the lectures on Transverse Particle Dynamics

The undepressed phase advance is defined as the phase advance of a particle in the absence of space-charge ($Q = 0$):

- Denote by σ_{0x} to distinguished from the "depressed" phase advance σ_x in the presence of space-charge

$$\boxed{\begin{gathered} w_{0x}'' + \kappa_x w_{0x} - \frac{1}{w_{0x}^3} = 0 \qquad w_{0x}(s + L_p) = w_{0x}(s) \\ w_{0x} > 0 \\ \sigma_{0x} = \int_{s_i}^{s_i + L_p} \frac{ds}{w_{0x}^2} \end{gathered}}$$

This can be equivalently calculated from the matched envelope with $Q = 0$:

$$\boxed{\begin{gathered} r_{0x}'' + \kappa_x r_{0x} - \frac{\varepsilon_x^2}{r_{0x}^3} = 0 \qquad r_{0x}(s + L_p) = r_{0x}(s) \\ r_{0x} > 0 \\ \sigma_{0x} = \varepsilon_x \int_{s_i}^{s_i + L_p} \frac{ds}{r_{0x}^2} \end{gathered}}$$

- Value of is arbitrary (answer for σ_{0x} is independent)

Equation of motion for x-plane "depressed" orbit in the presence of space-charge:

$$\boxed{x''(s) + \kappa_x(s)x(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_x(s)}x(s) = 0}$$

$$w_x''(s) + \kappa_x(s)w_x(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_x(s)}w_x(s) - \frac{1}{w_x^3(s)} = 0$$

$$\psi_x(s) = \psi_{xi} + \int_{s_i}^{s} \frac{d\tilde{s}}{w_x^2(\tilde{s})} \qquad w_x \equiv \frac{r_x}{\sqrt{\varepsilon_x}}$$

$$\mathrm{r}_x''(s) + \kappa_x(s)r_x(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0$$

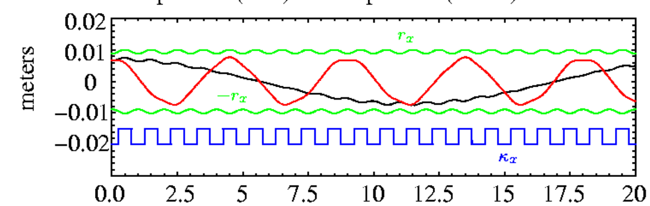
All particles have the *same value* of depressed phase advance (similar Eqns in *y*):

$$\boxed{\sigma_x \equiv \psi_x(s_i + L_p) - \psi_x(s_i) = \varepsilon_x \int_{s_i}^{s_i+L_p} \frac{ds}{r_x^2(s)}}$$

Depressed particle *x*-plane orbits within a matched KV beam in a periodic FODO quadrupole channel for the matched beams previously shown

Solenoidal Focusing (Larmor frame orbit):

Undepressed (Red) and Depressed (Black) Particle Orbits



x-plane orbit:

$y = 0 = y'$

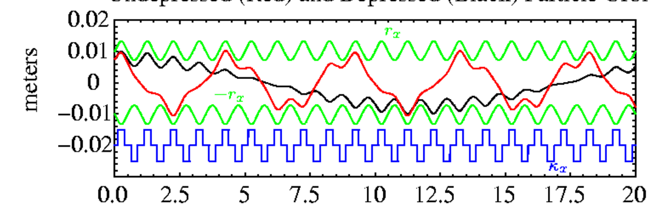
Both Problems Tuned for:

$\sigma_0 = 80^\circ$

$\frac{\sigma}{\sigma_0} = 0.2$

FODO Quadrupole Focusing: Lattice Periods

Undepressed (Red) and Depressed (Black) Particle Orbits



Lattice Periods

x-plane orbit:

$y = 0 = y'$

Clarification Comment on previous plots:

For the shown undepressed orbit (no beam space-charge), the particle is integrated from the same initial condition as the depressed orbit (in presence of space-charge). In this context the matched envelope which is shown including space-charge has no meaning.

- A beam rms "edge" envelope without space-charge r_{0x} could also be shown taking

$$r_{0x}(s) = \sqrt{\varepsilon_x}w_{0x}(s) = \sqrt{\varepsilon_x\beta_{0x}(s)}$$

- This envelope will be different than the depressed beam.

The undepressed particle orbit can be calculated using phase-amplitude methods or by simply integrating the ODE describing the particle moving in linear applied fields:

$$x'' + \kappa_x(s)x = 0$$

$$x(s = s_i) = x_i$$
$$x'(s = s_i) = x_i'$$

Same initial condition as depressed

Depressed particle phase advance provides a convenient measure of space-charge strength

For simplicity take (plane symmetry in average focusing and emittance)

$$\sigma_{0x} = \sigma_{0y} \equiv \sigma_0 \qquad \varepsilon_x = \varepsilon_y \equiv \varepsilon$$

Depressed phase advance of particles moving within a matched beam envelope:

$$\boxed{\sigma = \varepsilon \int_{s_i}^{s_i+L_p} \frac{ds}{r_x^2(s)} = \varepsilon \int_{s_i}^{s_i+L_p} \frac{ds}{r_y^2(s)}}$$

Limits:

1) $\lim_{Q\to 0} \sigma = \sigma_0$ Envelope just rescaled amplitude: $r_x^2 = \varepsilon w_{0x}^2$

2) $\lim_{\varepsilon\to 0} \sigma = 0$ Matched envelope exists with $\varepsilon = 0$. Then $\varepsilon = 0$ multiplying phase advance integral

Normalized space charge strength

$$\boxed{0 \leq \sigma/\sigma_0 \leq 1}$$

$\sigma/\sigma_0 \to 0$ Cold Beam (space-charge dominated) $\varepsilon \to 0$

$\sigma/\sigma_0 \to 1$ Warm Beam (kinetic dominated) $Q \to 0$

For example matched envelope presented earlier:

Undepressed phase advance: $\sigma_0 = 80^\circ$

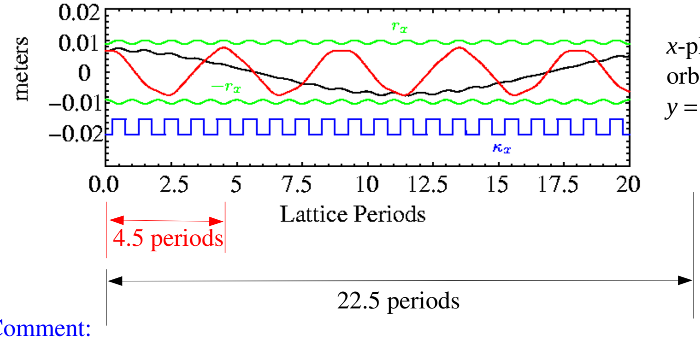
Depressed phase advance: $\sigma = 16^\circ \quad \rightarrow \sigma/\sigma_0 = 0.2$

repeat periods
4.5
22.5

Periods for 360 degree phase advance

Solenoidal Focusing (Larmor frame orbit):

Undepressed (Red) and Depressed (Black) Particle Orbits



x-plane orbit
$y = 0 = y'$

Comment:

All particles in the distribution will, of course, always move in response to both applied and self-fields. You cannot turn off space-charge for an undepressed orbit. It is a convenient conceptual construction to help understand focusing properties.

The rms equivalent beam model helps interpret general beam evolution in terms of an "equivalent" local KV distribution with uniform density

Real beams distributions in the lab will not be KV form. But the KV model can be applied to interpret arbitrary distributions via the concept of *rms equivalence*.

For the same focusing lattice, replace any beam charge $\rho(x,y)$ density by a uniform density KV beam of the same species (q, m) and energy (β_b) in each axial slice (s) using averages calculated from the actual "real" beam distribution with:

$$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \; \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \; f_\perp} \qquad f_\perp = \text{ real distribution}$$

rms equivalent beam (identical 1st and 2nd order moments):

Quantity	KV Equiv.	Calculated from Distribution
Perveance	Q	$= q^2 \int d^2x_\perp \int d^2x'_\perp \; f_\perp \; /[2\pi\epsilon_0\gamma_b^3\beta_b^2c^2]$
x-Env Rad	r_x	$= 2\langle x^2\rangle_\perp^{1/2}$
y-Env Rad	r_y	$= 2\langle y^2\rangle_\perp^{1/2}$
x-Env Angle	r'_x	$= 2\langle xx'\rangle_\perp/\langle x^2\rangle_\perp^{1/2}$
y-Env Angle	r'_y	$= 2\langle yy'\rangle_\perp/\langle y^2\rangle_\perp^{1/2}$
x-Emittance	ε_x	$= 4[\langle x^2\rangle_\perp\langle x'^2\rangle_\perp - \langle xx'\rangle_\perp]^{1/2}$
y-Emittance	ε_y	$= 4[\langle y^2\rangle_\perp\langle y'^2\rangle_\perp - \langle yy'\rangle_\perp]^{1/2}$

Comments on rms equivalent beam concept:

- The emittances will generally evolve in s
  - Means that the equivalence must be recalculated in every slice as the emittances evolve
  - This evolution is often small
- Concept is highly useful
  - Unfiorm density KV equilibrium properties well understood and are approximately correct to model lowest order "real" beam properties
  - See, Reiser, *Theory and Design of Charged Particle Beams* (1994, 2008) for a detailed and instructive discussion of rms equivalence

S7: Envelope Perturbations:

Lund and Bukh, PRSTAB **7**, 024801 (2004)

In the envelope equations take:

Envelope Perturbations:

$$r_x(s) = \boxed{r_{xm}(s)} + \boxed{\delta r_x(s)}$$
$$r_y(s) = \boxed{r_{ym}(s)} + \boxed{\delta r_y(s)}$$

Matched Envelope — Mismatch Perturbations

Driving Perturbations:

$$\kappa_x(s) \rightarrow \kappa_x(s) + \delta\kappa_x(s)$$
$$\kappa_y(s) \rightarrow \kappa_y(s) + \delta\kappa_y(s)$$
Focus

$$Q \rightarrow Q + \delta Q(s)$$
Perveance

$$\varepsilon_x \rightarrow \varepsilon_x + \delta\varepsilon_x(s)$$
$$\varepsilon_y \rightarrow \varepsilon_y + \delta\varepsilon_y(s)$$
Emittance

Perturbations in envelope radii are about a matched solution:

$$r_{xm}(s + L_p) = r_{xm}(s) \qquad r_{xm}(s) > 0$$
$$r_{ym}(s + L_p) = r_{ym}(s) \qquad r_{ym}(s) > 0$$

Perturbations in envelope radii are small relative to matched solution and driving terms are consistently ordered:

$$r_{xm}(s) \gg |\delta r_x(s)|$$
$$r_{ym}(s) \gg |\delta r_y(s)| \qquad \longleftarrow$$

Amplitudes defined in terms of producing small envelope perturbations

- Driving perturbations and distribution errors generate/pump envelope perturbations
  - Arise from many sources: focusing errors, lost particles, emittance growth, ...

The matched solution satisfies:

- Add subscript *m* to denote matched envelope solution and distinguish from other evolutions

$$r_x \rightarrow r_{xm} \qquad r_y \rightarrow r_{ym}$$

For matched beam envelope with periodicity of lattice

Assume a coasting beam with $\gamma_b\beta_b = \text{const}$ or that emittance is small and the lattice is retuned to compensate for acceleration to maintain periodic $\kappa_x,\ \kappa_y$

$$r''_{xm}(s) + \kappa_x(s) r_{xm}(s) - \frac{2Q}{r_{xm}(s)+r_{ym}(s)} - \frac{\varepsilon_x^2}{r_{xm}^3(s)} = 0$$

$$r''_{ym}(s) + \kappa_y(s) r_{ym}(s) - \frac{2Q}{r_{xm}(s)+r_{ym}(s)} - \frac{\varepsilon_y^2}{r_{ym}^3(s)} = 0$$

$$r_{xm}(s+L_p) = r_{xm}(s) \qquad r_{xm}(s) > 0$$

$$r_{ym}(s+L_p) = r_{ym}(s) \qquad r_{ym}(s) > 0$$

Matching is usually cast in terms of finding 4 "initial" envelope phase-space values where the envelope solution satisfies the periodicity constraint for specified focusing, perveance, and emittances:

$$r_{xm}(s_i) \qquad r'_{xm}(s_i)$$

$$r_{ym}(s_i) \qquad r'_{ym}(s_i)$$

Linearized Perturbed Envelope Equations: (steps on next slide)

- Neglect all terms of order δ^2 and higher: $(\delta r_x)^2,\ \ \delta r_x \delta r_y,\ \ \delta Q \delta r_x,\ \cdots$

$$\delta r''_x + \kappa_x \delta r_x + \frac{2Q}{(r_{xm}+r_{ym})^2}(\delta r_x + \delta r_y) + \frac{3\varepsilon_x^2}{r_{xm}^4}\delta r_x = -r_{xm}\delta\kappa_x + \frac{2}{r_{xm}+r_{ym}}\delta Q + \frac{2\varepsilon_x}{r_{xm}^3}\delta\varepsilon_x$$

$$\delta r''_y + \kappa_y \delta r_y + \frac{2Q}{(r_{xm}+r_{ym})^2}(\delta r_x + \delta r_y) + \frac{3\varepsilon_y^2}{r_{ym}^4}\delta r_y = -r_{ym}\delta\kappa_y + \frac{2}{r_{xm}+r_{ym}}\delta Q + \frac{2\varepsilon_y}{r_{ym}^3}\delta\varepsilon_y$$

Homogeneous Equations:

- Linearized envelope equations with driving terms set to zero

$$\delta r''_x + \kappa_x \delta r_x + \frac{2Q}{(r_{xm}+r_{ym})^2}(\delta r_x + \delta r_y) + \frac{3\varepsilon_x^2}{r_{xm}^4}\delta r_x = 0$$

$$\delta r''_y + \kappa_y \delta r_y + \frac{2Q}{(r_{xm}+r_{ym})^2}(\delta r_x + \delta r_y) + \frac{3\varepsilon_y^2}{r_{ym}^4}\delta r_y = 0$$

Derivation steps for terms in the linearized envelope equation:

