04. Orbit Stability and the Phase Amplitude Formulation*

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S5: Linear Transverse Particle Equations of Motion without Space-Charge, Acceleration, and Momentum Spread

S5A: Hill's Equation

Neglect:
- Space-charge e ects
- Nonlinear applied focusing and bends
- Acceleration
- Momentum spread e ects:

Then the transverse particle equations of motion reduce to Hill's Equation:

$$x''(s) + \kappa(s) x(s) = 0$$

x=⊥ particle coordinate
(i.e., x or y or possibly combinations of coordinates)

$s =$ Axial coordinate of reference particle

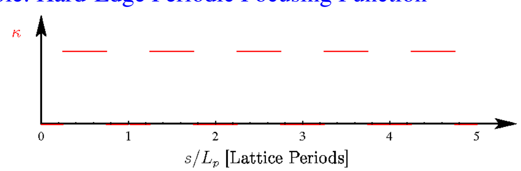
$' = \frac{d}{ds}$ Derivative with respect to axial coordinate

$\kappa(s) =$ Lattice focusing function (linear fields)

For a periodic lattice:

$$\kappa(s + L_p) = \kappa(s)$$
$$L_p = \text{Lattice Period}$$

/// Example: Hard-Edge Periodic Focusing Function



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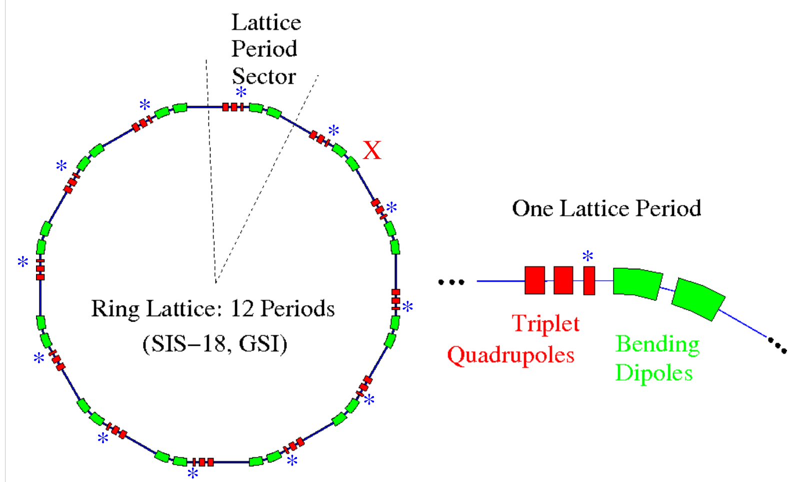
For a ring (i.e., circular accelerator), one also has the "superperiod" condition:

$$\kappa(s + \mathcal{C}) = \kappa(s)$$
$$\mathcal{C} = \mathcal{N} L_p = \text{Ring Circumfrance}$$
$$\mathcal{N} = \text{Superperiod Number}$$

- Distinction matters when there are (eld) construction errors in the ring
  - Repeat with superperiod but not lattice period
  - Will cover in lectures on: Particle Resonances

/// Example: Period and Superperiod distinctions for errors in a ring

* Magnet with systematic defect will be felt every lattice period
X Magnet with random (fabrication) defect felt once per lap



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S5B: Transfer Matrix Form of the Solution to Hill's Equation

Hill's equation is linear. The solution with initial condition:

$$x(s = s_i) = x(s_i)$$
$$x'(s = s_i) = x'(s_i)$$

$s = s_i =$ Axial location of initial condition

can be uniquely expressed in matrix form (**M** is the transfer matrix) as:

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \mathbf{M}(s|s_i) \cdot \begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix}$$
$$= \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \cdot \begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix}$$

Where $C(s|s_i)$ and $S(s|s_i)$ are "cosine-like" and "sine-like" principal trajectories satisfying:

$$C''(s|s_i) + \kappa(s)C(s|s_i) = 0 \qquad C(s_i|s_i) = 1 \qquad C'(s_i|s_i) = 0$$
$$S''(s|s_i) + \kappa(s)S(s|s_i) = 0 \qquad S(s_i|s_i) = 0 \qquad S'(s_i|s_i) = 1$$

This follows trivially because:

$$x(s) = x(s_i)C(s|s_i) + x'(s_i)S(s|s_i)$$

satis es the di erential equation:

$$x''(s) + \kappa(s)x(s) = 0$$

with initial condition:

$$x(s = s_i) = x(s_i)$$
$$x'(s = s_i) = x'(s_i)$$

Because:

$$x''(s) + \kappa(s)x(s) = x(s_i)\left[C''(s|s_i) + \kappa(s)C(s|s_i)\right]$$
$$+ x'(s_i)\left[S''(s|s_i) + \kappa(s)S(s|s_i)\right] = 0$$

since the terms in [...] vanish and the initial condition is satis ed:

$$x(s_i) = x(s_i)\overset{1}{C(s_i|s_i)} + x'(s_i)\overset{0}{S(s_i|s_i)} = x(s_i)$$
$$x'(s_i) = x(s_i)\overset{0}{C'(s_i|s_i)} + x'(s_i)\overset{1}{S'(s_i|s_i)} = x'(s_i)$$

Transfer matrices were covered in Hao lectures for a few simple focusing systems discussed with the additional assumption of piecewise constant $\kappa(s)$

1) Free Drift: $\kappa = 0$ $\qquad x'' = 0$

$$\mathbf{M}(s|s_i) = \begin{bmatrix} 1 & s - s_i \\ 0 & 1 \end{bmatrix}$$

2) Continuous Focusing: $\kappa = k_{\beta 0}^2 = \text{const} > 0$ $\qquad x'' + k_{\beta 0}^2 x = 0$

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cos[k_{\beta 0}(s - s_i)] & \frac{1}{k_{\beta 0}}\sin[k_{\beta 0}(s - s_i)] \\ -k_{\beta 0}\sin[k_{\beta 0}(s - s_i)] & \cos[k_{\beta 0}(s - s_i)] \end{bmatrix}$$

3) Solenoidal Focusing: $\kappa = \hat{\kappa} = \text{const} > 0$ $\qquad x'' + \hat{\kappa}x = 0$

Results are expressed within the rotating Larmor Frame
(same as continuous focusing with reinterpretation of variables)

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}(s - s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sin[\sqrt{\hat{\kappa}}(s - s_i)] \\ -\sqrt{\hat{\kappa}}\sin[\sqrt{\hat{\kappa}}(s - s_i)] & \cos[\sqrt{\hat{\kappa}}(s - s_i)] \end{bmatrix}$$

4) Quadrupole Focusing-Plane: $\kappa = \hat{\kappa} = \text{const} > 0$ $\qquad x'' + \hat{\kappa}x = 0$

(Obtain from continuous focusing case)

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}(s - s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sin[\sqrt{\hat{\kappa}}(s - s_i)] \\ -\sqrt{\hat{\kappa}}\sin[\sqrt{\hat{\kappa}}(s - s_i)] & \cos[\sqrt{\hat{\kappa}}(s - s_i)] \end{bmatrix}$$

5) Quadrupole DeFocusing-Plane: $\kappa = -\hat{\kappa} = \text{const} < 0$ $\qquad x'' - \hat{\kappa}x = 0$

(Obtain from quadrupole focusing case with $\sqrt{\hat{\kappa}} \rightarrow i\sqrt{\hat{\kappa}}$ $\quad i = \sqrt{-1}$)

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cosh[\sqrt{\hat{\kappa}}(s - s_i)] & \frac{1}{\sqrt{\hat{\kappa}}}\sinh[\sqrt{\hat{\kappa}}(s - s_i)] \\ \sqrt{\hat{\kappa}}\sinh[\sqrt{\hat{\kappa}}(s - s_i)] & \cosh[\sqrt{\hat{\kappa}}(s - s_i)] \end{bmatrix}$$

6) Thin Lens: $\kappa(s) = \frac{1}{f}\delta(s - s_0)$ $\qquad x'' + \frac{1}{f}\delta(s - s_0)x = 0$

$s_0 = \text{const} =$ Axial Location Lens
$f = \text{const} =$ Focal Length
$\delta(x) =$ Dirac-Delta Function

$$\mathbf{M}(s_0^+|s_0^-) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

S5C: Wronskian Symmetry of Hill's Equation

An important property of this linear motion is a Wronskian invariant/symmetry:

$$W(s|s_i) \equiv \det \mathbf{M}(s|s_i) = \det \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix}$$
$$= C(s|s_i)S'(s|s_i) - C'(s|s_i)S(s|s_i) = 1$$

/// Proof: Abbreviate Notation $C \equiv C(s|s_i)$ etc.

