05. The Courant Snyder Invariant and the Betatron Formulation*

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S7: Hill's Equation: The Courant-Snyder Invariant and Single Particle Emittance

S7A: Introduction

Constants of the motion can simplify the interpretation of dynamics in physics

- Desirable to identify constants of motion for Hill's equation for improved understanding of focusing in accelerators
- Constants of the motion are not immediately obvious for Hill's Equation due to s-varying focusing forces related to $\kappa(s)$ can add and remove energy from the particle
  - Wronskian symmetry is one useful symmetry
  - Are there other symmetries?

/// Illustrative Example: Continuous Focusing/Simple Harmonic Oscillator

Equation of motion:

$$x'' + k_{\beta 0}^2 x = 0 \qquad k_{\beta 0}^2 = \text{const} > 0$$

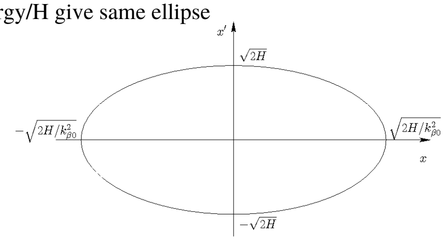
Constant of motion is the well-know Hamiltonian/Energy:

$$H = \frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2 \ = \text{const}$$

which shows that the particle moves on an ellipse in *x*-*x*' phase-space with:

- Location of particle on ellipse set by initial conditions
- All initial conditions with same energy/H give same ellipse

$$\text{Max/Min}[x] \Leftrightarrow x' = 0$$
$$\text{Max/Min}[x] = \pm\sqrt{2H/k_{\beta 0}^2}$$
$$\text{Max/Min}[x'] \Leftrightarrow x = 0$$
$$\text{Max/Min}[x'] = \pm\sqrt{2H}$$



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Question:

For Hill's equation:

$$x'' + \kappa(s)x = 0$$

does a quadratic invariant exist that can aid interpretation of the dynamics?

Answer we will find:

Yes, the Courant-Snyder invariant

Comments:

- Very important in accelerator physics
  - Helps interpretation of linear dynamics
- Named in honor of Courant and Snyder who popularized it's use in Accelerator physics while co-discovering alternating gradient (AG) focusing in a single seminal (and very elegant) paper:
  Courant and Snyder, *Theory of the Alternating Gradient Synchrotron*, Annals of Physics **3**, 1 (1958).
  - Christofolos also understood AG focusing in the same period using a more heuristic analysis
- Easily derived using phase-amplitude form of orbit solution
  - Can be much harder using other methods

S7B: Derivation of Courant-Snyder Invariant

The phase amplitude method described in S6 makes identi cation of the invariant elementary. Use the phase amplitude form of the orbit:

$$\boxed{\begin{aligned} x(s) &= A_i w(s)\cos\psi(s) \\ x'(s) &= A_i w'(s)\cos\psi(s) - \frac{A_i}{w(s)}\sin\psi(s) \end{aligned}}$$

$A_i,\ \psi_i = \psi(s_i)$ set by initial at $s = s_i$

where
$$w'' + \kappa(s)w - \frac{1}{w^3} = 0$$

Re-arrange the phase-amplitude trajectory equations:

$$\frac{x}{w} = A_i\cos\psi$$
$$wx' - w'x = A_i\sin\psi$$

square and add the equations to obtain the Courant-Snyder invariant:

$$\boxed{\begin{aligned} \left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 &= A_i^2(\cos^2\psi + \sin^2\psi) \\ &= A_i^2 = \text{const} \end{aligned}}$$

Comments on the Courant-Snyder Invariant:

- Simpli es interpretation of dynamics (will show how shortly)
- Extensively used in accelerator physics
- Quadratic structure in *x*-*x*' de nes a rotated ellipse in *x*-*x*' phase space.
- Because $$w^2\left(\frac{x}{w}\right)' = wx' - w'x$$
  the Courant-Snyder invariant can be alternatively expressed as:
  $$\left(\frac{x}{w}\right)^2 + \left[w^2\left(\frac{x}{w}\right)'\right]^2 = \text{const}$$
- *Cannot* be interpreted as a conserved energy!