Inertial: $$r''_x \rightarrow r''_{xm} + \delta r''_{xm}$$

Focusing: $$\kappa_x r_x \rightarrow (\kappa_x + \delta\kappa_x)(r_{xm} + \delta r_x) \simeq \kappa_x r_{xm} + \kappa_x \delta r_{xm} + \delta\kappa_x r_{xm} + \Theta(\delta^2)$$

Perveance: $$\frac{2Q}{r_x + r_y} \rightarrow \frac{2Q + 2\delta Q}{r_{xm} + r_{ym} + \delta r_x + \delta r_y} \simeq \frac{2Q}{r_{xm}+r_{ym}}\left[1 - \frac{\delta r_x + \delta r_y}{r_{xm}+r_{ym}}\right] + \frac{2\delta Q}{r_{xm}+r_{ym}} + \Theta(\delta^2)$$

Emittance: $$\frac{\varepsilon_x^2}{r_x^3} \rightarrow \frac{(\varepsilon_x + \delta\varepsilon_x)^2}{(r_{xm}+\delta r_x)^3} \simeq \frac{2\varepsilon_x \delta\varepsilon_x}{r_{xm}^3} + \frac{\varepsilon_x^2}{r_{xm}^3}\left[1 - 3\frac{\delta r_x}{r_{xm}}\right] + \Theta(\delta^2)$$

Collect all terms and neglect higher order:

$$r''_{xm}(s) + \kappa_x(s) r_{xm}(s) - \frac{2Q}{r_{xm}(s)+r_{ym}(s)} - \frac{\varepsilon_x^2}{r_{xm}^3(s)} \quad +$$

$$\delta r''_x + \kappa_x \delta r_x + \frac{2Q}{(r_{xm}+r_{ym})^2}(\delta r_x + \delta r_y) + \frac{3\varepsilon_x^2}{r_{xm}^4}\delta r_x = -r_{xm}\delta\kappa_x + \frac{2}{r_{xm}+r_{ym}}\delta Q + \frac{2\varepsilon_x}{r_{xm}^3}\delta\varepsilon_x$$

Use the matched beam constraint:

$$r''_{xm}(s) + \kappa_x(s) r_{xm}(s) - \frac{2Q}{r_{xm}(s)+r_{ym}(s)} - \frac{\varepsilon_x^2}{r_{xm}^3(s)} \quad = 0$$

Giving:

$$\delta r''_x + \kappa_x \delta r_x + \frac{2Q}{(r_{xm}+r_{ym})^2}(\delta r_x + \delta r_y) + \frac{3\varepsilon_x^2}{r_{xm}^4}\delta r_x = -r_{xm}\delta\kappa_x + \frac{2}{r_{xm}+r_{ym}}\delta Q + \frac{2\varepsilon_x}{r_{xm}^3}\delta\varepsilon_x$$

+ analogous equation in *y*-plane

Martix Form of the Linearized Perturbed Envelope Equations:

$$\boxed{\frac{d}{ds}\delta\mathbf{R} + \mathbf{K}\cdot\delta\mathbf{R} = \delta\mathbf{P}}$$

$$\delta\mathbf{R} \equiv \begin{pmatrix} \delta r_x \\ \delta r'_x \\ \delta r_y \\ \delta r'_y \end{pmatrix}$$ Coordinate vector

$$\mathbf{K} \equiv \begin{pmatrix} 0 & -1 & 0 & 0 \\ k_{xm} & 0 & k_{0m} & 0 \\ 0 & 0 & 0 & -1 \\ k_{0m} & 0 & k_{ym} & 0 \end{pmatrix}$$

Coefficient matrix

$$k_{0m} = \frac{2Q}{(r_{xm}+r_{ym})^2}$$

Has periodicity of the lattice period

$$k_{jm} = \kappa_j + 3\frac{\varepsilon_j^2}{r_{jm}^4} + k_{0m} \qquad j = x,\ y$$

$$\delta\mathbf{P} \equiv \begin{pmatrix} 0 \\ -\delta\kappa_x r_{xm} + 2\frac{\delta Q}{r_{xm}+r_{ym}} + 2\frac{\varepsilon_x\delta\varepsilon_x}{r_{xm}^3} \\ 0 \\ -\delta\kappa_y r_{ym} + 2\frac{\delta Q}{r_{xm}+r_{ym}} + 2\frac{\varepsilon_y\delta\varepsilon_y}{r_{ym}^3} \end{pmatrix}$$ Driving perturbation vector

Expand solution into homogeneous and particular parts:

$$\delta\mathbf{R} = \delta\mathbf{R}_h + \delta\mathbf{R}_p$$

$\delta\mathbf{R}_h =$ homogeneous solution

$\delta\mathbf{R}_p =$ particular solution

$$\frac{d}{ds}\delta\mathbf{R}_h + \mathbf{K}\cdot\delta\mathbf{R}_h = 0 \qquad \frac{d}{ds}\delta\mathbf{R}_p + \mathbf{K}\cdot\delta\mathbf{R}_p = \delta\mathbf{P}$$

Homogeneous Solution: Normal Modes

- Describes normal mode oscillations
- Original analysis by Struckmeier and Reiser [Part. Accel. **14**, 227 (1984)]

Particular Solution: Driven Modes

- Describes action of driving terms
- Characterize in terms of projections on homogeneous response (on normal modes)

Homogeneous solution expressible as a map:

$$\boxed{\begin{aligned} \delta\mathbf{R}(s) &= \mathbf{M}_e(s|s_i)\cdot\delta\mathbf{R}(s_i) \\ \delta\mathbf{R}(s) &= (\delta r_x, \delta r'_x, \delta r_y, \delta r'_y) \\ \mathbf{M}_e(s|s_i) &= \ 4\times 4 \text{ transfer map} \end{aligned}}$$

Now 4x4 system, but analogous to the 2x2 analysis of Hill's equation via transfer matrices: see S.M. Lund lectures on Transverse Particle Dynamics

Eigenvalues and eigenvectors of map through one period characterize normal modes and stability properties:

$$\boxed{\mathbf{M}_e(s_i + L_p|s_i)\cdot\mathbf{E}_n(s_i) = \lambda_n\mathbf{E}_n(s_i)}$$

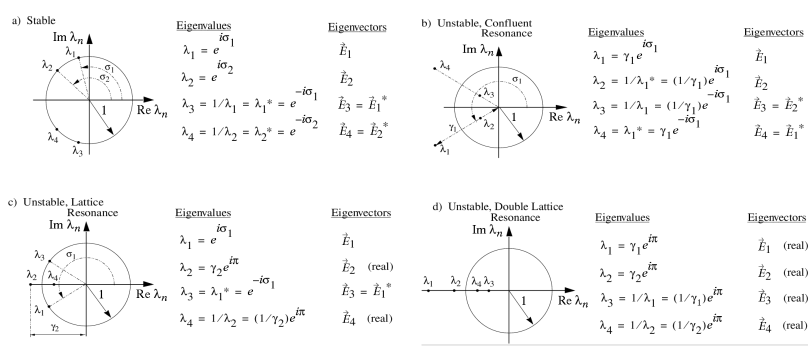
Stability Properties

$$\lambda_n = \gamma_n e^{i\sigma_n} \qquad \begin{aligned} &\sigma_n \rightarrow \text{mode phase advance (real)} \\ &\gamma_n \rightarrow \text{mode growth/damp factor (real)} \end{aligned}$$

Mode Expansion/Launching

$$\delta\mathbf{R}(s_i) = \sum_{n=1}^{4}\alpha_n\mathbf{E}_n(s_i)$$
$$\alpha_n = \text{const (complex)}$$

Eigenvalue/Eigenvector Symmetry Classes:



Symmetry classes of eigenvalues/eigenvectors:

- Determine normal mode symmetries
- Hamiltonian dynamics allow only 4 distinct classes of eigenvalue symmetries
  - See A. Dragt, Lectures on Nonlinear Orbit Dynamics, in Physics of High Energy Particle Accelerators, (AIP Conf. Proc. No. 87, 1982, p. 147)
- Envelope mode symmetries discussed fully in PRSTAB review
- Caution: Textbook by Reiser makes errors in quadrupole mode symmetries and mislabels/identifies dispersion characteristics and branch choices

Pure mode launching conditions:

Launching conditions for distinct normal modes corresponding to the eigenvalue classes illustrated:

$A_\ell =$ mode amplitude (real) $\qquad \ell =$ mode index

$\psi_\ell =$ mode launch phase (real) $\qquad C.C. =$ complex conjugate

Case	Mode	Launching Condition	Lattice Period Advance
(a) Stable	1 - Stable Osc.	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1}\mathbf{E}_1 + \text{C.C.}$	$\mathbf{M}_e\delta\mathbf{R}_1(\psi_1) = \delta\mathbf{R}_1(\psi_1+\sigma_1)$
	2 - Stable Osc.	$\delta\mathbf{R}_2 = A_2 e^{i\psi_2}\mathbf{E}_2 + \text{C.C.}$	$\mathbf{M}_e\delta\mathbf{R}_2(\psi_2) = \delta\mathbf{R}_2(\psi_2+\sigma_2)$
(b) Unstable	1 - Exp. Growth	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1}\mathbf{E}_1 + \text{C.C.}$	$\mathbf{M}_e\delta\mathbf{R}_1(\psi_1) = \gamma_1\delta\mathbf{R}_1(\psi_1+\sigma_1)$
Confluent Res.	2 - Exp. Damping	$\delta\mathbf{R}_2 = A_2 e^{i\psi_2}\mathbf{E}_2 + \text{C.C.}$	$\mathbf{M}_e\delta\mathbf{R}_2(\psi_2) = (1/\gamma_1)\delta\mathbf{R}_2(\psi_2+\sigma_1)$
(c) Unstable	1 - Stable Osc.	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1}\mathbf{E}_1 + \text{C.C.}$	$\mathbf{M}_e\delta\mathbf{R}_1(\psi_1) = \delta\mathbf{R}_1(\psi_1+\sigma_1)$
Lattice Res.	2 - Exp. Growth	$\delta\mathbf{R}_2 = A_2\mathbf{E}_2$	$\mathbf{M}_e\delta\mathbf{R}_2 = -\gamma_2\delta\mathbf{R}_2$
	3 - Exp. Damping	$\delta\mathbf{R}_3 = A_3\mathbf{E}_4$	$\mathbf{M}_e\delta\mathbf{R}_3 = -(1/\gamma_2)\delta\mathbf{R}_3$
(d) Unstable	1 - Exp. Growth	$\delta\mathbf{R}_1 = A_1\mathbf{E}_1$	$\mathbf{M}_e\delta\mathbf{R}_1 = -\gamma_1\delta\mathbf{R}_1$
Double Lattice	2 - Exp. Growth	$\delta\mathbf{R}_2 = A_2\mathbf{E}_2$	$\mathbf{M}_e\delta\mathbf{R}_2 = -\gamma_2\delta\mathbf{R}_2$
Resonance	3 - Exp. Damping	$\delta\mathbf{R}_3 = A_3\mathbf{E}_3$	$\mathbf{M}_e\delta\mathbf{R}_3 = -(1/\gamma_1)\delta\mathbf{R}_3$
	4 - Exp. Damping	$\delta\mathbf{R}_4 = A_4\mathbf{E}_4$	$\mathbf{M}_e\delta\mathbf{R}_4 = -(1/\gamma_2)\delta\mathbf{R}_4$

$$\delta\mathbf{R}_\ell \equiv \delta\mathbf{R}_\ell(s_i) \qquad \mathbf{E}_\ell \equiv \mathbf{E}_\ell(s_i) \qquad \mathbf{M}_e \equiv \mathbf{M}_e(s_i + L_p|s_i)$$

$$\delta\mathbf{R}(s) = \begin{cases} A_1[\mathbf{E}_1(s)e^{i\psi_1(s)} + \mathbf{E}_1^*(s)e^{-i\psi_1(s)}] + A_2[\mathbf{E}_2(s)e^{i\psi_2(s)} + \mathbf{E}_2^*(s)e^{-i\psi_2(s)}], & \text{cases (a) and (b)} \\ A_1[\mathbf{E}_1(s)e^{i\psi_1(s)} + \mathbf{E}_1^*(s)e^{-i\psi_1(s)}] + A_2\mathbf{E}_2(s) + A_3\mathbf{E}_4(s), & \text{case (c)} \\ A_1\mathbf{E}_1(s) + A_2\mathbf{E}_2(s) + A_3\mathbf{E}_3(s) + A_4\mathbf{E}_4(s), & \text{case (d)} \end{cases}$$

Decoupled Modes

In a continuous or periodic solenoidal focusing channel

$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

with a round matched-beam solution

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon = \text{const}$$
$$r_{xm}(s) = r_{ym}(s) \equiv r_m(s)$$

envelope perturbations are simply decoupled with:

Breathing Mode: $\delta r_+ \equiv \dfrac{\delta r_x + \delta r_y}{2}$

Quadrupole Mode: $\delta r_- \equiv \dfrac{\delta r_x - \delta r_y}{2}$

The resulting decoupled envelope equations are:

Breathing Mode: $[\cdots]\,\delta r_+ \equiv \kappa_+ \delta r_+$

$$\delta r_+'' + \left[\kappa + \frac{Q}{r_m^2} + \frac{3\varepsilon^2}{r_m^4}\right]\delta r_+ = -r_m\left(\frac{\delta\kappa_x + \delta\kappa_y}{2}\right) + \frac{1}{r_m}\delta Q + \frac{2\varepsilon}{r_m^3}\left(\frac{\delta\varepsilon_x + \delta\varepsilon_y}{2}\right)$$

Quadrupole Mode: $[\cdots]\,\delta r_- \equiv \kappa_- \delta r_-$

$$\delta r_-'' + \left[\kappa + \frac{3\varepsilon^2}{r_m^4}\right]\delta r_- = -r_m\left(\frac{\delta\kappa_x - \delta\kappa_y}{2}\right) + \frac{2\varepsilon}{r_m^3}\left(\frac{\delta\varepsilon_x - \delta\varepsilon_y}{2}\right)$$