Multiply Equations of Motion for C and S by $-S$ and C, respectively:

$$-S(C'' + \kappa C) = 0$$
$$+C(S'' + \kappa S) = 0$$

Add Equations:

$$CS'' - SC'' + \kappa(CS - SC) = 0$$

(with $CS - SC$ marked as equal to 0)

$$\implies \frac{dW}{ds} = \frac{d}{ds}(CS' - C'S) = CS'' - SC'' = 0$$
$$\implies W = \text{const}$$

Apply initial conditions:

$$W(s) = W(s_i) = C_i S'_i - C'_i S_i = 1 \cdot 1 - 0 \cdot 0 = 1$$ ///

/// Example: Continuous Focusing: Transfer Matrix and Wronskian

$$\kappa(s) = k_{\beta 0}^2 = \text{const} > 0$$

Principal orbit equations are simple harmonic oscillators with solution:

$$C(s|s_i) = \cos[k_{\beta 0}(s - s_i)] \qquad C'(s|s_i) = -k_{\beta 0}\sin[k_{\beta 0}(s - s_i)]$$
$$S(s|s_i) = \frac{\sin[k_{\beta 0}(s - s_i)]}{k_{\beta 0}} \qquad S'(s|s_i) = \cos[k_{\beta 0}(s - s_i)]$$

Transfer matrix gives the familiar solution:

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} \cos[k_{\beta 0}(s - s_i)] & \frac{\sin[k_{\beta 0}(s - s_i)]}{k_{\beta 0}} \\ -k_{\beta 0}\sin[k_{\beta 0}(s - s_i)] & \cos[k_{\beta 0}(s - s_i)] \end{bmatrix} \cdot \begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix}$$

Wronskian invariant is elementary:

$$W = \cos^2[k_{\beta 0}(s - s_i)] + \sin^2[k_{\beta 0}(s - s_i)] = 1$$

///

Can exploit Wronskian condition to check all matrices

- Most simple (3d phase-space) expression of dynamics being Hamiltonian/Symplectic

S5D: Stability of Solutions to Hill's Equation in a Periodic Lattice

The transfer matrix must be the same in any period of the lattice:

$$\mathbf{M}(s + L_p | s_i + L_p) = \mathbf{M}(s|s_i)$$

For a propagation distance $s - s_i$ satisfying

$$NL_p \leq s - s_i \leq (N+1)L_p \qquad N = 0, 1, 2, \cdots$$

the transfer matrix can be resolved as

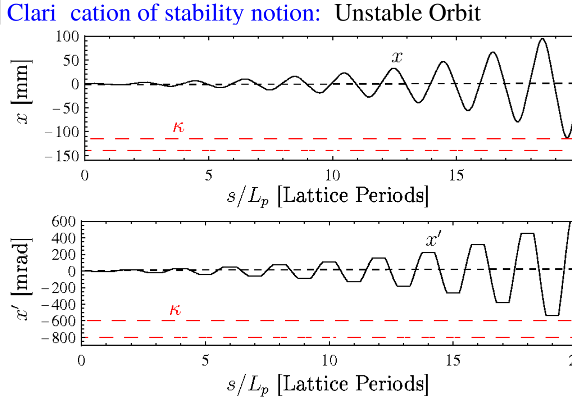
$$\mathbf{M}(s|s_i) = \mathbf{M}(s - NL_p|s_i) \cdot \mathbf{M}(s_i + NL_p|s_i)$$
$$= \mathbf{M}(s - NL_p|s_i) \cdot [\mathbf{M}(s_i + L_p|s_i)]^N$$

Residual — N Full Periods

For a lattice to have stable orbits, both $x(s)$ and $x'(s)$ should remain bounded on propagation through an arbitrary number N of lattice periods. This is equivalent to requiring that the elements of **M** remain bounded on propagation through any number of lattice periods: $\mathbf{M}^N \equiv [\mathbf{M}^N{}_{ij}]$

$$\lim_{N \to \infty} \left| \mathbf{M}^N{}_{ij} \right| < \infty \implies \text{Stable Motion}$$

Clari cation of stability notion: Unstable Orbit



$$L_p = 0.5 \text{ m}$$
$$\eta = 0.5$$

$$\kappa = \begin{cases} 48/\text{m}^2 & \text{where } \kappa \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$x(0) = 1 \text{ mm}$$
$$x'(0) = 0$$

For energetic particle: $H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 \sim \text{Large, but} \neq \text{const}$

where $|x'|$ small, $|x|$ large

where $|x|$ small, $|x'|$ large

The matrix criterion corresponds to our intuitive notion of stability: as the particle advances there are no large oscillation excursions in position and angle.

To analyze the stability condition, examine the eigenvectors/eigenvalues of **M** for transport through one lattice period:

$$\mathbf{M}(s_i + L_p|s_i) \cdot \mathbf{E} \equiv \lambda \mathbf{E}$$
$$\mathbf{E} = \text{Eigenvector}$$
$$\lambda = \text{Eigenvalue}$$

- Eigenvectors and Eigenvalues are generally complex
- Eigenvectors and Eigenvalues generally vary with s_i
- Two independent Eigenvalues and Eigenvectors
  - Degeneracies special case

Derive the two independent eigenvectors/eigenvalues through analysis of the characteristic equation: Abbreviate Notation

$$\mathbf{M}(s_i + L_p|s_i) = \begin{bmatrix} C(s_i + L_p|s_i) & S(s_i + L_p|s_i) \\ C'(s_i + L_p|s_i) & S'(s_i + L_p|s_i) \end{bmatrix} \equiv \begin{bmatrix} C & S \\ C' & S' \end{bmatrix}$$

Nontrivial solutions to $\mathbf{M} \cdot \mathbf{E} \equiv \lambda \mathbf{E}$ exist when (non-invertable coe matrix):

$$\det \begin{bmatrix} C - \lambda & S \\ C' & S' - \lambda \end{bmatrix} = \lambda^2 - (C + S')\lambda + (CS' - SC') = 0$$

But we can apply the Wronskian condition:

$$CS' - SC' = 1$$

and we make the notational de nition

$$C + S' = \text{Tr}\,\mathbf{M} \equiv 2\cos\sigma_0$$

Reminder:

$$\mathbf{M} = \begin{bmatrix} C & S \\ C' & S' \end{bmatrix}$$

The characteristic equation then reduces to:

$$\lambda^2 - 2\lambda\cos\sigma_0 + 1 = 0 \qquad \cos\sigma_0 \equiv \frac{1}{2}\text{Tr}\,\mathbf{M}(s_i + L_p|s_i)$$

The use of $2\cos\sigma_0$ to denote Tr **M** is in anticipation of later results (see S6) where σ_0 is identi ed as the phase-advance of a stable orbit

There are two solutions to the characteristic equation that we denote $\lambda_\pm$

$$\lambda_\pm = \cos\sigma_0 \pm \sqrt{\cos^2\sigma_0 - 1} = \cos\sigma_0 \pm i\sin\sigma_0 = e^{\pm i\sigma_0}$$
$$\mathbf{E}_\pm = \text{Corresponding Eigenvectors} \qquad i \equiv \sqrt{-1}$$

Note that: $\lambda_+\lambda_- = 1$
$\lambda_+ = 1/\lambda_-$ Reciprocal Symmetry

Consider a vector of initial conditions:

$$\begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} = \begin{bmatrix} x_i \\ x'_i \end{bmatrix}$$

The eigenvectors $\mathbf{E}_\pm$ span two-dimensional space. So any initial condition vector can be expanded as:

$$\begin{bmatrix} x_i \\ x'_i \end{bmatrix} = \alpha_+\mathbf{E}_+ + \alpha_-\mathbf{E}_-$$
$$\alpha_\pm = \text{Complex Constants}$$

Then using $\mathbf{M} \cdot \mathbf{E}_\pm = \lambda_\pm \mathbf{E}_\pm$

$$\mathbf{M}^N(s_i + L_p|s_i) \cdot \begin{bmatrix} x_i \\ x'_i \end{bmatrix} = \alpha_+\lambda_+^N\mathbf{E}_+ + \alpha_-\lambda_-^N\mathbf{E}_-$$

Therefore, if $\lim_{N\to\infty} \lambda_\pm^N$ is bounded, then the motion is stable. This will always be the case if $|\lambda_\pm| = |e^{\pm i\sigma_0}| \leq 1$, corresponding to σ_0 real with $|\cos\sigma_0| \leq 1$

This implies for stability or the orbit that we must have:

$$\frac{1}{2}|\text{Trace}\,\mathbf{M}(s_i + L_p|s_i)| = \frac{1}{2}|C(s_i + L_p|s_i) + S'(s_i + L_p|s_i)|$$
$$= |\cos\sigma_0| \leq 1$$

In a periodic focusing lattice, this important stability condition places restrictions on the lattice structure (focusing strength) that are generally interpreted in terms of phase advance limits (see: S6).