The point that the Courant-Snyder invariant is *not* a conserved energy should be elaborated. The equation of motion:

$$x'' + \kappa(s)x = 0$$

Is derivable from the Hamiltonian

$$H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 \implies \begin{aligned} \frac{d}{ds}x &= \frac{\partial H}{\partial x'} = x' \\ \frac{d}{ds}x' &= -\frac{\partial H}{\partial x} = -\kappa x \end{aligned} \implies x'' + \kappa x = 0$$

H is the energy:

$$H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 = T + V \qquad \begin{aligned} T &= \frac{1}{2}x'^2 &&= \text{Kinetic "Energy"} \\ V &= \frac{1}{2}\kappa x^2 &&= \text{Potential "Energy"} \end{aligned}$$

Apply the chain-Rule with *H* = *H*(*x*,*x*';*s*):

$$\frac{dH}{ds} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial x}\frac{dx}{ds} + \frac{\partial H}{\partial x'}\frac{dx'}{ds}$$

Apply the equation of motion in Hamiltonian form:

$$\frac{d}{ds}x = \frac{\partial H}{\partial x'} \qquad \frac{d}{ds}x' = -\frac{\partial H}{\partial x}$$

$$\frac{dH}{ds} = \frac{\partial H}{\partial s} \underbrace{- \frac{dx'}{ds}\frac{dx}{ds} + \frac{dx}{ds}\frac{dx'}{ds}}_{0} = \frac{\partial H}{\partial s} = \frac{1}{2}\kappa' x^2 \neq 0$$

$$\implies \boxed{H \neq \text{const}}$$

- Energy of a "kicked" oscillator with $\kappa(s) \neq \text{const}$ is not conserved
  - Lattice can source and sink particle energy
- Energy should not be confused with the Courant-Snyder invariant

/// Aside: Only for the special case of continuous focusing (i.e., a simple Harmonic oscillator) are the Courant-Snyder invariant and energy simply related:

Continuous Focusing: $\kappa(s) = k_{\beta 0}^2 = \text{const}$

$$\implies H = \frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2 = \text{const}$$

w equation: $$w'' + k_{\beta 0}^2 w - \frac{1}{w^3} = 0$$

$$\implies w = \sqrt{\frac{1}{k_{\beta 0}}} = \text{const}$$

Courant-Snyder Invariant: $$\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = \text{const}$$

$$\begin{aligned} \implies \left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 &= k_{\beta 0}x^2 + \frac{x'^2}{k_{\beta 0}} \\ &= \frac{2}{k_{\beta 0}}\left(\frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2\right) \\ &= \frac{2H}{k_{\beta 0}} = \text{const} \end{aligned}$$

///

Interpret the Courant-Snyder invariant:

$$\boxed{\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2 = \text{const}}$$

by expanding and isolating terms quadratic terms in *x*-*x*' phase-space variables:

$$\left[\frac{1}{w^2} + w'^2\right] x^2 + 2[-ww']xx' + [w^2]x'^2 = A_i^2 = \text{const}$$

The three coefficients in [...] are functions of *w* and *w*' only and therefore are *functions of the lattice only* (not particle initial conditions). The coefficients are commonly called "Twiss Parameters" and are denoted as:

$$\boxed{\begin{aligned} &\gamma x^2 + 2\alpha x x' + \beta x'^2 = A_i^2 = \text{const} \\ &\gamma(s) \equiv \frac{1}{w^2(s)} + [w'(s)]^2 = \frac{1+\alpha^2(s)}{\beta(s)} \\ &\beta(s) \equiv w^2(s) \\ &\alpha(s) \equiv -w(s)w'(s) \end{aligned}}$$

$$\gamma\beta = 1 + \alpha^2$$

- All Twiss "parameters" are specified by $w(s)$
- Given *w* and *w*' at a point (*s*) any 2 Twiss parameters give the 3rd

The area of the invariant ellipse is:

- Analytic geometry formulas: $\gamma x^2 + 2\alpha x x' + \beta x'^2 = A_i^2 \rightarrow \text{Area} = \pi A_i^2/\sqrt{\gamma\beta - \alpha^2}$
- For Courant-Snyder ellipse: $\gamma\beta = 1 + \alpha^2$

$$\boxed{\text{Phase-Space Area} = \int_{\text{ellipse}} dx dx' = \frac{\pi A_i^2}{\sqrt{\gamma\beta - \alpha^2}} = \pi A_i^2 \equiv \pi\epsilon}$$

Where ϵ is the single-particle emittance:

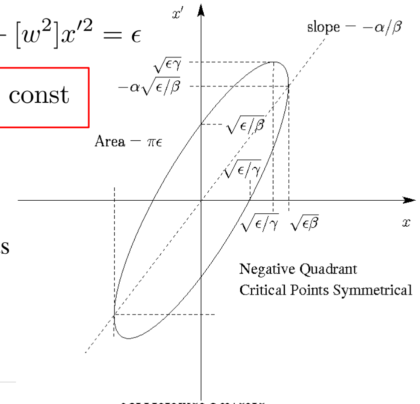
- Emittance is the area of the orbit in *x*-*x*' phase-space divided by π

$$[1/w^2 + w'^2]x^2 + 2[-ww']xx' + [w^2]x'^2 = \epsilon$$

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon = \text{const}}$$

See problem sets for critical point calculation

- Important to understand extents of bundle or particles with different initial conditions



/// Aside on Notation: Twiss Parameters and Emittance Units:

Twiss Parameters:

Use of $\alpha,\ \beta,\ \gamma$ should not create confusion with kinematic relativistic factors

- $\beta_b,\ \gamma_b$ are absorbed in the focusing function
- Contextual use of notation unfortunate reality .... not enough symbols!
- Notation originally due to Courant and Snyder, not Twiss, and might be more appropriately called "Courant-Snyder functions" or "lattice functions."

Emittance Units:

x has dimensions of length and *x*' is a dimensionless angle. So *x*-*x*' phase-space area has dimensions [[ϵ]] = length. A common choice of units is millimeters (mm) and milliradians (mrad), e.g.,

$$\epsilon = 10 \text{ mm-mrad}$$

The definition of the emittance employed is not unique and different workers use a wide variety of symbols. Some common notational choices:

$$\pi\epsilon \rightarrow \epsilon \qquad \epsilon \rightarrow \varepsilon \qquad \epsilon \rightarrow E$$

Write the emittance values in units with a π, e.g.,

$\epsilon = 10.5\ \pi - \text{mm-mrad}$ (seems falling out of favor but still common)

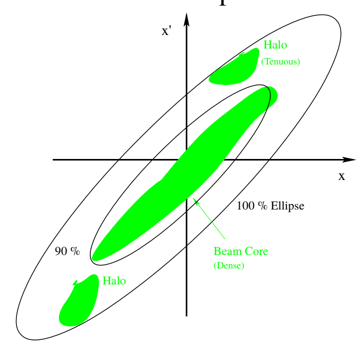
Use caution! Understand conventions being used before applying results! ///

Emittance is sometimes defined by the largest Courant-Snyder ellipse that will contain a specified fraction of the distribution of beam particles. Common choices are:

- 100%
- 95%
- 90%
- ....
- Depends emphasis

Comment:

Figure shows scaling of concentric ellipses for simplicity but can also define for smallest ellipse changing orientation



We will motivate (problems and later lectures) that the statistical measure

$$\boxed{\begin{aligned} \varepsilon_{\text{rms}} &= \left[\langle\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right]^{1/2} \qquad \langle\cdots\rangle = \text{Distribution Average} \\ &= \text{rms Statistical Emittance} \end{aligned}}$$

provides a weighted average measure of the beam phase-space area.

Properties of Courant-Snyder Invariant:

- The ellipse will rotate and change shape as the particle advances through the focusing lattice, but the instantaneous area of the ellipse ($\pi\epsilon = \text{const}$) remains constant.
- The location of the particle on the ellipse and the size (area) of the ellipse depends on the initial conditions of the particle.
- The orientation of the ellipse is independent of the particle initial conditions. All particles move on nested ellipses.
- Quadratic in the *x-x*' phase-space coordinates, but is *not* the transverse particle energy (which is not conserved).

See examples of Courant-Snyder Ellipse evolution in 04 lecture set

- Continuous Focusing
- Periodic Solenoid Focusing
- Periodic FODO Quadrupole Focusing

S7C: Lattice Maps

The Courant-Snyder invariant helps us understand the phase-space evolution of the particles. Knowing how the ellipse transforms (twists and rotates without changing area) is equivalent to knowing the dynamics of a *bundle* of particles.

To see this:

General *s*:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

Initial $s = s_i$

$$\gamma_i x_i^2 + 2\alpha_i x_i x_i' + \beta_i x_i'^2 = \epsilon$$

$$\beta_i \equiv \beta(s = s_i) \qquad x_i \equiv x(s = s_i)$$
$$\alpha_i \equiv \alpha(s = s_i) \qquad x_i' \equiv x'(s = s_i)$$
$$\gamma_i \equiv \gamma(s = s_i)$$

Apply the components of the transport matrix:

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \mathbf{M}(s|s_i) \cdot \begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \cdot \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$

Invert 2x2 matrix and apply det **M** = 1 (Wronskian):

$$\Longrightarrow \quad \begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \begin{bmatrix} S' & -S \\ -C' & C \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix} \qquad C \equiv C(s|s_i), \text{ etc.}$$