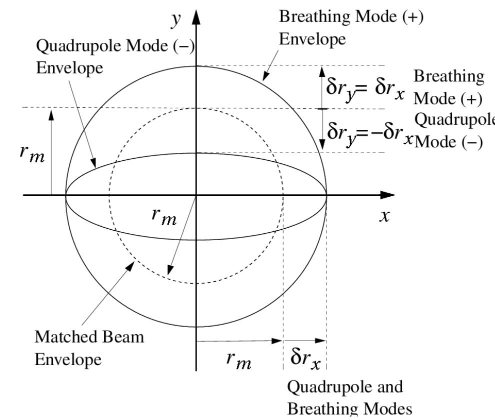
Graphical interpretation of mode symmetries:

Breathing Mode:

$$\delta r_+ = \frac{\delta r_x + \delta r_y}{2}$$

Quadrupole Mode:

$$\delta r_- = \frac{\delta r_x - \delta r_y}{2}$$



$$\kappa_+ = \kappa + \frac{Q}{r_m} + 3\frac{\varepsilon^2}{r_m^4}$$ Breathing Mode Linear Restoring Strength

$$\kappa_- = \kappa + 3\frac{\varepsilon^2}{r_m^4}$$ Quadrupole Mode Linear Restoring Strength

Decoupled Mode Properties:

Space charge terms ~ Q only directly expressed in equation for $\delta r_+(s)$

- Indirectly present in both equations from matched envelope $r_m(s)$

Homogeneous Solution:

- Restoring term for $\delta r_+(s)$ larger than for $\delta r_-(s)$
  - Breathing mode should oscillate faster than the quadrupole mode

$$\kappa_+ = \kappa + \frac{Q}{r_m} + 3\frac{\varepsilon^2}{r_m^4} > \kappa_- = \kappa + 3\frac{\varepsilon^2}{r_m^4}$$

Particular Solution:

- Misbalances in focusing and emittance driving terms can project onto either mode
  - nonzero perturbed $\kappa_x(s)$ + $\kappa_y(s)$ and $\varepsilon_x(s)$ + $\varepsilon_y(s)$ project onto breathing mode
  - nonzero perturbed $\kappa_x(s)$ - $\kappa_y(s)$ and $\varepsilon_x(s)$ - $\varepsilon_y(s)$ project onto quadrupole mode
- Perveance driving perturbations project *only* on breathing mode

Previous symmetry classes greatly reduce for decoupled modes:

Previous homogeneous 4x4 solution map:

$$\delta\mathbf{R}(s) = \mathbf{M}_e(s|s_i)\cdot\delta\mathbf{R}(s_i)$$
$$\delta\mathbf{R}(s) = (\delta r_x, \delta r_x', \delta r_y, \delta r_y')$$
$$\mathbf{M}_e(s|s_i) = 4\times 4 \text{ transfer map}$$

Reduces to two independent 2x2 maps with greatly simplified symmetries:

$$\delta\mathbf{R} \equiv (\delta r_+, \delta r_+', \delta r_-, \delta r_-')$$

$$\mathbf{M}_e(s_i + L_p|s_i) = \begin{bmatrix} \mathbf{M}_+(s_i + L_p|s_i) & 0 \\ 0 & \mathbf{M}_-(s_i + L_p|s_i) \end{bmatrix}$$

Here $\mathbf{M}_\pm$ denote the 2x2 map solutions to the uncoupled Hills equations for $\delta r_\pm$:

$$\delta r_\pm + \kappa_\pm \delta r_\pm = 0$$

$$\kappa_+ \equiv \kappa + \frac{Q}{r_m^2} + \frac{3\varepsilon^2}{r_m^4} \qquad \begin{pmatrix}\delta r_\pm \\ \delta r_\pm'\end{pmatrix} = \mathbf{M}_\pm(s|s_i)\cdot\begin{pmatrix}\delta r_\pm \\ \delta r_\pm'\end{pmatrix}_i$$

$$\kappa_- \equiv \kappa + \frac{3\varepsilon^2}{r_m^4}$$

The corresponding 2D eigenvalue problems:

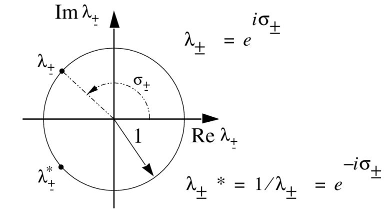
$$\boxed{\mathbf{M}_{\pm}(s_i + L_p|s_i) \cdot \mathbf{E}_n(s_i) = \lambda_{\pm}\mathbf{E}_n(s_i)}$$

Familiar results from analysis of Hills equation (see: S.M. Lund lectures on Transverse Particle Dynamics) can be immediately applied to the decoupled case, for example:

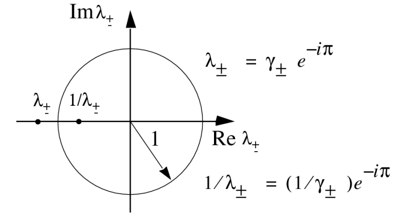
$$\frac{1}{2}|\mathrm{Tr}\ \mathbf{M}_{\pm}(s_i + L_p|s_i)| \leq 1 \qquad \Longleftrightarrow \qquad \text{mode stability}$$

Eigenvalue symmetries give decoupled mode launching conditions

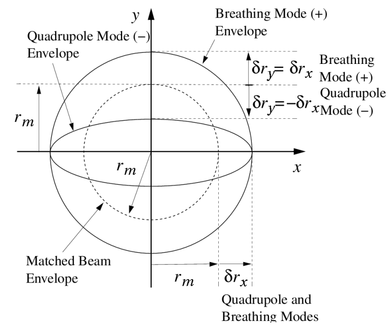
Eigenvalue Symmetry 1:
Stable



Eigenvalue Symmetry 2:
Unstable, Lattice Resonance



Launching Condition / Projections



General Envelope Mode Limits

Using phase-amplitude analysis can show for any linear focusing lattice:

1) Phase advance of any normal mode satisfies the zero space-charge limit:

$$\boxed{\lim_{Q\to 0} \sigma_\ell = 2\sigma_0}$$

2) Pure normal modes (not driven) evolve with a quadratic phase-space (Courant-Snyder) invariant in the normal coordinates of the mode

Simply expressed for decoupled modes with $\kappa_x = \kappa_y, \quad \varepsilon_x = \varepsilon_y$

$$\boxed{\left[\frac{\delta r_{\pm}(s)}{w_{\pm}(s)}\right]^2 + [w'_{\pm}(s)\delta r_{\pm}(s) - w_{\pm}(s)\delta r'_{\pm}(s)]^2 = \text{const}}$$

where

$$w''_{+} + \kappa w_{+} + \frac{Q}{r_m^2}w_{+} + \frac{3\varepsilon^2}{r_m^4}w_{+} - \frac{1}{w_{+}^3} = 0$$

$$w''_{-} + \kappa w_{-} + \frac{3\varepsilon^2}{r_m^4}w_{-} - \frac{1}{w_{-}^3} = 0$$

$$w_{\pm}(s + L_p) = w_{\pm}(s)$$

Analogous results for coupled modes [See Edwards and Teng, IEEE Trans Nuc. Sci. **20**, 885 (1973)]

- But typically much more complex expression due to coupling

S8: Envelope Modes in Continuous Focusing

Lund and Bukh, PRSTAB **7**, 024801 (2004)

Focusing: $\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \left(\frac{\sigma_0}{L_p}\right)^2 = \text{const}$

Matched beam: $\varepsilon_x = \varepsilon_y = \varepsilon = \text{const}$

symmetric beam: $r_{xm}(s) = r_{ym}(s) = r_m = \text{const}$

matched envelope: $k_{\beta 0}^2 r_m - \frac{Q}{r_m} - \frac{\varepsilon^2}{r_m^3} = 0$

depressed phase advance: $\sigma = \sqrt{\sigma_0^2 - \frac{Q}{(r_m/L_p)^2}} = \frac{\varepsilon L_p}{r_m^2}$

one parameter needed for scaled solution: $\frac{k_{\beta 0}^2 \varepsilon^2}{Q^2} = \frac{\sigma_0^2 \varepsilon^2}{Q^2 L_p^2} = \frac{(\sigma/\sigma_0)^2}{[1 - (\sigma/\sigma_0)^2]^2}$

Decoupled Modes:

$$\delta r_{\pm}(s) = \frac{\delta r_x(s) \pm \delta r_y(s)}{2}$$

Envelope equations of motion become:

$$L_p^2 \frac{d^2}{ds^2}\left(\frac{\delta r_+}{r_m}\right) + \sigma_+^2 \left(\frac{\delta r_+}{r_m}\right) = -\frac{\sigma_0^2}{2}\left(\frac{\delta\kappa_x}{k_{\beta 0}^2} + \frac{\delta\kappa_y}{k_{\beta 0}^2}\right) + (\sigma_0^2 - \sigma^2)\frac{\delta Q}{Q} + \sigma^2\left(\frac{\delta\varepsilon_x}{\varepsilon} + \frac{\delta\varepsilon_y}{\varepsilon}\right)$$

$$L_p^2 \frac{d^2}{ds^2}\left(\frac{\delta r_-}{r_m}\right) + \sigma_-^2 \left(\frac{\delta r_-}{r_m}\right) = -\frac{\sigma_0^2}{2}\left(\frac{\delta\kappa_x}{k_{\beta 0}^2} - \frac{\delta\kappa_y}{k_{\beta 0}^2}\right) + \sigma^2\left(\frac{\delta\varepsilon_x}{\varepsilon} - \frac{\delta\varepsilon_y}{\varepsilon}\right)$$

$\sigma_+ \equiv \sqrt{2\sigma_0^2 + 2\sigma^2}$ "breathing" mode phase advance

$\sigma_- \equiv \sqrt{\sigma_0^2 + 3\sigma^2}$ "quadrupole" mode phase advance

Homogeneous equations for normal modes:

$$\frac{d^2}{ds^2}\delta r_\pm + \left(\frac{\sigma_\pm}{L_p}\right)^2 \delta r_\pm = 0$$

- Simple harmonic oscillator equation

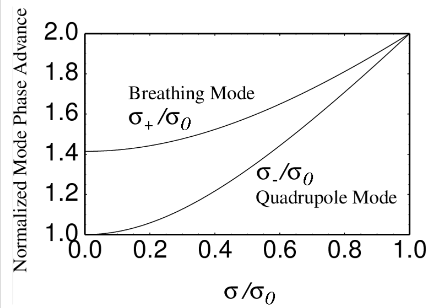
Homogeneous Solution (normal modes):

$$\delta r_\pm(s) = \delta r_\pm(s_i)\cos\left(\sigma_\pm \frac{s - s_i}{L_p}\right) + \frac{\delta r'_\pm(s_i)}{\sigma_\pm/L_p}\sin\left(\sigma_\pm\frac{s - s_i}{L_p}\right)$$

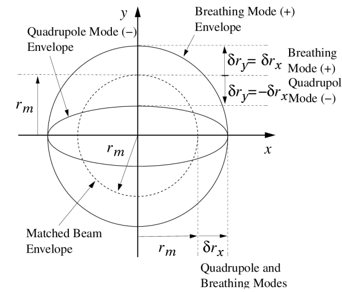
$\delta r_\pm(s_i)$, $\delta r'_\pm(s_i)$ mode initial conditions

Properties of continuous focusing homogeneous solution: Normal Modes

Mode Phase Advances



Mode Projections



$\sigma_+ \equiv \sqrt{2\sigma_0^2 + 2\sigma^2}$ Breathing Mode: $\delta r_+ \equiv \dfrac{\delta r_x + \delta r_y}{2}$

$\sigma_- \equiv \sqrt{\sigma_0^2 + 3\sigma^2}$ Quadrupole Mode: $\delta r_- \equiv \dfrac{\delta r_x - \delta r_y}{2}$

Particular Solution (driving perturbations):

Green's function form of solution derived using projections onto normal modes

- See proof that this is a valid solution is given in Appendix A

$$\frac{\delta r_\pm(s)}{r_m} = \frac{1}{L_p^2}\int_{s_i}^{s} d\tilde{s}\; G_\pm(s,\tilde{s})\delta p_\pm(\tilde{s})$$

$$\delta p_+(s) = -\frac{\sigma_0^2}{2}\left[\frac{\delta\kappa_x(s)}{k_{\beta 0}^2} + \frac{\delta\kappa_y(s)}{k_{\beta 0}^2}\right] + (\sigma_0^2 - \sigma^2)\frac{\delta Q(s)}{Q} + \sigma^2\left[\frac{\delta\varepsilon_x(s)}{\varepsilon} + \frac{\delta\varepsilon_y(s)}{\varepsilon}\right]$$

$$\delta p_-(s) = -\frac{\sigma_0^2}{2}\left[\frac{\delta\kappa_x(s)}{k_{\beta 0}^2} - \frac{\delta\kappa_y(s)}{k_{\beta 0}^2}\right] + \sigma^2\left[\frac{\delta\varepsilon_x(s)}{\varepsilon} - \frac{\delta\varepsilon_y(s)}{\varepsilon}\right]$$

$$G_\pm(s,\tilde{s}) = \frac{1}{\sigma_\pm/L_p}\sin\left(\sigma_\pm\frac{s-\tilde{s}}{L_p}\right)$$

Green's function solution is *fully general*. Insight gained from simplified solutions for specific classes of driving perturbations:

- Adiabatic
- Sudden

covered in these lectures

- Ramped
- Harmonic

covered in PRSTAB Review article

Continuous Focusing – adiabatic particular solution

For driving perturbations $\delta p_+(s)$ and $\delta p_-(s)$ slow on quadrupole mode (slower mode) wavelength $\sim 2\pi L_p/\sigma_-$ the Green function solution reduces to:

Focusing

Perveance

$$\frac{\delta r_+(s)}{r_m} = \frac{\delta p_+(s)}{\sigma_+^2}$$
$$= -\left[\frac{1}{2}\frac{1}{1+(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\kappa_x(s)}{k_{\beta 0}^2} + \frac{\delta\kappa_y(s)}{k_{\beta 0}^2}\right) + \left[\frac{1}{2}\frac{1-(\sigma/\sigma_0)^2}{1+(\sigma/\sigma_0)^2}\right]\frac{\delta Q(s)}{Q}$$
$$+ \left[\frac{(\sigma/\sigma_0)^2}{1+(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\varepsilon_x(s)}{\varepsilon} + \frac{\delta\varepsilon_y(s)}{\varepsilon}\right),$$

Emittance

Coefficients of adiabatic terms in square brackets "[ ]"

Focusing

$$\frac{\delta r_-(s)}{r_m} = \frac{\delta p_-(s)}{\sigma_-^2}$$
$$= -\left[\frac{1}{1+3(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\kappa_x(s)}{k_{\beta 0}^2} - \frac{\delta\kappa_y(s)}{k_{\beta 0}^2}\right)$$
$$+ \left[\frac{2(\sigma/\sigma_0)^2}{1+3(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\varepsilon_x(s)}{\varepsilon} - \frac{\delta\varepsilon_y(s)}{\varepsilon}\right).$$

Emittance

$\sigma_+ \equiv \sqrt{2\sigma_0^2 + 2\sigma^2}$

$\sigma_- \equiv \sqrt{\sigma_0^2 + 3\sigma^2}$

Derivation of Adiabatic Solution:

- Several ways to derive, show more “mechanical” procedure here ....