- Accelerator lattices almost always tuned for single particle stability to maintain beam control
  - Even for intense beams, beam centroid approximately obeys single particle equations of motion when image charges are negligible
- Space-charge and nonlinear applied elds can further limit particle stability
  - Resonances: see: Particle Resonances ....
  - Envelope Instability: see: Transverse Centroid and Envelope ....
  - Higher Order Instability: see: Transverse Kinetic Stability
- We will show (see: S6) that for stable orbits σ_0 can be interpreted as the phase-advance of single particle oscillations

/// Example: Continuous Focusing Stability

$$\kappa(s) = k_{\beta 0}^2 = \text{const} > 0$$

Principal orbit equations are simple harmonic oscillators with solution:

$$C(s|s_i) = \cos[k_{\beta 0}(s - s_i)] \qquad C'(s|s_i) = -k_{\beta 0}\sin[k_{\beta 0}(s - s_i)]$$

$$S(s|s_i) = \frac{\sin[k_{\beta 0}(s - s_i)]}{k_{\beta 0}} \qquad S'(s|s_i) = \cos[k_{\beta 0}(s - s_i)]$$

Stability bound then gives:

$$\frac{1}{2}|\text{Trace}\,\mathbf{M}(s_i + L_p|s_i)| = \frac{1}{2}|C(s_i + L_p|s_i) + S'(s_i + L_p|s_i)|$$
$$= |\cos[k_{\beta 0}(s - s_i)]| \leq 1$$

- Always satis ed for real $k_{\beta 0}$
- Con rms known result using formalism: continuous focusing stable
  - Energy not pumped into or out of particle orbit ///

The simplest example of the stability criterion applied to periodic lattices will be given in the problem sets: Stability of a periodic thin lens lattice

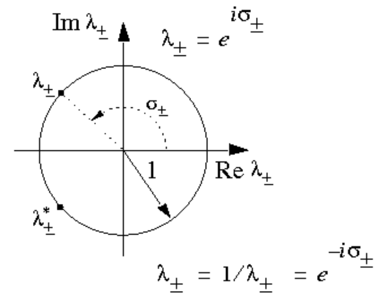
- Analytically nd that lattice unstable when focusing kicks su ciently strong

More advanced treatments

- See: Dragt, *Lectures on Nonlinear Orbit Dynamics*, AIP Conf Proc 87 (1982)

show that symplectic 2x2 transfer matrices associated with Hill's Equation have only two possible classes of eigenvalue symmetries:

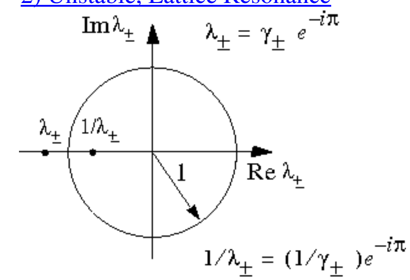
1) Stable



Occurs for:

$$0 \leq \sigma_0 \leq 180^\circ/\text{period}$$

2) Unstable, Lattice Resonance



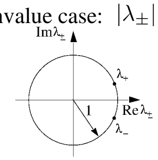
Occurs in bands when focusing strength is increased beyond $\sigma_0 = 180^\circ/\text{period}$

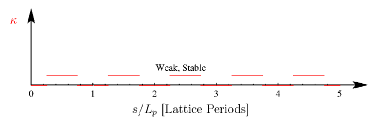
- Limited class of possibilities simpli es analysis of focusing lattices

Eigenvalue structure as focusing strength is increased

Weak Focusing:

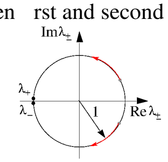
- Make κ as small as needed (low phase advance σ_0)
- Always rst eigenvalue case: $|\lambda_\pm| = 1, \quad \lambda_+ = 1/\lambda_- = \lambda_-^*$

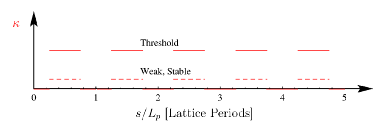




Stability Threshold:

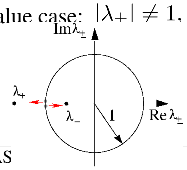
- Increase κ to stability limit (phase advance $\sigma_0 = 180^\circ/\text{Period}$)
- Transition between rst and second eigenvalue case: $\lambda_\pm = -1$

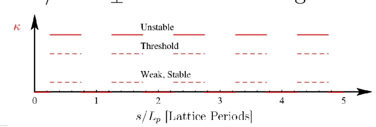




Instability:

- Increase κ beyond threshold (phase advance $\sigma_0 = 180^\circ/\text{Period}$)
- Second eigenvalue case: $|\lambda_+| \neq 1, \quad \lambda_+ = 1/\lambda_- \quad \lambda_\pm$ both real and negative





Comments:

- As κ becomes stronger and stronger it is not necessarily the case that instability persists. There can be (typically) narrow ranges of stability within a mostly unstable range of parameters.
  - Example: Stability/instability bands of the Matheiu equation commonly studied in mathematical physics which is a special case of Hills' equation.
- Higher order regions of stability past the rst instability band likely make little sense to exploit because they require higher eld strength (to generate larger κ) and generally lead to larger particle oscillations than for weaker elds below the rst stability threshold.

S6: Hill's Equation: Floquet's Theorem and the Phase-Amplitude Form of the Particle Orbit
S6A: Introduction

In this section we consider Hill's Equation:

$$x''(s) + \kappa(s)x(s) = 0$$

subject to a periodic applied focusing function

$$\kappa(s + L_p) = \kappa(s)$$
$$L_p = \text{Lattice Period}$$

- Many results will also hold in more complicated form for a non-periodic $\kappa(s)$
  - Results less clean in this case
  (initial conditions not removable to same degree as periodic case)

S6B: Floquet's Theorem

Floquet's Theorem (proof: see standard Mathematics and Mathematical Physics Texts)

The solution to Hill's Equation *x*(*s*) can be written in terms of two linearly independent solutions expressible as:

$$x_1(s) = w(s)e^{i\mu s}$$
$$x_2(s) = w(s)e^{-i\mu s}$$

$$i = \sqrt{-1}$$
$$\mu = \frac{1}{2}\text{Tr}\,\mathbf{M}(s_i + L_p|s_i) = \cos\sigma_0$$
$$= \text{const} = \text{Characteristic Exponent}$$

Where *w*(*s*) is a periodic function:

$$w(s + L_p) = w(s)$$

- Theorem as written only applies for **M** with non-degenerate eigenvalues. But a similar theorem applies in the degenerate case.
- A similar theorem is also valid for non-periodic focusing functions
  - Expression not as simple but has analogous form

S6C: Phase-Amplitude Form of Particle Orbit

As a consequence of Floquet's Theorem, any (stable or unstable) nondegenerate solution to Hill's Equation can be expressed in phase-amplitude form as:

$$x(s) = A(s)\cos\psi(s) \qquad A(s) = \text{Real-Valued Amplitude Function}$$
$$A(s + L_p) = A(s) \qquad \psi(s) = \text{Real-Valued Phase Function}$$

- Have not done anything yet: replace one function *x*(*s*) by two $A(s),\ \psi(s)$
- Floquet's theorem tells us we lose nothing in doing this

Derive equations of motion for $A,\ \psi$ by taking derivatives of the phase-amplitude form for *x*(*s*):

$$x = A\cos\psi$$
$$x' = A'\cos\psi - A\psi'\sin\psi$$
$$x'' = A''\cos\psi - 2A'\psi'\sin\psi - A\psi''\sin\psi - A\psi'^2\cos\psi$$

then substitute in Hill's Equation and isolate coe cients of $\sin\psi,\ \cos\psi$:

$$x'' + \kappa x = \left[A'' + \kappa A - A\psi'^2\right]\cos\psi - \left[2A'\psi' + A\psi''\right]\sin\psi = 0$$

$$x'' + \kappa x = \left[A'' + \kappa A - A\psi'^2\right]\cos\psi - \left[2A'\psi' + A\psi''\right]\sin\psi = 0$$

We are *free to introduce an additional constraint* between *A* and ψ:

- Two functions A, ψ to represent one function *x* allows a constraint

Choose:

Eq. (1) $$2A'\psi' + A\psi'' = 0 \qquad \Longrightarrow \qquad \text{Coefficient of } \sin\psi \text{ zero}$$

Then to satisfy Hill's Equation for all ψ , the coe cient of $\cos\psi$ must also vanish giving:

Eq. (2) $$A'' + \kappa A - A\psi'^2 = 0 \qquad \Longrightarrow \qquad \text{Coefficient of } \cos\psi \text{ zero}$$