Use:

$$\frac{\delta r_\pm(s)}{r_m} = \frac{1}{L_p^2}\int_{s_i}^{s} d\tilde{s}\; G_\pm(s,\tilde{s})\delta p_\pm(\tilde{s})$$

$$G_\pm(s,\tilde{s}) = \frac{1}{\sigma_\pm/L_p}\sin\left(\sigma_\pm\frac{s-\tilde{s}}{L_p}\right) = \frac{1}{(\sigma_\pm/L_p)^2}\frac{d}{d\tilde{s}}\cos\left(\sigma_\pm\frac{s-\tilde{s}}{L_p}\right)$$

Gives:

$$\frac{\delta r_\pm(s)}{r_m} = \int_{s_i}^{s} d\tilde{s}\left[\frac{d}{d\tilde{s}}\cos\left(\sigma_\pm\frac{s-\tilde{s}}{L_p}\right)\right]\frac{\delta p_\pm(\tilde{s})}{\sigma_\pm^2}$$

$$= \int_{s_i}^{s} d\tilde{s}\,\frac{d}{d\tilde{s}}\left[\cos\left(\sigma_\pm\frac{s-\tilde{s}}{L_p}\right)\frac{\delta p_\pm(\tilde{s})}{\sigma_\pm^2}\right] - \int_{s_i}^{s} d\tilde{s}\,\cos\left(\sigma_\pm\frac{s-\tilde{s}}{L_p}\right)\frac{d}{d\tilde{s}}\frac{\delta p_\pm(\tilde{s})}{\sigma_\pm^2}$$

Adiabatic $\rightarrow 0$

$$= \cos\left(\sigma_\pm\frac{s-\tilde{s}}{L_p}\right)\frac{\delta p_\pm(\tilde{s})}{\sigma_\pm^2}\Bigg|_{\tilde{s}=s_i}^{\tilde{s}=s} = \frac{\delta p_\pm(s)}{\sigma_\pm^2} - \cos\left(\sigma_\pm\frac{s-s_i}{L_p}\right)\frac{\delta p_\pm(s_i)}{\sigma_\pm^2}$$

No Initial Perturbation $\rightarrow 0$

$$= \frac{\delta p_\pm(s)}{\sigma_\pm^2}$$

Comments on Adiabatic Solution:

- Adiabatic response is essentially a slow adaptation in the matched envelope to perturbations (solution does not oscillate due to slow changes)
- Slow envelope frequency σ_- sets the scale for slow variations required

Replacements in adiabatically adapted match:

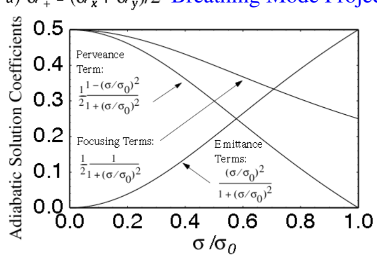
$$\boxed{\begin{aligned} r_x &= r_m \rightarrow r_m + \delta r_+ + \delta r_- \\ r_y &= r_m \rightarrow r_m + \delta r_- - \delta r_+ \end{aligned}}$$

Parameter replacements in rematched beam (no longer axisymmetric):

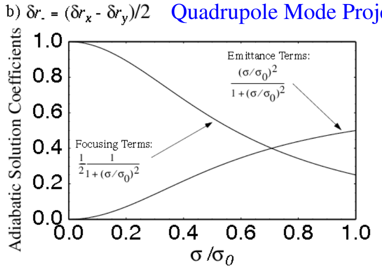
$$\boxed{\begin{aligned} \kappa_x &= k_{\beta 0}^2 \rightarrow k_{\beta 0}^2 + \delta\kappa_x(s) \\ \kappa_y &= k_{\beta 0}^2 \rightarrow k_{\beta 0}^2 + \delta\kappa_y(s) \\ Q &\rightarrow Q + \delta Q(s) \\ \varepsilon_x &= \varepsilon \rightarrow \varepsilon + \delta\varepsilon_x(s) \\ \varepsilon_y &= \varepsilon \rightarrow \varepsilon + \delta\varepsilon_y(s) \end{aligned}}$$

Continuous Focusing – adiabatic solution coefficients

a) $\delta r_+ = (\delta r_x + \delta r_y)/2$ Breathing Mode Projection



b) $\delta r_- = (\delta r_x - \delta r_y)/2$ Quadrupole Mode Projection



Relative strength of:

- Space-Charge (Perveance)
- Applied Focusing
- Emittance

terms vary with space-charge depression (σ/σ_0) for both breathing and quadrupole mode projections

Plots allow one to read off the relative importance of various contributions to beam mismatch as a function of space-charge strength

Continuous Focusing – sudden particular solution

For sudden, step function driving perturbations of form:

$$\boxed{\delta p_\pm(s) = \widehat{\delta p}_\pm\Theta(s-s_p)}$$

$s = s_p =$ axial coordinate perturbation applied

Hat quantities are constant amplitudes

with amplitudes:

$$\widehat{\delta p}_+ = -\frac{\sigma_0^2}{2}\left[\frac{\widehat{\delta\kappa_x}}{k_{\beta 0}^2} + \frac{\widehat{\delta\kappa_y}}{k_{\beta 0}^2}\right] + (\sigma_0^2-\sigma^2)\frac{\widehat{\delta Q}}{Q} + \sigma^2\left[\frac{\widehat{\delta\varepsilon_x}}{\varepsilon} + \frac{\widehat{\delta\varepsilon_y}}{\varepsilon}\right] = \text{const}$$

$$\widehat{\delta p}_- = -\frac{\sigma_0^2}{2}\left[\frac{\widehat{\delta\kappa_x}}{k_{\beta 0}^2} - \frac{\widehat{\delta\kappa_y}}{k_{\beta 0}^2}\right] + \sigma^2\left[\frac{\widehat{\delta\varepsilon_x}}{\varepsilon} - \frac{\widehat{\delta\varepsilon_y}}{\varepsilon}\right] = \text{const}$$

The solution is given by the substitution in the expression for the adiabatic solution:

- Manipulate Green's function solution to show (similar to Adiabatic case steps)

$$\frac{\delta r_\pm(s)}{r_m} = \frac{\delta p_\pm(s)}{\sigma_\pm^2}$$

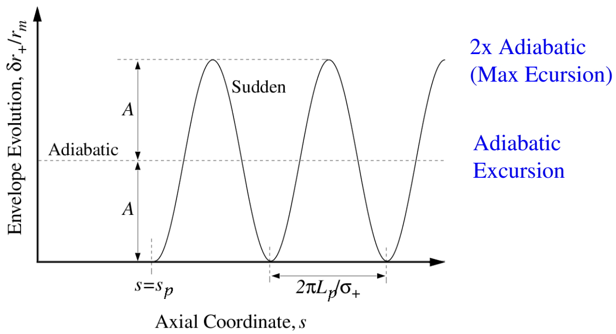
with

$$\delta p_\pm(s) \rightarrow \widehat{\delta p}_\pm\left[1-\cos\left(\sigma_\pm\frac{s-s_p}{L_p}\right)\right]\Theta(s-s_p)$$

Sudden perturbation solution, substitute in pervious adiabatic expressions:

$$\boxed{\frac{\delta r_\pm(s)}{r_m} = \frac{\widehat{\delta p_\pm}}{\sigma_\pm^2}\left[1-\cos\left(\sigma_\pm \frac{s-s_p}{L_p}\right)\right]\Theta(s-s_p)}$$

Illustration of solution properties for a sudden $\delta p_+(s)$ perturbation term



For the same amplitude of total driving perturbations, sudden perturbations result in 2x the envelope excursion that adiabatic perturbations produce

Continuous Focusing – Driven perturbations on a continuously focused matched equilibrium (summary)

Adiabatic Perturbations:

- Essentially a rematch of equilibrium beam if the change is slow relative to quadrupole envelope mode oscillations (phase advance σ_-)

Sudden Perturbations:

- Projects onto breathing and quadrupole envelope modes with 2x adiabatic amplitude oscillating from zero to max amplitude

Ramped Perturbations: (see PRSTAB article; based on Green's function)

- Can be viewed as a superposition between the adiabatic and sudden form perturbations

Harmonic Perturbations: (see PRSTAB article; based on Green's function)

- Can build very general cases of driven perturbations by linear superposition
- Results may be less "intuitive" (expressed in complex form)

Cases covered in class illustrate a range of common behavior and help build intuition on what can drive envelope oscillations and the relative importance of various terms as a function of space-charge strength

Appendix A: Particular Solution for Driven Envelope Modes

Lund and Bukh, PRSTAB **7**, 024801 (2004)

Following Wiedemann (Particle Accelerator Physics, 1993, pp 106) first, consider more general *Driven Hill's Equation*

$$\boxed{x'' + \kappa(s)x = p(s)}$$

The corresponding homogeneous equation:

$$\boxed{x'' + \kappa(s)x = 0}$$

has principal solutions

$$x(s) = C_1\mathcal{C}(s) + C_2\mathcal{S}(s) \qquad C_1, C_2 = \text{constants}$$

where

Cosine-Like Solution	Sine-Like Solution
$\mathcal{C}'' + \kappa(s)\mathcal{C} = 0$	$\mathcal{S}'' + \kappa(s)\mathcal{S} = 0$
$\mathcal{C}(s=s_i) = 1$	$\mathcal{S}(s=s_i) = 0$
$\mathcal{C}'(s=s_i) = 0$	$\mathcal{S}'(s=s_i) = 1$

Recall that the homogeneous solutions have the Wronskian symmetry:

- See S.M. Lund lectures on Transverse Dynamics, S5C

$$W(s) = \mathcal{C}(s)\mathcal{S}'(s) - \mathcal{C}'(s)\mathcal{S}(s) = 1$$

A particular solution to the *Driven Hill's Equation* can be constructed using a Greens' function method:

$$\boxed{\begin{aligned} x(s) &= \int_{s_i}^{s} d\tilde{s}\; G(s,\tilde{s})p(\tilde{s}) \\ G(s,\tilde{s}) &= \mathcal{S}(s)\mathcal{C}(\tilde{s}) - \mathcal{C}(s)\mathcal{S}(\tilde{s}) \end{aligned}}$$

Demonstrate this works by first taking derivatives:

$$x = \mathcal{S}(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \; - \; \mathcal{C}(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s})$$

$$\begin{aligned} x' &= \mathcal{S}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \; - \; \mathcal{C}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s}) \\ &\quad + p(s)\,\underbrace{[\mathcal{S}(s)\mathcal{C}(s) - \mathcal{S}(s)\mathcal{C}(s)]}_{0} \\ &= \mathcal{S}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \; - \; \mathcal{C}'(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s}) \end{aligned}$$

$$\begin{aligned} x'' &= \mathcal{S}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \; - \; \mathcal{C}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s}) \\ &\quad + p(s)\,\underbrace{[\mathcal{S}'(s)\mathcal{C}(s) - \mathcal{C}'(s)\mathcal{S}(s)]}_{1} \quad \text{Wronskian Symmetry} \\ &= p(s) \; + \; \mathcal{S}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{C}(\tilde{s})p(\tilde{s}) \; - \; \mathcal{C}''(s)\int_{s_i}^{s} d\tilde{s}\; \mathcal{S}(\tilde{s})p(\tilde{s}) \end{aligned}$$

Insert these results in the *Driven Hill's Equation*:

Definition of Principal Orbit Functions

$$x'' + \kappa(s)x = p(s) + [\mathcal{S}'' \cancel{+ \kappa\mathcal{S}}]^{0} \int_{s_i}^{s} d\tilde{s}\, \mathcal{C}(\tilde{s})p(\tilde{s}) - [\mathcal{C}'' \cancel{+ \kappa\mathcal{C}}]^{0} \int_{s_i}^{s} d\tilde{s}\, \mathcal{S}(\tilde{s})p(\tilde{s})$$

$$= p(s)$$

Thereby proving we have a valid particular solution. The general solution to the *Driven Hill's Equation* is then:

- Choose constants $C_1,\ C_2$ consistent with particle initial conditions at $s = s_i$

$$\boxed{\begin{aligned} &x(s) = x(s_i)\mathcal{C}(s) + x'(s_i)\mathcal{S}(s) + \int_{s_i}^{s} d\tilde{s}\, G(s,\tilde{s})p(\tilde{s}) \\ &G(s,\tilde{s}) = \mathcal{S}(s)\mathcal{C}(\tilde{s}) - \mathcal{C}(s)\mathcal{S}(\tilde{s}) \end{aligned}}$$

Apply these results to the driven perturbed envelope equation:

$$\boxed{\frac{d^2}{ds^2}\delta r_\pm + \frac{\sigma_\pm^2}{L_p^2}\delta r_\pm = \frac{r_m}{L_p^2}\delta p_\pm}$$

The homogeneous equations can be solved exactly for continuous focusing:

$$\boxed{\begin{aligned} \mathcal{C}(s) &= \cos\left(\sigma_\pm \frac{s - s_i}{L_p}\right) \\ \mathcal{S}(s) &= \frac{L_p}{\sigma_\pm}\sin\left(\sigma_\pm \frac{s - s_i}{L_p}\right) \end{aligned}}$$

and the Green's function can be simplified as:

$$\begin{aligned} G(s,\tilde{s}) &= \mathcal{S}(s)\mathcal{C}(\tilde{s}) - \mathcal{C}(s)\mathcal{S}(\tilde{s}) \\ &= \frac{L_p}{\sigma_\pm}\left\{\sin\left(\sigma_\pm \frac{s - s_i}{L_p}\right)\cos\left(\sigma_\pm \frac{\tilde{s} - s_i}{L_p}\right) - \cos\left(\sigma_\pm \frac{s - s_i}{L_p}\right)\sin\left(\sigma_\pm \frac{\tilde{s} - s_i}{L_p}\right)\right\} \\ &= \frac{L_p}{\sigma_\pm}\sin\left(\sigma_\pm \frac{s - \tilde{s}}{L_p}\right) \end{aligned}$$

Using these results the particular solution for the driven perturbed envelope equation can be expressed as:

- Here we rescale the Green's function to put in the form given in S8

$$\boxed{\begin{aligned} \frac{\delta r_\pm(s)}{r_m} &= \frac{1}{L_p^2}\int_{s_i}^{s} d\tilde{s}\, G_\pm(s,\tilde{s})\delta p_\pm(\tilde{s}) \\ G_\pm(s,\tilde{s}) &= \frac{1}{\sigma_\pm/L_p}\sin\left(\sigma_\pm \frac{s - \tilde{s}}{L_p}\right) \end{aligned}}$$

Simplified Treatment of Envelope Modes in Continuous Focusing Channels

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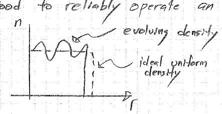
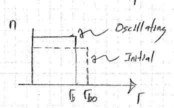
<u>Envelope Modes</u>: Introduction to collective beam behavior

Space charge also results in many collective waves and instabilities which must be understood to reliably operate an accelerator with intense space charge.



⇒ ripples produce nonlinear force.



lowest order collective mode:
self-similarly oscillate uniform density
⇒ "envelope mode"

Let's examine this situation using <u>statistical averages</u> over the beam

$$\boxed{\langle \{\cdots\} \rangle = \frac{\sum_i \{\cdots\}_i}{N} = \frac{\iint \{\cdots\} f(\vec{x}_\perp, \vec{x}'_\perp, s)\, d^2x_\perp d^2x'_\perp}{\iint_{\text{phase-space}} f(\vec{x}_\perp, \vec{x}'_\perp, s)\, d^2x_\perp d^2x'_\perp}}$$

some quantity; Average over n particles; continuum approx.

$f(\vec{x}_\perp, \vec{x}'_\perp, s)$ = beam distribution

$\rho(\vec{x}_\perp, s) = q\int f(\vec{x}, \vec{x}'_\perp, s)\, d^2x'_\perp$ = charge density

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Relate the beam edge to a statistical average over the uniform density beam:

$$\boxed{r_b = 2\langle x^2 \rangle^{1/2}}$$

Proof:

$$\langle x^2 \rangle = \frac{\iint_{\text{phase-space}} x^2 f\, d^2x\, d^2x'}{\iint_{\text{phase space}} f\, d^2x\, d^2x'}$$

Take norm $\int_{\text{angles}} f\, d^2x' = n(x, y, s)$ = density

$$\langle x^2 \rangle = \frac{\int_{\text{round beam}} x^2 n\, d^2x}{\int_{\text{round beam}} n\, d^2x} = \frac{\int_0^{2\pi}\cos^2\theta\, d\theta \int_0^{r_b} r^2 \cancel{n}\, r\, dr}{\cancel{n}\,\pi r_b^2}$$

$$= \frac{\pi r^4 \big|_0^{r_b}}{4\pi r_b^2} = \frac{r_b^2}{4}$$

$$\Rightarrow\ r_b = 2\langle x^2 \rangle^{1/2}$$

Next differentiate this equation for r_b twice to obtain an evolution equation for r_b

$$\frac{d}{ds} r_b = \frac{d}{ds} 2\langle x^2 \rangle^{1/2} = \frac{2\langle xx' \rangle}{\langle x^2 \rangle^{1/2}}$$

$$\frac{d^2}{ds^2} r_b = \frac{2\langle x'^2 \rangle + 2\langle xx'' \rangle}{\langle x^2 \rangle^{1/2}} - \frac{2\langle xx' \rangle^2}{\langle x^2 \rangle^{3/2}} \qquad ' = \frac{d}{ds}$$

Or

$$\boxed{\frac{d^2}{ds^2} r_b - \frac{2\langle xx'' \rangle}{\langle x^2 \rangle^{1/2}} - \frac{2\left[\langle x^2 \rangle\langle x'^2 \rangle - \langle xx' \rangle^2\right]}{\langle x^2 \rangle^{3/2}} = 0}$$

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Next, apply the uniform density beam equation of motion.

$$x'' + k_{\beta 0}^2 x - \frac{Q}{r_b^2} x = 0$$

to simplify

$$\langle x x''\rangle = -k_{\beta 0}^2 \langle x^2\rangle + \frac{Q}{r_b^2}\langle x^2\rangle$$

$$\Rightarrow \frac{d^2}{ds^2} r_b + 2k_{\beta 0}^2 \frac{\langle x^2\rangle}{\langle x^2\rangle^{1/2}} - \frac{2Q}{r_b^2}\langle x^2\rangle^{1/2} - \frac{2[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2]}{\langle x^2\rangle^{3/2}} = 0$$

Use:

$$r_b = 2\langle x^2\rangle^{1/2}$$

$$\varepsilon_{x,rms} = [\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2]^{1/2}$$ = rms emittance.

Statistical measure beam phase-space area.

rms Envelope equation

$$\Rightarrow \boxed{\frac{d^2}{ds^2} r_b + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{16\varepsilon_{x,rms}^2}{r_b^3} = 0}$$

More advanced studies show that

$\varepsilon_{x,rms}$ = const for linear space-charge forces

$\varepsilon_{x,rms}$ varies little for most "reasonable" distributions in linear transport channels.

For r_b = const, $\varepsilon_{x,rms}$ = const

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} = \frac{16\varepsilon_{x,rms}^2}{r_b^3} \Rightarrow \boxed{k_\beta^2 = k_{\beta 0}^2 - \frac{Q}{r_b^2} = \frac{16\varepsilon_{x,rms}^2}{r_b^2}}$$

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Try to better understand small-amplitude oscillations about equilibrium "matched" beam

$$\boxed{\begin{array}{l} r_b = r_{b0} = \text{const} \Rightarrow \text{"Matched" Equilibrium} \\ k_{\beta 0}^2 r_{b0} - \frac{Q}{r_{b0}} - \frac{16\varepsilon_{x,rms}^2}{r_{b0}^3} = 0 \quad (*) \end{array}}$$

(*) constrains parameters of matched core in "force balance"

Set: $\boxed{r_b = r_{b0} + \delta r_b \quad ; \quad |\delta r_b|/r_{b0} \ll 1}$ "Breathing" Oscillation.

$$\rightarrow \frac{d^2}{ds^2}\delta r_b + k_{\beta 0}^2[r_{b0} + \delta r_b] - \frac{Q}{r_{b0}+\delta r_b} - \frac{16\varepsilon_{x,rms}^2}{(r_{b0}+\delta r_b)^3} = 0$$

Linearize:

$$\frac{d^2}{ds^2}\delta r_b + k_{\beta 0}^2 r_{b0} + k_{\beta 0}^2 \delta r_b - \frac{Q}{r_{b0}}\left[1 - \frac{\delta r_b}{r_{b0}}\right] - \frac{16\varepsilon_{x,rms}^2}{r_{b0}^3}\left[1 - 3\frac{\delta r_b}{r_{b0}}\right] = 0$$

$$\frac{d^2}{ds^2}\delta r_b + \underbrace{\left[k_{\beta 0}^2 r_{b0} - \frac{Q}{r_{b0}} - \frac{16\varepsilon_{x,rms}^2}{r_{b0}^3}\right]}_{= 0 \text{ from } *} + k_{\beta 0}^2\delta r_b + \underbrace{\frac{Q}{r_{b0}^2}}_{\frac{Q}{r_{b0}^2} = k_{\beta 0}^2 - k_\beta^2}\delta r_b + 3\underbrace{\frac{16\varepsilon_{x,rms}^2}{r_{b0}^4}}_{= k_\beta^2 = k_{\beta 0}^2 - \frac{Q}{r_{b0}^2} = \frac{16\varepsilon_{x,rms}^2}{r_{b0}^4} \text{ from } *}\delta r_b = 0$$

$$\frac{d^2}{ds^2}\delta r_b + k_{\beta 0}^2\delta r_b + (k_{\beta 0}^2 - k_\beta^2)\delta r_b + 3k_\beta^2\delta r_b = 0$$

$$\boxed{\frac{d^2}{ds^2}\delta r_b + (2k_{\beta 0}^2 + 2k_\beta^2)\delta r_b = 0} \Rightarrow \boxed{\frac{d^2}{ds^2}\delta r_b + k_+^2\delta r_b = 0}$$

* simple harmonic oscillator equation with restoring wavenumber $k_+ = \sqrt{2k_{\beta 0}^2 + 2k_\beta^2}$

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Try to better understand small-amplitude oscillations about equilibrium "matched" beam

$$\boxed{\begin{array}{l} r_b = r_{b0} = \text{const} \Rightarrow \text{"Matched" Equilibrium} \\ k_{\beta 0}^2 r_{b0} - \frac{Q}{r_{b0}} - \frac{16\varepsilon_{x,rms}^2}{r_{b0}^3} = 0 \quad (*) \end{array}}$$

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Set: $\boxed{r_b = r_{b0} + \delta r_b \quad ; \quad |\delta r_b|/r_{b0} \ll 1}$ "Breathing" Oscillation.

$$\rightarrow \frac{d^2}{ds^2}\delta r_b + k_{\beta 0}^2[r_{b0} + \delta r_b] - \frac{Q}{r_{b0}+\delta r_b} - \frac{16\varepsilon_{x,rms}^2}{(r_{b0}+\delta r_b)^3} = 0$$

Linearize:

$$\frac{d^2}{ds^2}\delta r_b + k_{\beta 0}^2 r_{b0} + k_{\beta 0}^2 \delta r_b - \frac{Q}{r_{b0}}\left[1 - \frac{\delta r_b}{r_{b0}}\right] - \frac{16\varepsilon_{x,rms}^2}{r_{b0}^3}\left[1 - 3\frac{\delta r_b}{r_{b0}}\right] = 0$$

$$\frac{d^2}{ds^2}\delta r_b + \underbrace{\left[k_{\beta 0}^2 r_{b0} - \frac{Q}{r_{b0}} - \frac{16\varepsilon_{x,rms}^2}{r_{b0}^3}\right]}_{= 0 \text{ from } *} + k_{\beta 0}^2\delta r_b + \underbrace{\frac{Q}{r_{b0}^2}}_{\frac{Q}{r_{b0}^2} = k_{\beta 0}^2 - k_\beta^2}\delta r_b + 3\underbrace{\frac{16\varepsilon_{x,rms}^2}{r_{b0}^4}}_{= k_\beta^2 = k_{\beta 0}^2 - \frac{Q}{r_{b0}^2} = \frac{16\varepsilon_{x,rms}^2}{r_{b0}^4} \text{ from } *}\delta r_b = 0$$

$$\frac{d^2}{ds^2}\delta r_b + k_{\beta 0}^2\delta r_b + (k_{\beta 0}^2 - k_\beta^2)\delta r_b + 3k_\beta^2\delta r_b = 0$$

$$\boxed{\frac{d^2}{ds^2}\delta r_b + (2k_{\beta 0}^2 + 2k_\beta^2)\delta r_b = 0} \Rightarrow \boxed{\frac{d^2}{ds^2}\delta r_b + k_+^2\delta r_b = 0}$$

* simple harmonic oscillator equation with restoring wavenumber $k_+ = \sqrt{2k_{\beta 0}^2 + 2k_\beta^2}$

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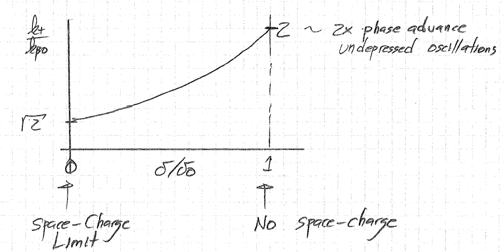
$$\frac{k_+}{k_{\beta 0}} = \sqrt{2\frac{k_{\beta 0}^2}{k_{\beta 0}^2} + 2\frac{k_\beta^2}{k_{\beta 0}^2}} = \sqrt{2 + 2(\sigma/\sigma_0)^2}$$



* Dispersion curve gives freq. of lowest collective mode response.
* If frequency breathing mode resonates with errors etc, trouble can result.
  - Mode can be destabilized
  - Find it is key in generating beam halo. — large amplitude tenuous distribution of particles outside the core.

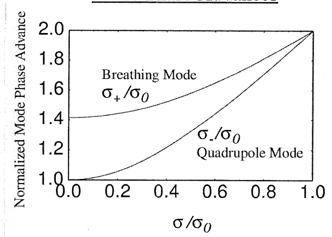
What if different perturbations along x- and y-
⇒ 2 Modes ! Breathing + Quadupole
High Freq Low freq.