Eq. (1) Analysis (coe cient of $\sin\psi$): $2A'\psi' + A\psi'' = 0$

Simplify:

$$2A'\psi' + A\psi'' = \frac{(A^2\psi')'}{A} = 0$$

$$\implies (A^2\psi')' = 0$$

Assume for moment: $A \neq 0$

Will show later that this assumption met for all *s*

Integrate once:

$$A^2\psi' = \text{const}$$

One commonly rescales the amplitude *A*(*s*) in terms of an auxiliary amplitude function *w*(*s*):

$$A(s) = A_i w(s) \qquad A_i = \text{const} = \text{Initial Amplitude}$$

such that

$$w^2\psi' \equiv 1$$

Note:
- [[A_i]] = [[w]] = sqrt(meters)
- [[A]] = meters and [[A]] ≠ [[A_i]]

This equation can then be integrated to obtain the phase-function of the particle:

$$\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})} \qquad \psi_i = \text{const} = \text{Initial Phase} \qquad w \neq 0$$

Eq. (2) Analysis (coe cient of $\cos\psi$): $A'' + \kappa A - A\psi'^2 = 0$

With the choice of amplitude rescaling, $A = A_i w$ and $w^2\psi' = 1$, Eq. (2) becomes:

$$w'' + \kappa w - \frac{1}{w^3} = 0$$

Floquet's theorem tells us that we are *free to restrict w to be a periodic solution*:

$$w(s + L_p) = w(s)$$

Reduced Expressions for *x* and *x*':

Using $A = A_i w$ and $w^2\psi' = 1$:

$$x = A\cos\psi$$
$$x' = A'\cos\psi - A\psi'\sin\psi$$

$$\implies \begin{aligned} x &= A_i w \cos\psi \\ x' &= A_i w'\cos\psi - \frac{A_i}{w}\sin\psi \end{aligned}$$

Phase-Space form of orbit in phase-amplitude form

S6D: Summary: Phase-Amplitude Form of Solution to Hill's Eqn

$$x(s) = A_i w(s)\cos\psi(s)$$

$$x'(s) = A_i w'(s)\cos\psi(s) - \frac{A_i}{w(s)}\sin\psi(s)$$

$A_i = \text{const} = \text{Initial Amplitude}$

$\psi_i = \text{const} = \text{Initial Phase}$

where *w*(*s*) and $\psi(s)$ are amplitude- and phase-functions satisfying:

Amplitude Equations

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$
$$w(s + L_p) = w(s)$$
$$w(s) > 0$$

Phase Equations

$$\psi'(s) = \frac{1}{w^2(s)}$$
$$\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$
$$\psi(s) = \psi_i + \Delta\psi(s)$$

Initial ($s = s_i$) amplitude and phase are constrained by the particle initial conditions as:

$$x(s = s_i) = A_i w_i \cos\psi_i$$
$$x'(s = s_i) = A_i w_i' \cos\psi_i - \frac{A_i}{w_i}\sin\psi_i$$

or

$$A_i\cos\psi_i = x(s = s_i)/w_i \qquad w_i \equiv w(s = s_i)$$
$$A_i\sin\psi_i = x(s = s_i)w_i' - x'(s = s_i)w_i \qquad w_i' \equiv w'(s = s_i)$$

S6E: Points on the Phase-Amplitude Formulation

1) *w*(*s*) can be taken as positive de nite

$$w(s) > 0$$

/// Proof: Sign choices in *w*:

Let *w*(*s*) be positive at some point. Then the equation:

$$w'' + \kappa w - \frac{1}{w^3} = 0$$

Insures that w can never vanish or change sign. This follows because whenever *w* becomes small, $w'' \simeq 1/w^3 \gg 0$ can become arbitrarily large to turn *w* before it reaches zero. Thus, to x phases, we conveniently require that *w* > 0. ///

- Proof veri es assumption made in analysis that $A = A_i w \neq 0$
- Conversely, one could choose *w* negative and it would always remain negative for analogous reasons. This choice is *not* commonly made.
- Sign choice removes ambiguity in relating initial conditions $x(s_i)$, $x'(s_i)$ to A_i, ψ_i

2) $w(s)$ is a unique periodic function

- Can be proved using a connection between w and the principal orbit functions C and S (see: Appendix A and S7)
- $w(s)$ can be regarded as a special, periodic function describing the lattice focusing function $\kappa(s)$

3) The amplitude parameters

$$w_i = w(s = s_i)$$
$$w'_i = w'(s_i)$$

depend *only* on the periodic lattice properties and are *independent* of the particle initial conditions $x(s_i),\ x'(s_i)$

4) The change in phase

$$\Delta\psi(s) = \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

depends on the choice of initial condition s_i . However, the phase-advance through one lattice period $\Delta\psi(s_i + L_p) = \int_{s_i}^{s_i+L_p} \frac{d\tilde{s}}{w^2(\tilde{s})}$

is independent of s_i since w is a periodic function with period L_p

- Will show later that (see S6F)

$$\Delta\psi(s_i + L_p) \equiv \sigma_0$$

is the undepressed phase advance of particle oscillations. This will help us interpret the lattice focusing strength.

5) $w(s)$ has dimensions [[w]] = Sqrt[meters]

- Can prove inconvenient in applications and motivates the use of an alternative "betatron" function β

$$\beta(s) \equiv w^2(s)$$

with dimension [[β]] = meters (see: S7 and S8)

6) On the surface, what we have done: Transform the linear Hill's Equation to a form where a solution to nonlinear axillary equations for w and ψ are needed via the phase-amplitude method seems insane ..... why do it?

- Method will help identify the useful Courant-Snyder invariant which will aid interpretation of the dynamics (see: S7)
- Decoupling of initial conditions in the phase-amplitude method will help simplify understanding of bundles of particles in the distribution

S6F: Relation between Principal Orbit Functions and Phase-Amplitude Form Orbit Functions

The transfer matrix **M** of the particle orbit can be expressed in terms of the principal orbit functions C and S as (see: S4):

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \mathbf{M}(s|s_i)\cdot\begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix}\cdot\begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix}$$

Use of the phase-amplitude forms and some algebra identi es (see problem sets):

$$C(s|s_i) = \frac{w(s)}{w_i}\cos\Delta\psi(s) - w'_i w(s)\sin\Delta\psi(s)$$
$$S(s|s_i) = w_i w(s)\sin\Delta\psi(s)$$
$$C'(s|s_i) = \left(\frac{w'(s)}{w_i} - \frac{w'_i}{w(s)}\right)\cos\Delta\psi(s) - \left(\frac{1}{w_i w(s)} + w'_i w'(s)\right)\sin\Delta\psi(s)$$
$$S'(s|s_i) = \frac{w_i}{w(s)}\cos\Delta\psi(s) + w_i w'(s)\sin\Delta\psi(s)$$
$$\Delta\psi(s) \equiv \int_{s_i}^{s}\frac{d\tilde{s}}{w^2(\tilde{s})} \qquad w_i \equiv w(s=s_i) \qquad w'_i \equiv w'(s=s_i)$$

// Aside: Some steps in derivation: $\psi = \psi_i + \Delta\psi \qquad \Delta\psi(s=s_i) = 0$

$$x = A_i w\cos\psi \qquad = A_i w\cos(\Delta\psi + \psi_i)$$
$$x' = A_i w'\cos\psi - \frac{A_i}{w}\sin\psi \qquad = A_i w'\cos(\Delta\psi+\psi_i) - \frac{A_i}{w}\sin(\Delta\psi+\psi_i) \qquad (*)$$

Initially: $x_i = A_i w\cos\psi_i$

$$x'_i = A_i w'_i\cos\psi_i - \frac{A_i}{w_i}\sin\psi_i = w'_i\frac{x_i}{w_i} - \frac{A_i}{w_i}\sin\psi_i$$

Or:

$$A_i\cos\psi_i = x_i/w_i$$
$$A_i\sin\psi_i = x_i w'_i - x'_i w_i \qquad (2)$$

Use trigonometric formulas:

$$\cos(\Delta\psi+\psi_i) = \cos\Delta\psi\cos\psi_i - \sin\Delta\psi\sin\psi_i$$
$$\sin(\Delta\psi+\psi_i) = \sin\Delta\psi\cos\psi_i + \cos\Delta\psi\sin\psi_i \qquad (1)$$

Insert (1) and (2) in (*) for x and then rearrange and compare to $x = Cx_i + Sx'_i$ to obtain: $[\cdots] = C(s|s_i) \qquad [\cdots] = S(s|s_i)$

$$\mathrm{x} = \left[\frac{w}{w_i}\cos\Delta\psi - w'_i w\sin\Delta\psi\right]x_i + [w_i w\sin\Delta\psi]\,x'_i$$

Add steps and repeat with particle angle x' to complete derivation //

/// Aside: Alternatively, it can be shown (see: Appendix A) that $w(s)$ can be related to the principal orbit functions calculated over one Lattice period by:

$$w^2(s) = \beta(s) = \sin\sigma_0 \frac{S(s|s_i)}{S(s_i+L_p|s_i)} + \frac{S(s_i+L_p|s_i)}{\sin\sigma_0}\left[C(s|s_i) + \frac{\cos\sigma_0 - C(s|s_i)}{S(s_i+L_p|s_i)}S(s|s_i)\right]^2$$

$$\sigma_0 \equiv \int_{s_i}^{s_i+L_p} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