17/

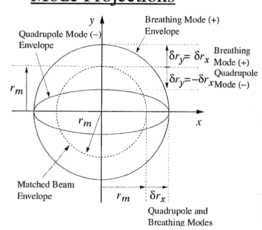
Properties of continuous focusing homogeneous solution: Normal Modes

From Lund, USPAS

Mode Phase Advances



Mode Projections



$\sigma_+ \equiv \sqrt{2\sigma_0^2 + 2\sigma^2}$ Breathing Mode: $\delta r_+ \equiv \dfrac{\delta r_x + \delta r_y}{2}$

$\sigma_- \equiv \sqrt{\sigma_0^2 + 3\sigma^2}$ Quadrupole Mode: $\delta r_- \equiv \dfrac{\delta r_x - \delta r_y}{2}$

S9: Envelope Modes in Periodic Focusing Channels

Lund and Bukh, PRSTAB **7**, 024801 (2004)

Overview

- Much more complicated than continuous focusing results
  - Lattice can couple to oscillations and destabilize the system
  - Broad parametric instability bands can result
- Instability bands calculated will exclude wide ranges of parameter space from machine operation
  - Exclusion region depends on focusing type
  - Will find that alternating gradient quadrupole focusing tends to have more instability than high occupancy solenoidal focusing due to larger envelope flutter driving stronger, broader instability
- Results in this section are calculated numerically and summarized parametrically to illustrate the full range of normal mode characteristics
  - Driven modes not considered but should be mostly analogous to CF case
  - Results presented in terms of phase advances and normalized space-charge strength to allow broad applicability
  - Coupled 4x4 eigenvalue problem and mode symmetries identified in S6 are solved numerically and analytical limits are verified
  - Carried out for piecewise constant lattices for simplicity (fringe changes little)
- More information on results presented can be found in the PRSTAB review

Procedure

1) Specify periodic lattice to be employed and beam parameters

2) Calculate undepressed phase advance σ_0 and characterize focusing strength in terms of σ_0

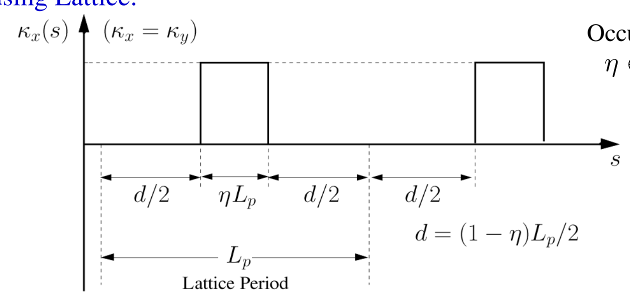
3) Find matched envelope solution to the KV envelope equation and depressed phase advance σ to estimate space-charge strength
   - Procedures described in: Lund, Chilton and Lee, PRSTAB **9**, 064201 (2006) can be applied to greatly simplify analysis, particularly where lattice is unstable
     - Instabilities complicate calculation of matching conditions

4) Calculate 4x4 envelope perturbation transfer matrix $\mathbf{M}_e(s_i + L_p|s_i)$ through one lattice period and calculate 4 eigenvalues

5) Analyze eigenvalues using symmetries to characterize mode properties
   - Instabilities
   - Stable mode characteristics and launching conditions

1st Example: Envelope Stability for Periodic Solenoid Focusing

Focusing Lattice:



Occupancy η

$\eta \in (0, 1]$

$d = (1-\eta)L_p/2$

Matched Envelope Equation:

$$\kappa_x(s) = \kappa_y(s) = \kappa(s) \qquad \varepsilon_x = \varepsilon_y = \varepsilon$$

$$r_x(s) = r_y(s) = r_m(s)$$

$$r''_m(s) + \kappa(s) r_m(s) - \frac{Q}{r_m(s)} - \frac{\varepsilon^2}{r_m^3(s)} = 0$$

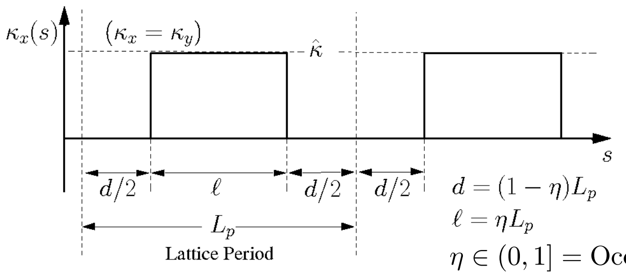
$$r_m(s + L_p) = r_m(s)$$

Using a transfer matrix approach on undepressed single-particle orbits set the strength of the focusing function for specified undepressed particle phase advance by solving:

- See: S.M. Lund, lectures on Transverse Particle Dynamics
- Particle phase-advance is measured in the rotating Larmor frame

Solenoidal Focusing - piecewise constant focusing lattice

$$\boxed{\cos\sigma_0 = \cos(2\Theta) - \frac{1-\eta}{\eta}\Theta\sin(2\Theta) \qquad \Theta \equiv \frac{\sqrt{\hat{\kappa}}L_p}{2}}$$



$d = (1-\eta)L_p$

$\ell = \eta L_p$

$\eta \in (0, 1] =$ Occupancy

Flutter scaling of the matched beam envelope varies for quadrupole and solenoidal focusing

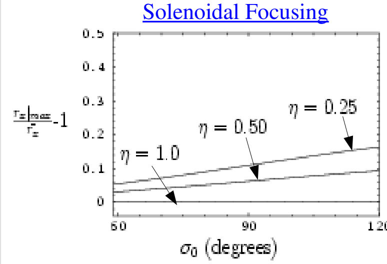
In both cases depends little on space charge with theory showing:

$$\frac{r_x|_{\max}}{\bar{r_x}} - 1 \simeq \begin{cases} (1-\cos\sigma_0)\frac{(1-\eta)(1-\eta/2)}{6} & \text{Solenoidal Focusing} \\ (1-\cos\sigma_0)^{1/2}\frac{(1-\eta/2)}{2^{3/2}(1-2\eta/3)^{1/2}} & \text{Quadrupole Focusing} \end{cases}$$

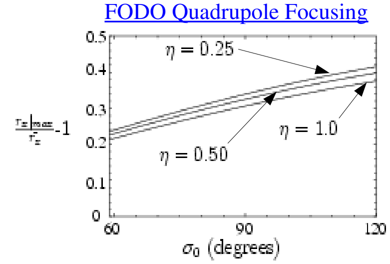
Based on: E.P. Lee, Phys. Plasmas, **9** 4301 (2002)
for limit $\sigma/\sigma_0 \to 0$

- Solenoids:
  - Varies significant in both σ_0 and η
- Quadrupoles:
  - Phase advance σ_0 variation significant
  - Occupancy η variation weak

Solenoidal Focusing

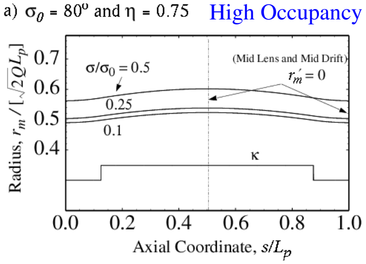


FODO Quadrupole Focusing

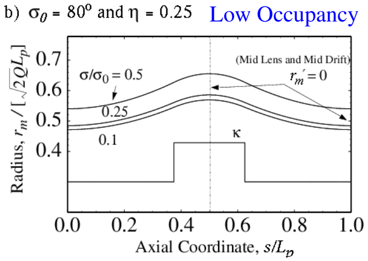


Solenoidal Focusing – Matched Envelope Solution

a) σ_0 = 80° and η = 0.75 High Occupancy



b) σ_0 = 80° and η = 0.25 Low Occupancy



Focusing:

$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

$$\kappa(s+L_p) = \kappa(s)$$

Matched Beam:

$$\varepsilon_x = \varepsilon_y = \varepsilon = \text{const}$$

$$r_{xm}(s) = r_{ym}(s) = r_m(s)$$

$$r_m(s+L_p) = r_m(s)$$

Comments:

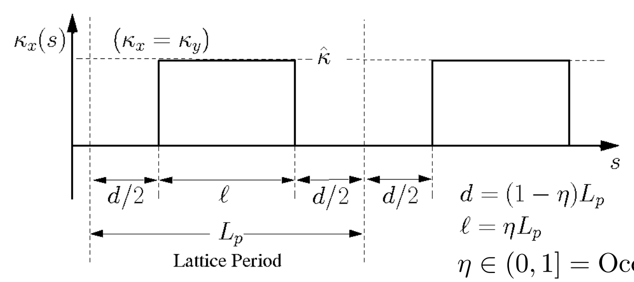
- Envelope flutter a strong function of occupancy η
  - Flutter also increases with higher values of σ_0
- Space-charge expands envelope but does not strongly modify periodic flutter

Using a transfer matrix approach on undepressed single-particle orbits set the strength of the focusing function for specified undepressed particle phase advance by solving:

- See: S.M. Lund, lectures on Transverse Particle Dynamics
- Particle phase-advance is measured in the rotating Larmor frame

Solenoidal Focusing - piecewise constant focusing lattice

$$\boxed{\cos\sigma_0 = \cos(2\Theta) - \frac{1-\eta}{\eta}\Theta\sin(2\Theta) \qquad \Theta \equiv \frac{\sqrt{\hat{\kappa}}L_p}{2}}$$

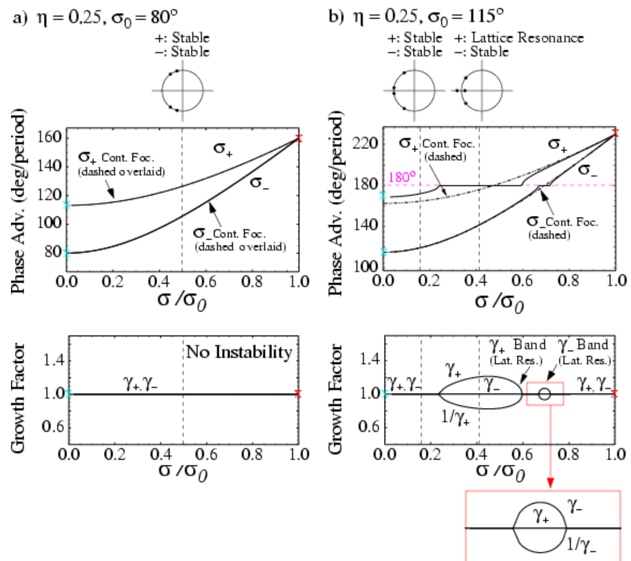


$d = (1-\eta)L_p$

$\ell = \eta L_p$

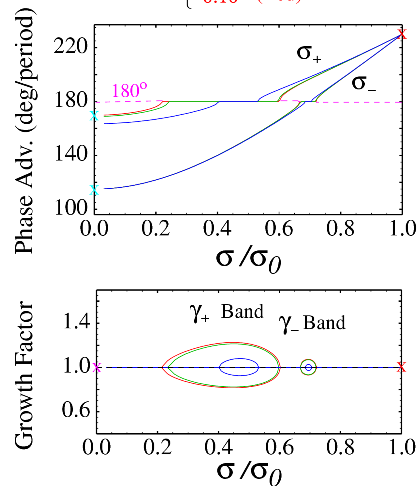
$\eta \in (0, 1] =$ Occupancy

Solenoidal Focusing – parametric plots of breathing and quadrupole envelope mode phase advances two values of undepressed phase advance



Solenoidal Focusing – mode instability bands become wider and stronger for smaller occupancy

$$\eta = \begin{cases} 0.75 & \text{(Blue)} \\ 0.25 & \text{(Green)} \\ 0.10 & \text{(Red)} \end{cases} \qquad \sigma_0 = 115^\circ$$



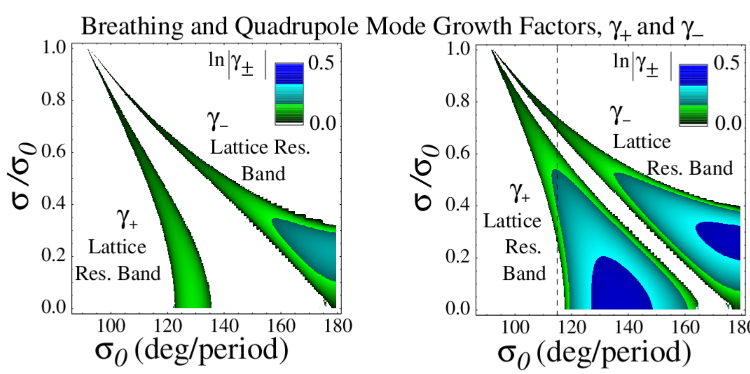
Comments:

- Mode phase advance in instability band 180 degrees per lattice period
- Significant deviations from continuous model even outside the band of instability when space-charge is strong
- Instability band becomes stronger/broader for low occupancy and weaker/narrower for high occupancy
  - Disappears at full occupancy (continuous limit)

Solenoidal Focusing – broad ranges of parametric instability are found for the breathing and quadrupole bands that must be avoided in machine operation:

Contour unstable parameters for breathing and quadrupole modes to clarify

$\eta = 0.75$ $\qquad$ $\eta = 0.25$



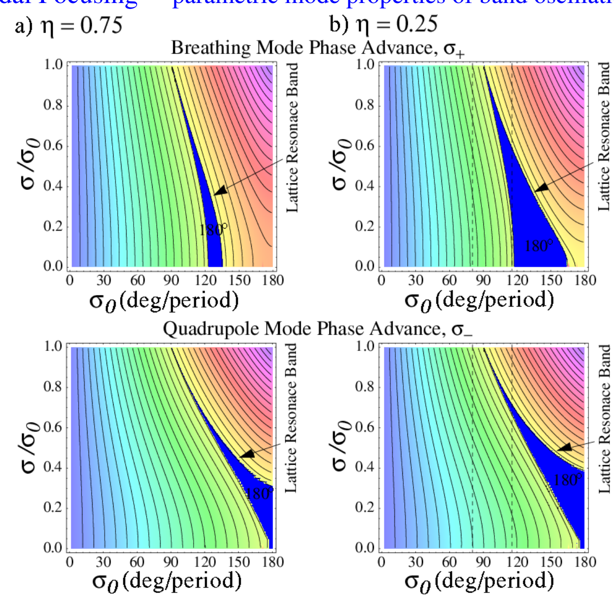
Eigenvalues in unstable regions:

$$\lambda_\pm = \gamma_\pm e^{i\pi}$$

$\gamma_\pm > 1$ for unstable growing mode

$\ln \gamma_\pm = e$-folds of growth per period

Solenoidal Focusing – parametric mode properties of band oscillations



Parametric scaling of the boundary of the region of instability

Solenoid instability bands identified as a Lattice Resonance Instability corresponding to a 1/2-integer parametric resonance between the mode oscillation frequency and the lattice

Estimate normal mode frequencies for weak focusing from continuous focusing theory:

$$\sigma_+ \simeq \sqrt{2\sigma_0^2 + 2\sigma^2}$$

$$\sigma_- \simeq \sqrt{\sigma_0^2 + 3\sigma^2}$$

This gives (measure phase advance in degrees):