The formula for σ_0 in terms of principal orbit functions is useful:

- σ_0 (phase advance, see: S6G) is often speci ed for the lattice and the focusing function $\kappa(s)$ is tuned to achieve the speci ed value
- Shows that $w(s)$ can be constructed from two principal orbit integrations over one lattice period
  - Integrations must generally be done numerically for C and S
  - No root nding required for initial conditions to construct periodic $w(s)$
  - s_i can be anywhere in the lattice period and $w(s)$ will be independent of the speci c choice of s_i

- The form of $w^2(s)$ suggests an underlying Courant-Snyder Invariant (see: S7 and Appendix A)
- $w^2 = \beta$ can be applied to calculate max beam particle excursions in the absence of space-charge e ects (see: S8)
  - Useful in machine design
  - Exploits Courant-Snyder Invariant
- Techniques to map lattice functions from one point in lattice to another are also presented in Appendix A and S7C
  - Include e cient Lee Algebra derived expressions in S7C

///

S6G: Undepressed Particle Phase Advance

We can now concretely connect σ_0 for a stable orbit to the change in particle oscillation phase $\Delta\psi$ through one lattice period:

From S5D:

$$\cos\sigma_0 \equiv \frac{1}{2}\text{Tr }\mathbf{M}(s_i+L_p|s_i)$$

Apply the principal orbit representation of **M**

$$\mathbf{M} \equiv \begin{bmatrix} C & S \\ C' & S' \end{bmatrix}$$

$$\text{Tr }\mathbf{M}(s_i+L_p|s_i) = C(s_i+L_p|s_i) + S'(s_i+L_p|s_i)$$

and use the phase-amplitude identi cations of C and S' calculated in S6F:

$$\frac{1}{2}\text{Tr }\mathbf{M}(s_i+L_p|s_i) = \frac{1}{2}\left[\frac{w(s_i+L_p)}{w_i} + \frac{w_i}{w(s_i+L_p)}\right]\cos\Delta\psi(s_i+L_p) + \frac{1}{2}\left[w_i w'(s_i+L_p) - w'_i w(s_i+L_p)\right]\sin\Delta\psi(s_i+L_p)$$

By periodicity:

$$w(s_i+L_p) = w(s_i) = w_i \qquad w'(s_i+L_p) = w'(s_i) = w'_i \quad\Longrightarrow\quad \text{coefficient of }\cos\Delta\psi = 1, \quad \text{coefficient of }\sin\Delta\psi = 0$$

Applying these results gives:

$$\cos\sigma_0 = \cos\Delta\psi(s_i+L_p) = \frac{1}{2}\text{Tr }\mathbf{M}(s_i+L_p|s_i)$$

Thus, σ_0 is identi ed as the phase advance of a stable particle orbit through one lattice period:

$$\sigma_0 = \Delta\psi(s_i+L_p) = \int_{s_i}^{s_i+L_p} \frac{ds}{w^2(s)}$$

- Again veri es that σ_0 is independent of s_i since w(s) is periodic with period L_p
- The stability criterion (see: S5)

$$\frac{1}{2}|\text{Tr }\mathbf{M}(s_i+L_p|s_i)| = |\cos\sigma_0| \leq 1$$

is concretely connected to the particle phase advance through one lattice period providing a useful physical interpretation

Consequence:
Any periodic lattice with undepressed phase advance satisfying

$$\sigma_0 < \pi/\text{period} = 180^\circ/\text{period}$$

will have stable single particle orbits.

Discussion:

The phase advance σ_0 is an extremely useful dimensionless measure to characterize the focusing strength of a periodic lattice. Much of conventional accelerator physics centers on focusing strength and the suppression of resonance e ects. The phase advance is a natural parameter to employ in many situations to allow ready interpretation of results in a generalizable manner.

We present phase advance formulas for several simple classes of lattices to help build intuition on focusing strength:

1) Continuous Focusing
2) Periodic Solenoidal Focusing
3) Periodic Quadrupole Doublet Focusing
   - FODO Quadrupole Limit

One of these will be derived in the problem sets

- Lattices analyzed as "hard-edge" with piecewise-constant $\kappa(s)$ and lattice period L_p
- Results are summarized only with derivations guided in the problem sets.

4) Thin Lens Limits
   - Useful for analysis of scaling properties

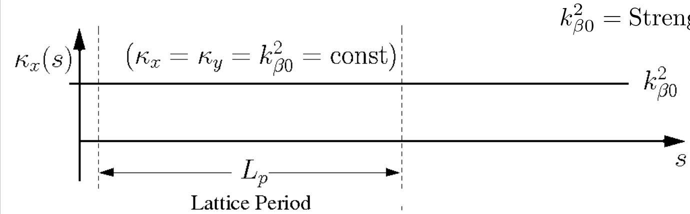
1) Continuous Focusing $x'' + \kappa_x x = x'' + k_{\beta 0}^2 x = 0$

"Lattice period" L_p is an arbitrary length for phase accumulation

$\kappa(s) = k_{\beta 0}^2 = \text{const} > 0$

Parameters:

$L_p = \text{Lattice Period}$
$k_{\beta 0}^2 = \text{Strength}$



Apply phase advance formulas:

$$w'' + k_{\beta 0}^2 w - \frac{1}{w^3} = 0 \quad \Longrightarrow \quad w = \frac{1}{\sqrt{k_{\beta 0}}}$$

$$\sigma_0 = \int_{s_i}^{s_i+L_p} \frac{ds}{w^2} = k_{\beta 0} L_p$$

$$\boxed{\sigma_0 = k_{\beta 0} L_p}$$

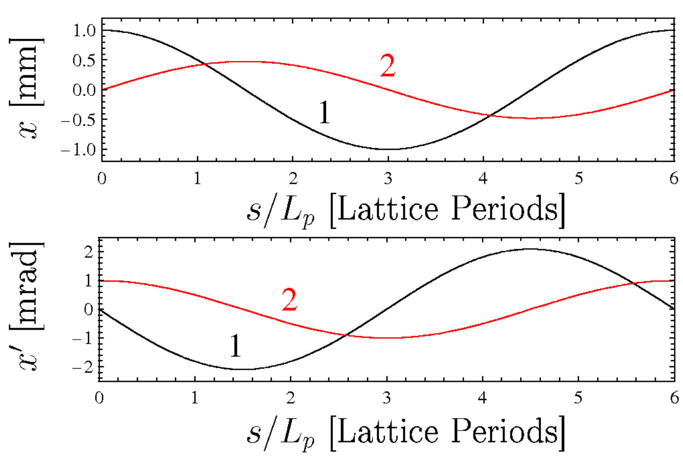
- Always stable
  - Energy cannot pump into or out of particle orbit

Rescaled Principal Orbit Evolution:

$L_p = 0.5$ m
$\sigma_0 = \pi/3 = 60°$
$k_{\beta 0} = (\pi/6)$ rad/m

Cosine-Like
1: $x(0) = 1$ mm
$x'(0) = 0$ mrad

Sine-Like
2: $x(0) = 0$ mm
$x'(0) = 1$ mrad



Phase-Space Evolution (see also S7):

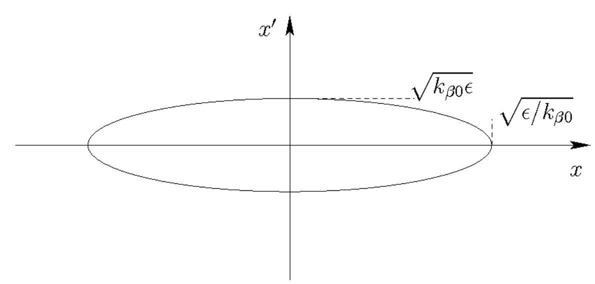
- Phase-space ellipse stationary and aligned along x, x' axes for continuous focusing

$$w = \sqrt{1/k_{\beta 0}} = \text{const} \qquad \gamma = \frac{1}{w^2} = k_{\beta 0} = \text{const}$$

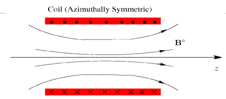
$$w' = 0 \qquad \alpha = -ww' = 0$$

$$\beta = w^2 = 1/k_{\beta 0} = \text{const}$$

$$k_{\beta 0} x^2 + x'^2/k_{\beta 0} = \epsilon = \text{const}$$



2) Periodic Solenoidal Focusing

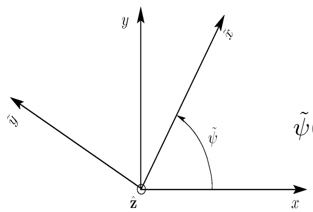


$$x''(s) - \frac{B'_{z0}(s)}{2[B\rho]}y(s) - \frac{B_{z0}(s)}{[B\rho]}y'(s) = 0$$

$$y''(s) + \frac{B'_{z0}(s)}{2[B\rho]}x(s) + \frac{B_{z0}(s)}{[B\rho]}x'(s) = 0$$

$$B_{z0}(s) = B_z(r=0, z=s) = \text{On-Axis Field}$$

Results are interpreted in the rotating Larmor frame (see Solenoid Focusing)

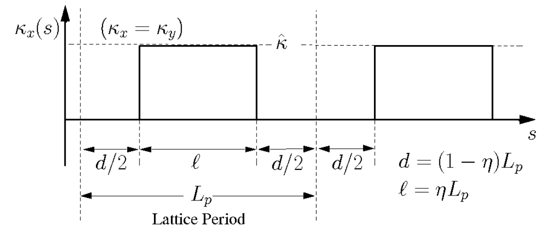


$$\tilde{x} = x\cos\tilde{\psi}(s) + y\sin\tilde{\psi}(s)$$
$$\tilde{y} = -x\sin\tilde{\psi}(s) + y\cos\tilde{\psi}(s)$$

$$\tilde{\psi}(s) = -\int_{s_i}^{s} d\bar{s}\; k_L(\bar{s}) \qquad k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]} = \text{Larmor Wavenumber}$$

$$[B\rho] \equiv \frac{p}{q} \simeq \frac{\gamma_b\beta_b mc}{q} \equiv \text{Rigidity}$$

$$\tilde{x}''(s) + \kappa(s)\tilde{x}(s) = 0 \qquad \kappa(s) = \left(\frac{B_0(s)}{2[B\rho]}\right)^2 = (k_L)^2$$
$$\tilde{y}''(s) + \kappa(s)\tilde{y}(s) = 0$$



<u>Parameters:</u>

L_p = Lattice Period

$\eta \in (0,1]$ = Occupancy

$\hat{\kappa}$ = Strength

<u>Characteristics:</u>

ηL_p = Optic Length

$(1-\eta)L_p$ = Drift Length

Calculation (in problem sets) gives:

$$\cos\sigma_0 = \cos(2\Theta) - \frac{1-\eta}{\eta}\Theta\sin(2\Theta) \qquad \Theta \equiv \frac{\eta}{2}\sqrt{\hat{\kappa}}L_p$$

- Can be unstable when $\hat{\kappa}$ becomes large
  - Energy can pump into or out of particle orbit

Rescaled Larmor-Frame Principal Orbit Evolution Solenoid Focusing:

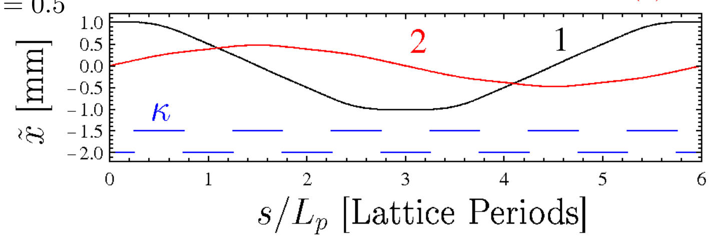
$L_p = 0.5$ m

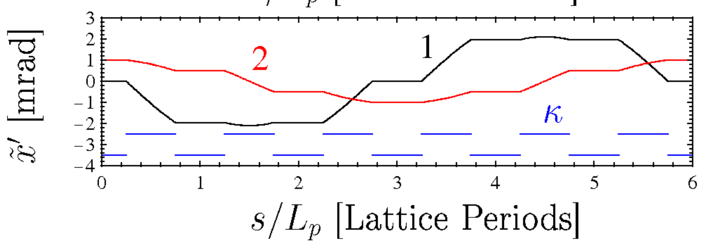
$\sigma_0 = \pi/3 = 60^\circ$ ($\kappa = 8.558\ \text{m}^{-2}$)

$\eta = 0.5$

Cosine-Like
1: $\tilde{x}(0) = 1$ mm
$\tilde{x}'(0) = 0$ mrad

Sine-Like
2: $\tilde{x}(0) = 0$ mm
$\tilde{x}'(0) = 1$ mrad



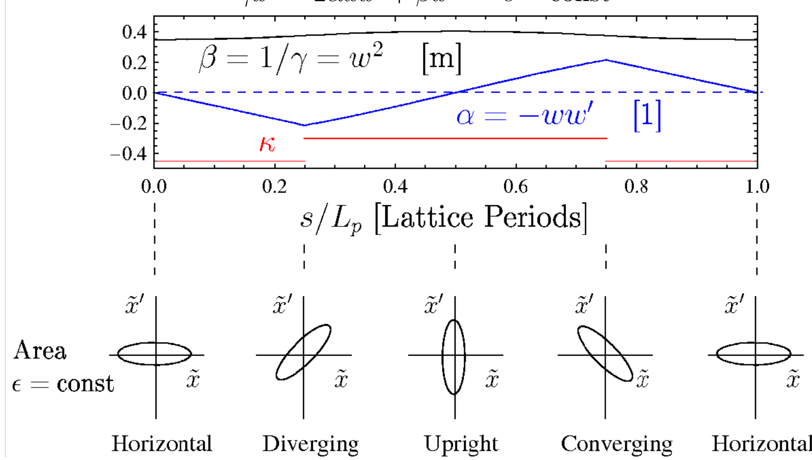


- Principal orbits in $\tilde{y} - \tilde{y}'$ phase-space are identical

Phase-Space Evolution in the Larmor frame (see also: S7):

- Phase-Space ellipse rotates and evolves in periodic lattice
  $\tilde{y} - \tilde{y}'$ phase-space properties same as in $\tilde{x} - \tilde{x}'$
  - Phase-space structure in *x-x'*, *y-y'* phase space is complicated

$$\gamma\tilde{x}^2 - 2\alpha\tilde{x}\tilde{x}' + \beta\tilde{x}'^2 = \epsilon = \text{const}$$

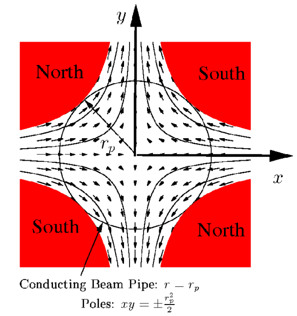


Comments on periodic solenoid results:

- Larmor frame analysis greatly simpli es results
  - 4D coupled orbit in *x-x'*, *y-y'* phase-space will be much more intricate in structure
- Phase-Space ellipse rotates and evolves in periodic lattice
- Periodic structure of lattice changes orbits from simple harmonic

3) Periodic Quadrupole FODO Lattice

$$x''(s) + \kappa(s)x(s) = 0$$
$$y''(s) - \kappa(s)y(s) = 0$$



For a magnetic quadrupole, we have

$$\kappa(s) = \frac{qG(s)}{m\gamma_b\beta_b c} = \frac{G(s)}{[B\rho]}$$

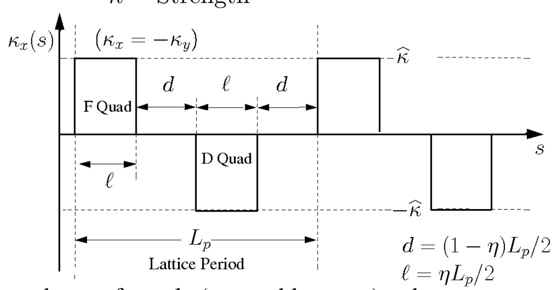
$$G(s) = \left.\frac{\partial B_x^a}{\partial y}\right|_{r=0} = \left.\frac{\partial B_y^a}{\partial x}\right|_{r=0} = \frac{B_q}{r_p}$$
$$= \text{Field Gradient}$$

$B_p =$ Pole Tip Field
$r_p =$ Clear Bore Aperture

$$[B\rho] \equiv \frac{p}{q} \simeq \frac{\gamma_b\beta_b mc}{q} \equiv \text{Rigidity}$$

<u>Parameters:</u>
$L_p =$ Lattice Period
$\eta \in (0,1] =$ Occupancy
$\hat{\kappa} =$ Strength

<u>Characteristics:</u>
$\eta L_p/2 = \ell =$ F/D Len
$(1-\eta)L_p/2 = d =$ Drift Len