Breathing Band:

$$\sigma_+ = 180^\circ$$

$$\Longrightarrow \quad \sqrt{2\sigma_0^2 + 2\sigma^2} = 180^\circ$$

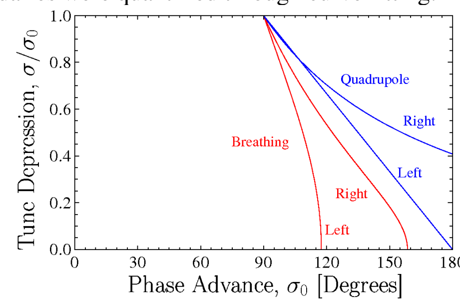
Quadrupole Band:

$$\sigma_- = 180^\circ$$

$$\Longrightarrow \quad \sqrt{\sigma_0^2 + 3\sigma^2} = 180^\circ$$

- Predictions poor due to inaccurate mode frequency estimates
  - Predictions nearer to left edge of band rather than center (expect resonance strongest at center)
- Simple resonance condition cannot predict width of band
  - Important to characterize width to avoid instability in machine designs
  - Width of band should vary strongly with solenoid occupancy η

To provide an approximate guide on the location/width of the breathing and quadrupole envelope bands, many parametric runs were made and the instability band boundaries were quantified through curve fitting:



Breathing Band Boundaries:

$$\sigma^2 + f\sigma_0^2 = (90^\circ)^2(1+f)$$

$$f = f(\sigma_0, \eta) = \begin{cases} 1.113 - 0.413\eta + 0.00348\sigma_0, & \text{left-edge} \\ 1.046 + 0.318\eta - 0.00410\sigma_0, & \text{right-edge} \end{cases}$$

Quadrupole Band Boundaries:

Left: $\sigma/\sigma_0 + g\dfrac{\sigma_0}{90^\circ} = 1 + g$

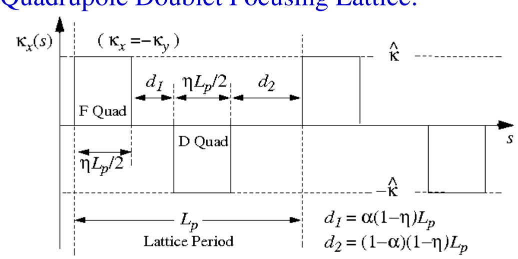
Right: $\sigma + g\sigma_0 = 90^\circ(1+g)$

$$g = g(\eta) = \begin{cases} 1, & \text{left-edge} \\ 0.227 - 0.173\eta, & \text{right-edge} \end{cases}$$

- Breathing band: maximum errors ~5 /~2 degrees on left/right boundaries
- Quadrupole band: maximum errors ~8/~3 degrees on left/right boundaries

2nd Example: Env Stability for Periodic Quadrupole Focusing

Quadrupole Doublet Focusing Lattice:



$\eta \in (0, 1]$ Occupancy

$\alpha \in [0, 1/2]$ Syncopation Factor

$\alpha = 1/2 \rightarrow$ FODO

Matched Envelope Equation:

$$r''_{xm}(s) + \kappa_x(s) r_{xm}(s) - \frac{2Q}{r_{xm}(s) + r_{ym}(s)} - \frac{\varepsilon_x^2}{r_{xm}^3(s)} = 0$$

$$r''_{ym}(s) + \kappa_y(s) r_{ym}(s) - \frac{2Q}{r_{xm}(s) + r_{ym}(s)} - \frac{\varepsilon_y^2}{r_{ym}^3(s)} = 0$$

$$r_{xm}(s + L_p) = r_{xm}(s) \qquad r_{xm}(s) > 0$$

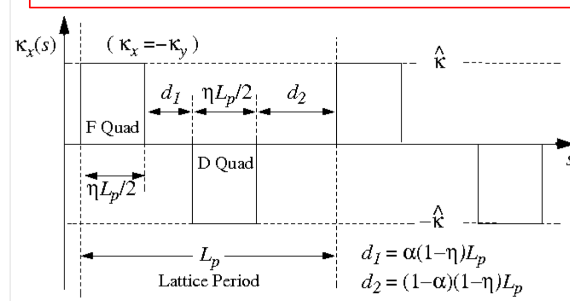
$$r_{ym}(s + L_p) = r_{ym}(s) \qquad r_{ym}(s) > 0$$

Using a transfer matrix approach on undepressed single-particle orbits set the strength of the focusing function for specified undepressed particle phase advance by solving:

- See: S.M. Lund, lectures on Transverse Particle Dynamics

Quadrupole Doublet Focusing - piecewise constant focusing lattice

$$\boxed{\begin{aligned} \cos\sigma_0 = \cos\Theta\cosh\Theta + \frac{1-\eta}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta) \\ - 2\alpha(1-\alpha)\frac{(1-\eta)^2}{\eta^2}\Theta^2\sin\Theta\sinh\Theta \end{aligned} \qquad \Theta \equiv \frac{\sqrt{|\hat{\kappa}|}L_p}{2}}$$



$\eta \in (0, 1]$ Occupancy

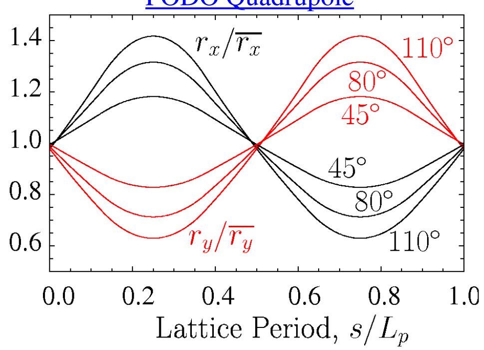
$\alpha \in [0, 1/2]$ Syncopation Factor

$\alpha = 1/2 \rightarrow$ FODO

Envelope Flutter Scaling of Matched Envelope Solution

For FODO quadrupole transport, plot relative matched beam envelope excursions for a fixed form focusing lattice and fixed beam perveance as the strength of applied focusing strength increases as measured by σ_0

FODO Quadrupole



$$\overline{r_x} = \int_0^{L_p} \frac{ds}{L_p} r_x(s)$$

$\eta = 0.5 \quad L_p = 0.5$ m

$Q = 5 \times 10^{-4}$

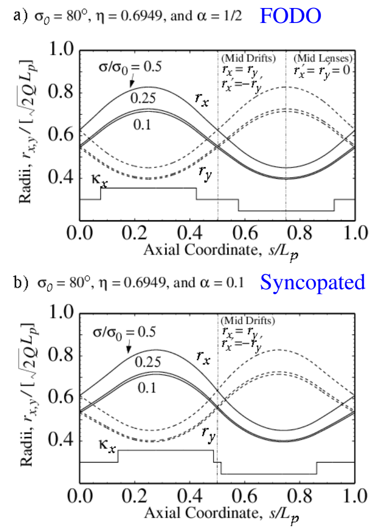
$\varepsilon_x = \varepsilon_y = 50$ mm-mrad

σ_0	σ/σ_0
45°	0.20
80°	0.26
110°	0.32

- Larger matched envelope "flutter" corresponds to larger σ_0
  - More flutter results in higher prospects for instability due to transfer of energy from applied focusing
- Little dependence of flutter on quadrupole occupancy η

Quadrupole Doublet Focusing – Matched Envelope Solution

FODO and Syncopated Lattices



Focusing:

$$\kappa_x(s) = -\kappa_y(s) = \kappa(s)$$

$$\kappa(s + L_p) = \kappa(s)$$

Matched Beam:

$$\varepsilon_x = \varepsilon_y = \varepsilon = \text{const}$$

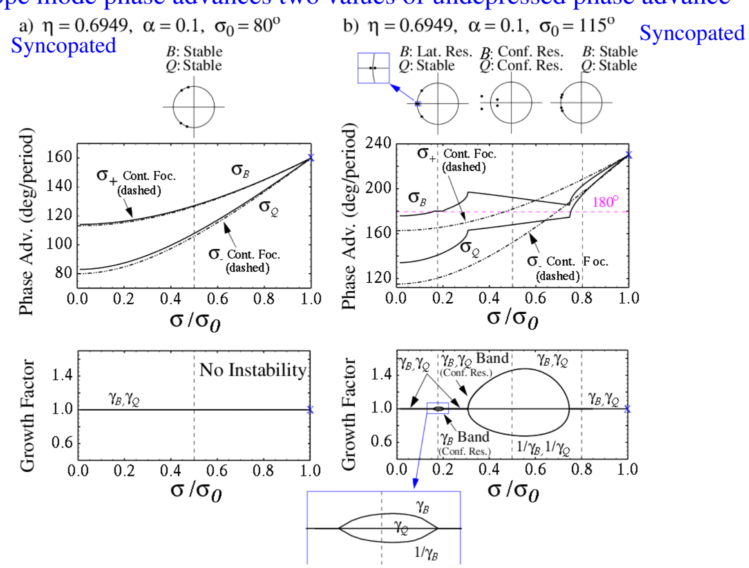
$$r_{xm}(s + L_p) = r_{xm}(s)$$

$$r_{ym}(s + L_p) = r_{ym}(s)$$

Comments:

- Envelope flutter a *weak* function of occupancy η
- Syncopation factors $\alpha \neq 1/2$ reduce envelope symmetry and can drive more instabilities
- Space-charge expands envelope

Quadrupole Focusing – parametric plots of breathing and quadrupole envelope mode phase advances two values of undepressed phase advance



Important point:

For quadrupole focusing the normal mode coordinates are *NOT*

$$\delta r_\pm = \frac{\delta r_x \pm \delta r_y}{2} \qquad \begin{array}{lll} \delta r_+ & \Leftrightarrow & \text{Breathing Mode} \\ \delta r_- & \Leftrightarrow & \text{Quadrupole Mode} \end{array}$$

- Only works for axisymmetric focusing $(\kappa_x = \kappa_y = \kappa)$ with an axisymmetric matched beam $(\varepsilon_x = \varepsilon_y = \varepsilon)$

However, for low σ_0 we will find that the two stable modes correspond closely in frequency with continuous focusing model breathing and quadrupole modes even though they have different symmetry properties in terms of normal mode coordinates. Due to this, we denote:

Subscript B <==> Breathing Mode
Subscript Q <==> Quadrupole Mode

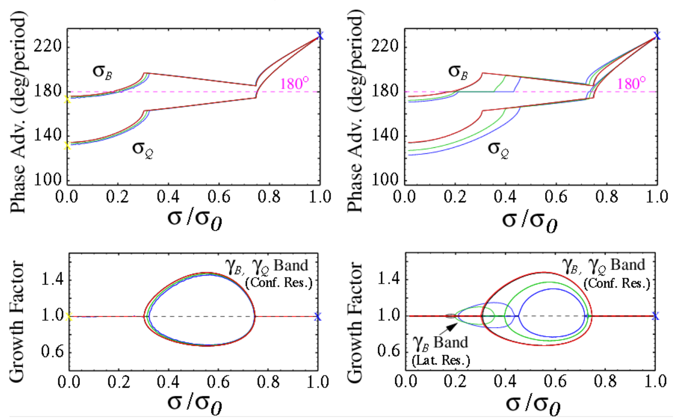
- Label branches breathing and quadrupole in terms of low σ_0 branch frequencies corresponding to breathing and quadrupole frequencies from continuous theory
- Continue label to larger values of σ_0 where frequency correspondence with continuous modes breaks down

Quadrupole Focusing – mode instability bands vary little/strongly with occupancy for FODO/syncopated lattices

a) $\alpha = 1/2$ (FODO), $\sigma_0 = 115^\circ$ — FODO

b) $\alpha = 0.1$, $\sigma_0 = 115^\circ$ — Syncopated

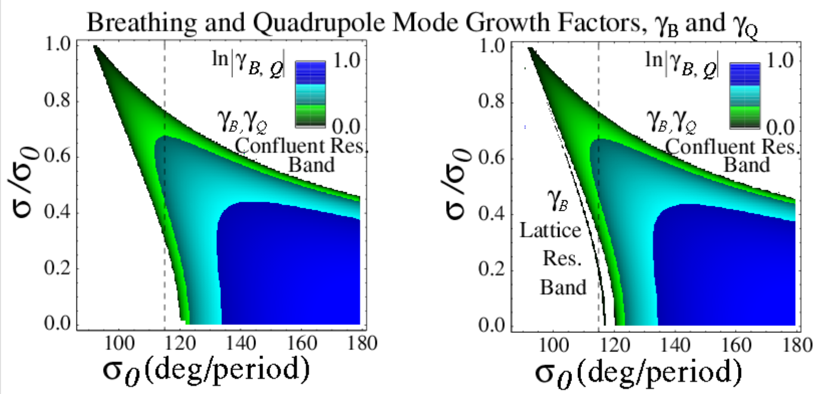
$\eta = $ 0.90 (Blue), 0.6949 (Black), 0.25 (Green), 0.10 (Red)



Quadrupole Focusing – broad ranges of parametric instability are found for the breathing and quadrupole bands that must be avoided in machine operation: Contour parameter ranges of instability to clarify

FODO Lattice $\eta = 0.6949, \quad \alpha = 1/2$

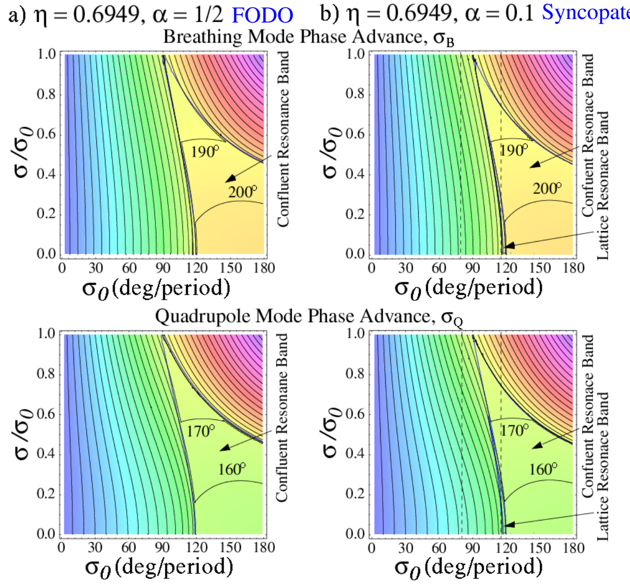
Syncopated Lattice $\eta = 0.6949, \quad \alpha = 0.1$