Phase advance formula (see problem sets) reduces to:

$$\cos\sigma_0 = \cos\Theta\cosh\Theta + \frac{1-\eta}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta) - \frac{(1-\eta)^2}{2\eta^2}\Theta^2\sin\Theta\sinh\Theta$$

$$\Theta \equiv \frac{\eta}{2}\sqrt{|\hat{\kappa}|}L_p$$

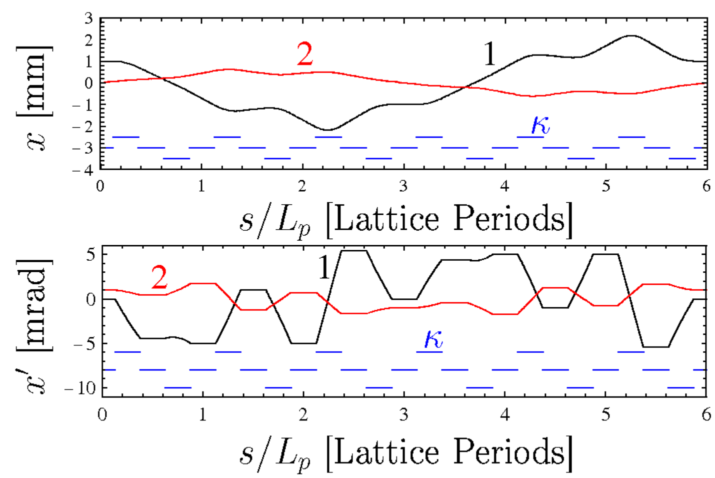
- Analysis shows FODO provides stronger focus for same integrated eld gradients than asymmetric doublet (see following) due to symmetry

Rescaled Principal Orbit Evolution FODO Quadrupole:

$L_p = 0.5$ m
$\sigma_0 = \pi/3 = 60^\circ$ $(\kappa = 39.24\ \text{m}^{-2})$
$\eta = 0.5$

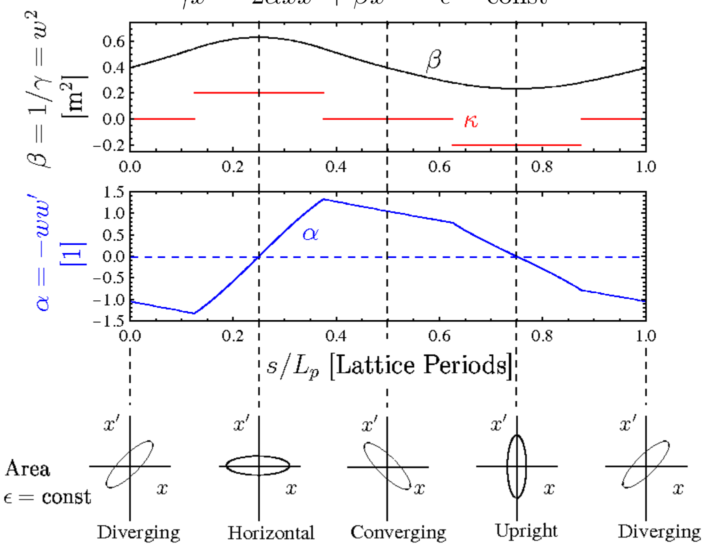
Cosine-Like
1: $x(0) = 1$ mm
$x'(0) = 0$ mrad

Sine-Like
2: $x(0) = 0$ mm
$x'(0) = 1$ mrad



Phase-Space Evolution (see also: S7):

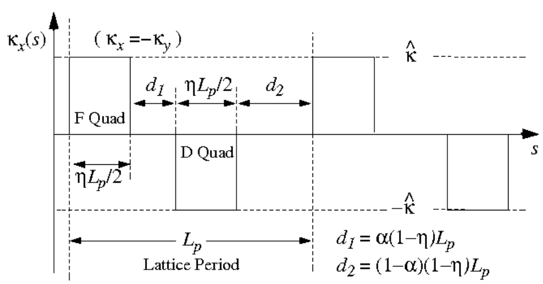
$$\gamma x^2 - 2\alpha x x' + \beta x'^2 = \epsilon = \text{const}$$



Comments on periodic FODO quadrupole results:

- Phase-Space ellipse rotates and evolves in periodic lattice
  - Evolution more intricate for Alternating Gradient (AG) focusing than for solenoidal focusing in the Larmor frame
- Harmonic content of orbits larger for AG focusing than solenodial focusing
- Orbit and phase space evolution analogous in *y-y'* plane
  - Simply related by an shift in *s* of the lattice

Extra: FODO drift symmetry relaxed: Periodic Quadrupole Doublet Focusing



Parameters:

L_p = Lattice Period

$\eta \in (0, 1]$ = Occupancy

$\alpha \in [0, 1]$ = Syncopation

$\hat{\kappa}$ = Strength

Characteristics:

$\eta L_p/2$ = F/D Len

$\alpha(1-\eta)L_p$ = Drift Len d_1

$(1-\alpha)(1-\eta)L_p$ = Drift Len d_2

Calculation gives:

$$\cos\sigma_0 = \cos\Theta\cosh\Theta + \frac{1-\eta}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta) - 2\alpha(1-\alpha)\frac{(1-\eta)^2}{\eta^2}\Theta^2\sin\Theta\sinh\Theta$$

$$\Theta \equiv \frac{\eta}{2}\sqrt{|\hat{\kappa}|}L_p$$

- Can be unstable when $\hat{\kappa}$ becomes large
  - Energy can pump into or out of particle orbit

Comments on Parameters:

- The "syncopation" parameter α measures how close the Focusing (F) and DeFocusing (D) quadrupoles are to each other in the lattice

$\alpha \in [0, 1]$

$$\alpha = 0 \implies d_1 = 0 \qquad d_2 = (1-\eta)L_p$$

$$\alpha = 1 \implies d_1 = (1-\eta)L_p \qquad d_2 = 0$$

The range $\alpha \in [1/2, 1]$ can be mapped to $\alpha \in [0, 1/2]$ by simply relabeling quantities. Therefore, we can take:

$\alpha \in [0, 1/2]$

- The special case of a doublet lattice with $\alpha = 1/2$ corresponds to equal drift lengths between the F and D quadrupoles and is called a FODO lattice

$$\alpha = 1/2 \implies d_1 = d_2 \equiv d = (1-\eta)L_p/2$$

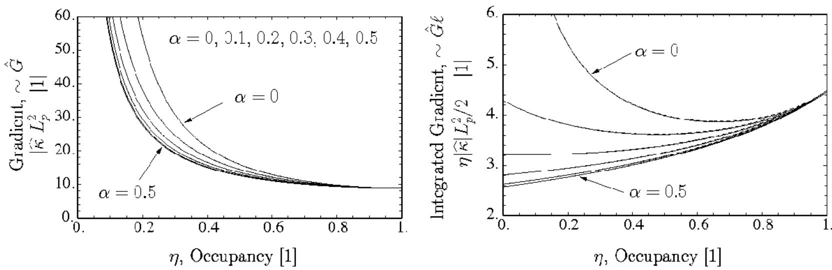
Phase advance constraint will be derived for FODO case in problems (algebra much simpler than doublet case)

Using these results, plot the Field Gradient and Integrated Gradient for quadrupole doublet focusing needed for $\sigma_0 = 80^\circ$ per lattice period

Gradient ~ $|\hat{\kappa}|L_p^2 \sim \hat{G}$

Integrated Gradient ~ $\eta|\hat{\kappa}|L_p^2/2 \sim \hat{G}\ell$

$\sigma_0 = 80^\circ$ /(Lattice Period) Quadrupole Doublet



- Exact solutions plotted dashed almost overlay with approx thin lens (next sec)
- Gradient and integrated gradient required depend only weakly on syncopation factor α when α is near or larger than ½
- Stronger gradient required for low occupancy η but integrated gradient varies comparatively less with η except for small α ///

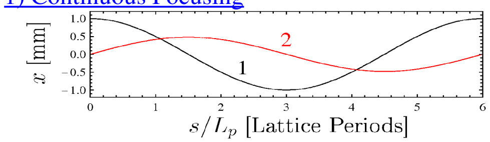
Contrast of Principal Orbits for di erent focusing:

- Use previous examples with "equivalent" focusing strength $\sigma_0 = 60^\circ$
- Note that periodic focusing adds harmonic structure: increasing for AG focus

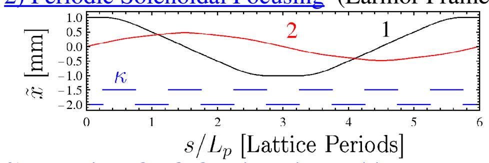
1) Continuous Focusing

2) Periodic Solenoidal Focusing (Larmor Frame)

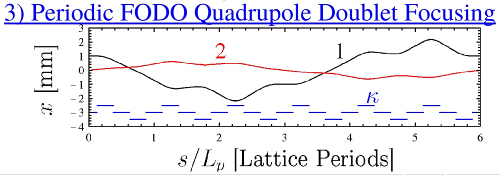
3) Periodic FODO Quadrupole Doublet Focusing



Simple Harmonic Oscillator



Simple harmonic oscillations modi ed with additional harmonics due to periodic focus



Simple harmonic oscillations more strongly modi ed due to periodic AG focus

4) Thin Lens Limits

Convenient to simply understand analytic scaling

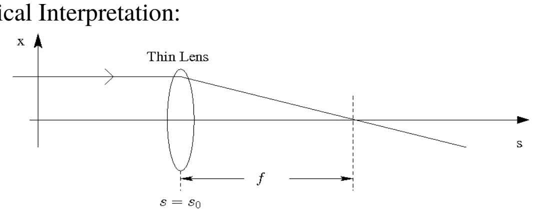
$$\kappa_x(s) = \frac{1}{f}\delta(s - s_0)$$

$$s_0 = \text{Optic Location} = \text{const}$$
$$f = \text{focal length} = \text{const}$$

Transfer Matrix:

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{s=s_0^+} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix}_{s=s_0^-}$$

Graphical Interpretation:



The thin lens limit of "thick" hard-edge solenoid and quadrupole focusing lattices presented can be obtained by taking:

Solenoids: $\hat{\kappa} \equiv \dfrac{1}{\eta f L_p}$ then take $\lim_{\eta \to 0}$

Quadrupoles: $\hat{\kappa} \equiv \dfrac{2}{\eta f L_p}$ then take $\lim_{\eta \to 0}$

This obtains when applied in the previous formulas:

$$\cos\sigma_0 = \begin{cases} 1 - \frac{1}{2}\frac{L_p}{f}, & \text{thin-lens periodic solenoid} \\ 1 - \frac{\alpha}{2}(1-\alpha)\left(\frac{L_p}{f}\right)^2, & \text{thin-lens quadrupole doublet} \\ & \alpha = \frac{1}{2} \Longrightarrow \text{FODO} \end{cases}$$

These formulas can also be derived directly from the drift and thin lens transfer matrices as

Periodic Solenoid

$$\cos\sigma_0 = \frac{1}{2}\text{Tr}\begin{bmatrix} 1 & L_p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = 1 - \frac{1}{2}\frac{L_p}{f}$$

Periodic FODO Quadrupole Doublet

$$\cos\sigma_0 = \frac{1}{2}\text{Tr}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & \alpha L_p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & (1-\alpha)L_p \\ 0 & 1 \end{bmatrix} = 1 - \frac{\alpha}{2}(1-\alpha)\left(\frac{L_p}{f}\right)^2$$

$\color{blue}{\text{Expanded phase advance formulas}}$ (thin lens type limit and similar) can be useful in system design studies

- Desirable to derive simple formulas relating magnet parameters to σ_0
  - Clear analytic scaling trends clarify design trade-o s
- For hard edge periodic lattices, expand formula for $\cos\sigma_0$ to leading order in $\Theta = \sqrt{|\hat{\kappa}|\eta L_p/2}$

/// Example: Periodic Quadrupole Doublet Focusing:

Expand previous phase advance formula for syncopated quadrupole doublet to obtain:

$$\cos\sigma_0 = 1 - \frac{(\eta\hat{\kappa}L_p^2)^2}{32}\left[\left(1-\frac{2}{3}\eta\right) - 4\left(\alpha - \frac{1}{2}\right)^2(1-\eta)^2\right]$$

where:

$$\hat{\kappa} = \begin{cases} \frac{\hat{G}}{[B\rho]}, & \text{Magnetic Quadrupoles} \\ \frac{\hat{G}}{\beta_b c[B\rho]}, & \text{Electric Quadrupoles} \end{cases} \qquad \hat{G} = \text{Hard-Edge Field Gradient}$$

Appendix A: Calculation of *w*(*s*) from Principal Orbit Functions

Evaluate principal orbit expressions of the transfer matrix through one lattice period using

$$w(s_i + L_p) = w_i$$
$$w'(s_i + L_p) = w'_i$$

and

$$\Delta\psi(s_i + L_p) = \int_{s_i}^{s_i+L_p} \frac{ds}{w^2(s)} = \sigma_0$$

to obtain (see S6F for principal orbit formulas in phase-amplitude form):

Example: $C(s|s_i) = \frac{w(s)}{w_i}\cos\Delta\psi(s) - w_i w(s)\sin\Delta\psi(s)$

$$\Longrightarrow \quad C(s_i + L_p|s_i) = \cos\sigma_0 - w_i w'_i \sin\sigma_0$$
$$S(s_i + L_p|s_i) = w_i^2 \sin\sigma_0$$
$$C'(s_i + L_p|s_i) = -\left(\frac{1}{w_i^2} + w_i w'_i\right)\sin\sigma_0$$
$$S'(s_i + L_p|s_i) = \cos\sigma_0 + w_i w'_i \sin\sigma_0$$

A1

Giving:

$$w_i = \sqrt{\frac{S(s_i + L_p|s_i)}{\sin\sigma_0}} \qquad \text{Apply } C(s|s_i) \text{ Eqn.}$$

$$w'_i = \frac{\cos\sigma_0 - C(s_i + L_p|s_i)}{\sqrt{S(s_i + L_p|s_i)\sin\sigma_0}} \qquad \text{Apply } S(s|s_i) \text{ Eqn.} + w_i \text{ Result Above}$$

Or in terms of the betatron formulation (see: S7 and S8) with $\beta = w^2, \ \beta' = 2ww'$

$$\beta_i = w_i^2 = \frac{S(s_i + L_p|s_i)}{\sin\sigma_0}$$
$$\beta'_i = 2w_i w'_i = \frac{2[\cos\sigma_0 - C(s_i + L_p|s_i)]}{\sin\sigma_0}$$

Next, calculate w from the principal orbit expression (S6F) in phase-amplitude forn

$$\frac{S}{w_i w} = \sin\Delta\psi \qquad S \equiv S(s|s_i) \ \text{etc.}$$
$$\frac{w_i}{w}C + \frac{w'_i}{w}S = \cos\Delta\psi$$

A2

Square and add equations:

$$\left(\frac{S}{w_i w}\right)^2 + \left(\frac{w_i C}{w} + \frac{w'_i S}{w}\right)^2 = 1$$

- This result re ects the structure of the underlying Courant-Snyder invariant (see: S7)

Gives:

$$w^2 = \left(\frac{S}{w_i}\right)^2 + (w_i C + w'_i S)^2$$

Use $w_i, \ w'_i$ previously identi ed and write out result:

$$w^2(s) = \beta(s) = \sin\sigma_0 \frac{S^2(s|s_i)}{S(s_i + L_p|s_i)} + \frac{S(s_i + L_p|s_i)}{\sin\sigma_0}\left[C(s|s_i) + \frac{\cos\sigma_0 - C(s_i + L_p|s_i)}{S(s_i + L_p|s_i)}S(s|s_i)\right]^2$$

- Formula shows that for a given σ_0 (used to specify lattice focusing strength), *w*(*s*) is given by two linear principal orbits calculated over one lattice period
  - Easy to apply numerically

A3

An alternative way to calculate $w(s)$ is as follows. 1[st] apply the phase-amplitude formulas for the principal orbit functions with:

$$s_i \rightarrow s$$
$$s \rightarrow s + L_p$$

$$\Longrightarrow \quad \begin{aligned} C(s+L_p|s) &= \cos\sigma_0 - w(s)w'(s)\sin\sigma_0 \\ S(s+L_p|s) &= w^2(s)\sin\sigma_0 \end{aligned}$$

$$\boxed{w^2(s) = \beta(s) = \frac{S(s+L_p|s)}{\sin\sigma_0} = \frac{\mathbf{M}_{12}(s+L_p|s)}{\sin\sigma_0}}$$

- Formula requires calculation of $S(s+L_p|s)$ at every value of s within lattice period
- Previous formula requires one calculation of $C(s|s_i),\ S(s|s_i)$ for $s_i \leq s \leq s_i + L_p$ and any value of s_i

A4

Matrix algebra can be applied to simplify this result:

$$\begin{aligned} \mathbf{M}(s+L_p|s) &= \mathbf{M}(s+L_p|s_i+L_p)\cdot\mathbf{M}(s_i+L_p|s) \\ &= \mathbf{M}(s|s_i)\cdot\mathbf{M}(s_i+L_p|s)\cdot[\mathbf{M}(s|s_i)\cdot\mathbf{M}^{-1}(s|s_i)] \\ &= \mathbf{M}(s|s_i)\cdot\mathbf{M}(s_i+L_p|s_i)\cdot\mathbf{M}^{-1}(s|s_i) \end{aligned}$$

$$\boxed{\mathbf{M}(s+L_p|s) = \mathbf{M}(s|s_i)\cdot\mathbf{M}(s_i+L_p|s_i)\cdot\mathbf{M}^{-1}(s|s_i)}$$

- Using this result with the previous formula allows the transfer matrix to be calculated only once per period from any initial condition
- Using:

$$\mathbf{M} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \qquad \mathbf{M}^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

Apply Wronskian condition: $\det\mathbf{M} = 1$

The matrix formula can be shown to the equivalent to the previous one

- Methodology applied in: Lund, Chilton, and Lee, PRSTAB **9** 064201 (2006) to construct a fail-safe iterative matched envelope including space-charge

A5

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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Please provide corrections with respect to the present archived version at:

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