$\ln|\gamma_{B,Q}| = e$-folds of growth per period of unstable mode

Quadrupole Focusing – parametric mode properties of band oscillations

a) $\eta = 0.6949$, $\alpha = 1/2$ FODO b) $\eta = 0.6949$, $\alpha = 0.1$ Syncopated



Parametric scaling of the boundary of the region of instability

Quadrupole instability bands identified:

- Confluent Band: 1/2-integer parametric resonance between *both* breathing and quadrupole modes and the lattice
- Lattice Resonance Band (Syncopated lattice only): 1/2-integer parametric resonance between *one* envelope mode and the lattice

Estimate mode frequencies for weak focusing from continuous focusing theory:

$$\sigma_B = \sigma_+ = \sqrt{2\sigma_0^2 + 2\sigma^2}$$

$$\sigma_Q = \sigma_- = \sqrt{\sigma_0^2 + 3\sigma^2}$$

This gives (measure phase advance in degrees here):

Confluent Band:

$$(\sigma_+ + \sigma_-)/2 = 180^\circ$$

$$\Longrightarrow \quad \sqrt{2\sigma_0^2 + 2\sigma^2} + \sqrt{\sigma_0^2 + 3\sigma^2} = 360^\circ$$

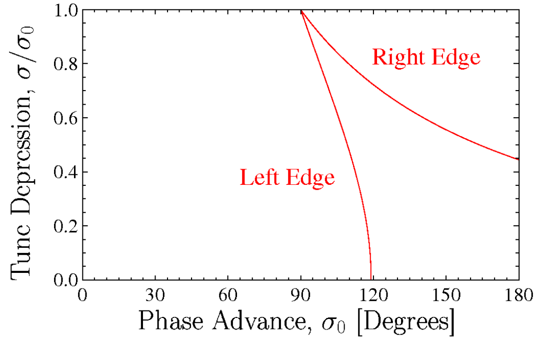
Lattice Resonance Band:

$$\sigma_+ = 180^\circ$$

$$\Longrightarrow \quad \sqrt{2\sigma_0^2 + 2\sigma^2} = 180^\circ$$

- Predictions poor due to inaccurate mode frequency estimates from continuous model
  - Predictions nearer to edge of band rather than center (expect resonance strongest at center)
- Cannot predict width of band
  - Important to characterize to avoid instability

To provide a rough guide on the location/width of the important FODO confluent instability band, many parametric runs were made and the instability region boundary was quantified through curve fitting:



Left Edge Boundary:

$$\sigma^2 + f(\eta)\sigma_0^2 = (90^\circ)^2[1 + f(\eta)]$$

$$f(\eta) = \frac{4}{3}$$

Right Edge Boundary:

$$\sigma + g(\eta)\sigma_0 = 90^\circ[1 + g(\eta)]$$

$$g(\eta) = \frac{1}{9}$$

- Negligible variation in quadrupole occupancy η is observed
- Formulas have a maximum error ~5 and ~2 degrees on left and right boundaries

Pure mode launching conditions for quadrupole focusing

Launching a pure breathing (B) or quadrupole (Q) mode in alternating gradient quadrupole focusing requires specific projections that generally require an eigenvalue/eigenvector analysis of symmetries to carry out

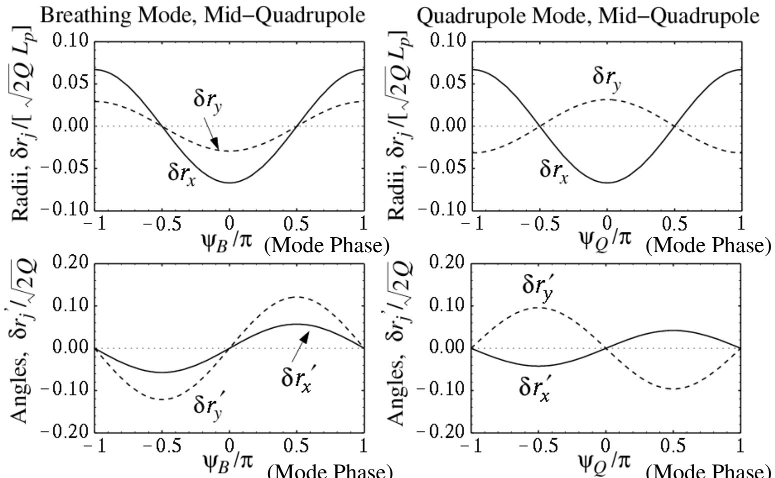
- See eignenvalue symmetries given in S6

Show example launch conditions for:

FODO Lattice

$$\eta = 0.6949$$
$$\sigma_0 = 80^\circ$$
$$\sigma/\sigma_0 = 0.2$$

Quadrupole Focusing – projections of perturbations on pure modes varies strongly with mode phase and the location in the lattice (FODO example)

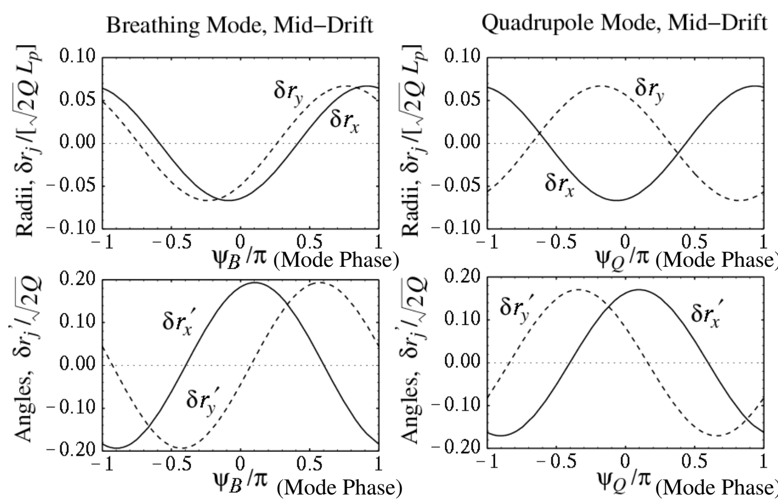


$\delta r_x \neq \delta r_y$

generally not exact breathing symmetry

$\delta r_x \neq -\delta r_y$

generally not exact quadrupole symmetry



$\delta r_x \neq \delta r_y$

generally not exact breathing symmetry

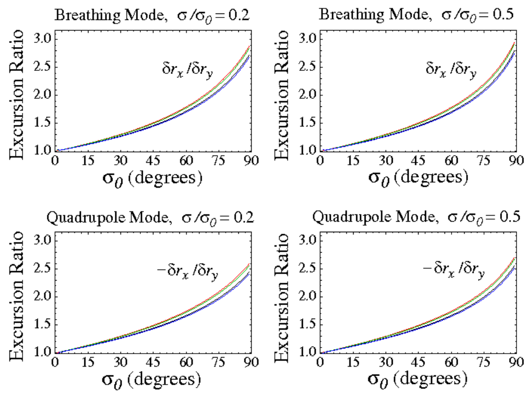
$\delta r_x \neq -\delta r_y$

generally not exact quadrupole symmetry

As a further guide in pure mode launching, summarize FODO results for:

- Mid-axial location of an x-focusing quadrupole with the additional choice $\delta r'_j = 0$
- Specify ratio of $\delta r_x/\delta r_y$ to launch pure mode
- Plot as function of σ_0 for $\sigma_0 < 90^\circ$
  - Results vary little with occupancy η or σ/σ_0

$\eta = \begin{cases} 0.90 & \text{(Blue)} \\ 0.6949 & \text{(Black)} \\ 0.25 & \text{(Green)} \\ 0.10 & \text{(Red)} \end{cases}$



Specific mode phase in this case due to the choice $\delta r'_x = 0 = \delta r'_y$ at launch location

Comments:

- For quadrupole transport using the axisymmetric equilibrium projections on the breathing (+) mode and quadrupole (-) mode will *NOT* generally result in nearly pure mode projections:

$$\delta r_+ \equiv \frac{\delta r_x + \delta r_y}{2} \quad \neq \text{Breathing Mode Projection}$$

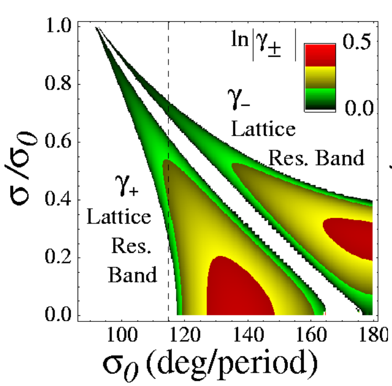
$$\delta r_- \equiv \frac{\delta r_x - \delta r_y}{2} \quad \neq \text{Quadrupole Mode Projection}$$

  - Mistake can be commonly found in research papers and can confuse analysis of Suppositly pure classes of envelope oscillations which are not.
  - Recall: reason denoted generalization of breathing mode with a subscript B and quadrupole mode with a subscript Q was an attempt to avoid confusion by overgeneralization
- Must solve for eigenvectors of 4x4 envelope transfer matrix through one lattice period calculated from the launch location in the lattice and analyze symmetries to determine proper projections (see S6)
- Normal mode coordinates can be found for the quadrupole and breathing modes in AG quadrupole focusing lattices through analysis of the eigenvectors but the expressions are typically complicated
  - Modes have underlying Courant-Snyder invariant but it will be a complicated

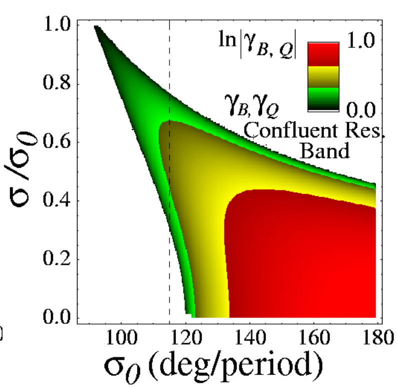
Summary: Envelope band instabilities and growth rates for periodic solenoidal and quadrupole doublet focusing lattices have been described

Envelope Mode Instability Growth Rates

Solenoid (η = 0.25)



Quadrupole FODO (η = 0.70)



Summary Discussion: Envelope modes in periodic focusing lattices

- Envelope modes are low order collective oscillations and since beam mismatch always exists, instabilities and must be avoided for good transport
- KV envelope equations faithfully describe the low order force balance acting on a beam and can be applied to predict locations of envelope instability bands in periodic focusing
- Absence of envelope instabilities for a machine operating point is a *necessary condition* but not *sufficient condition* for a good operating point
  - Higher order kinetic instabilities possible: see lectures on Transverse Kinetic Theory
- Launching pure modes in alternating gradient periodic focusing channels requires analysis of the mode eigenvalues/eigenvectors
  - Even at symmetrical points in lattices, launching conditions can be surprisingly complex
- Driven modes for periodic focusing will be considerably more complex than for continuous focusing
  - Can be analyzed paralleling the analysis given for continuous focusing and likely have similar characteristics where the envelope is stable.

References: For more information see:

These course notes are posted with updates, corrections, and supplemental material at:

https://people.nscl.msu.edu/~lund/uspas/bpisc_2017

Materials associated with previous and related versions of this course are archived at:

JJ Barnard and SM Lund, *Beam Physics with Intense Space-Charge,* USPAS:
- https://people.nscl.msu.edu/~lund/uspas/bpisc_2015 2015 Version
- http://hifweb.lbl.gov/USPAS_2011 2011 Lecture Notes + Info
- http://uspas.fnal.gov/programs/past-programs.shtml (2008, 2006, 2004)

JJ Barnard and SM Lund, *Interaction of Intense Charged Particle Beams with Electric and Magnetic Fields*, UC Berkeley, Nuclear Engineering NE290H
- http://hifweb.lbl.gov/NE290H 2009 Lecture Notes + Info

References: Continued (2):

Image charge couplings:

E.P. Lee, E. Close, and L. Smith, "SPACE CHARGE EFFECTS IN A BENDING MAGNET SYSTEM," Proc. Of the 1987 Particle Accelerator Conf., 1126 (1987)

Seminal work on envelope modes:

J. Struckmeier and M. Reiser, "Theoretical Studies of Envelope Oscillations and Instabilities of Mismatched Intense Charged-Particle Beams in Periodic Focusing Channels," Particle Accelerators **14,** 227 (1984)

M. Reiser, *Theory and Design of Charged Particle Beams* (John Wiley, 1994, 2008)

Extensive review on envelope instabilities:

S.M. Lund and B. Bukh, "Stability properties of the transverse envelope equations describing intense ion beam transport," PRSTAB **7** 024801 (2004)

Efficient, Fail-Safe Generation of Matched Envelope Solutions:

S.M. Lund and S.H. Chilton, and E.P. Lee, "Efficient computation of matched solutions of the Kapchinskij-Vladimirskij envelope equations," PRSTAB **9,** 064201(2006)

A highly flexible Mathematica -based implementation is archived on the course web site with these lecture notes. This was used to generated many plots in this course.

KV distribution:

F. Sacherer, *Transverse Space-Charge Effects in Circular Accelerators*, Univ. of California Berkeley, Ph.D Thesis (1968)

I. Kaphinskij and V. Vladimirskij, in *Proc. Of the Int. Conf. On High Energy Accel. and Instrumentation* (CERN Scientific Info. Service, Geneva, 1959) p. 274

S.M. Lund, T. Kikuchi, and R.C. Davidson, ",Generation of initial kinetic distributions for simulation of long-pulse charged particle beams with high space-charge intensity," PRSTAB **12**, 114801 (2009)

Symmetries and phase-amplitude methods:

A. Dragt, *Lectures on Nonlinear Orbit Dynamics* in *Physics of High Energy Particle Accelerators,* (American Institute of Physics, 1982), AIP Conf. Proc. No. 87, p. 147

E. D. Courant and H. S. Snyder, "Theory of the Alternating-Gradient Synchrotron," Annals of Physics **3**, 1 (1958)

Analytical analysis of matched envelope solutions and transport scaling:

E. P. Lee, "Precision matched solution of the coupled beam envelope equations for a periodic quadrupole lattice with space-charge," Phys. Plasmas **9**, 4301 (2005)

O.A. Anderson, "Accurate Iterative Analytic Solution of the KV Envelope Equations for a Matched Beam," PRSTAB, **10** 034202 (2006)

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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https://people.nscl.msu.edu/~lund/msu/phy905_2018

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