Initial $s = s_i$ General *s*:

$$\gamma_i x_i^2 + 2\alpha_i x_i x_i' + \beta_i x_i'^2 = \epsilon = \text{const} = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

Insert expressions for $x_i,\ x_i'$ in the initial ellipse expression, collect factors of $x^2, \ \ xx' \ \ x'^2$ and equate to general *s* ellipse expression:

$$\begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \begin{bmatrix} S' & -S \\ -C' & C \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix}$$

$$x_i = S'x - Sx'$$
$$x_i' = -C'x + Cx'$$

$$\Longrightarrow \quad \gamma_i x_i^2 = [\gamma_i S'^2]x^2 + 2[-\gamma_i S S']x x' + [\gamma_i S^2]x'^2$$

+ Similar steps for other two terms for $2\alpha_i x_i x_i'^2, \ \ \beta_i x_i'^2$

Completing this procedure and gathering terms:

$$\begin{aligned} &[\gamma_i S'^2 - 2\alpha_i S'C' + \beta_i C'^2]x^2 \\ &+2[-\gamma_i S S' + \alpha_i(CS' + SC') - \beta_i CC']xx' \\ &+[\gamma_i S^2 - 2\alpha_i SC + \beta_i C^2]x'^2 \\ &= \gamma x^2 + 2\alpha x x' + \beta x'^2 \end{aligned}$$

Collect coe cients of x^2, xx', and x'^2 and summarize in matrix form:

$$\boxed{\begin{bmatrix} \gamma \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} S'^2 & -2C'S' & C'^2 \\ -SS' & CS' + SC' & -CC' \\ S^2 & -2CS & C^2 \end{bmatrix} \cdot \begin{bmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{bmatrix}}$$

See steps on next page

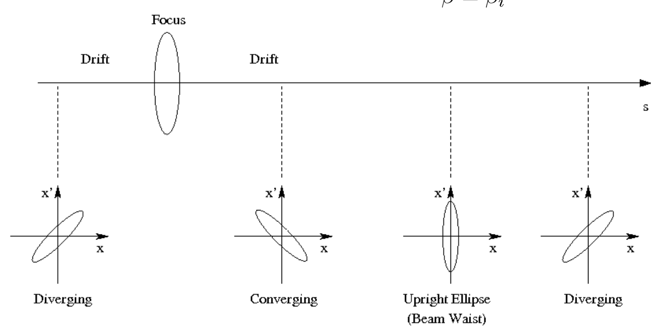
This result can be applied to illustrate how a bundle of particles will evolve from an initial location in the lattice subject to the linear focusing optics in the machine using *only* the principal orbit functions *C, S, C'*, and *S'*

- Principal orbits will generally need to be calculated numerically
  - Intuition can be built up using simple analytical results (hard edge etc)
- Can express C, S, C', S' in terms of CS-ellipse functions using S6F results and de nitions for $\beta, \ \alpha$

/// Example: Ellipse Evolution in a simple kicked focusing lattice

Drift:
$$\begin{bmatrix} C & S \\ C' & S' \end{bmatrix} = \begin{bmatrix} 1 & s-s_i \\ 0 & 1 \end{bmatrix} \qquad \begin{aligned} \gamma &= \gamma_i \\ \alpha &= -\gamma_i(s-s_i) + \alpha_i \\ \beta &= \gamma_i(s-s_i)^2 - 2\alpha_i(s-s_i) + \beta_i \end{aligned}$$

Thin Lens:
focal length f
$$\begin{bmatrix} C & S \\ C' & S' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \qquad \begin{aligned} \gamma &= \gamma_i + 2\alpha_i/f + \beta_i/f^2 \\ \alpha &= -\beta_i/f + \alpha_i \\ \beta &= \beta_i \end{aligned}$$



For further examples of phase-space ellipse evolutions in standard lattices, see previous examples given in: S6G ///

Rather than use a 3x3 advance matrix for $\gamma,\ \alpha,\ \beta$, we can alternatively derive an expression based on the usual 2x2 transfer matrix ***M*** which will help further clarify the underlying structure of the linear dynamics.

Recall in S6F

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \mathbf{M}(s|s_i)\cdot\begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix}\cdot\begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix}$$

Identi ed

$$C(s|s_i) = \frac{w(s)}{w_i}\cos\Delta\psi(s) - w_i' w(s)\sin\Delta\psi(s)$$
$$S(s|s_i) = w_i w(s)\sin\Delta\psi(s)$$
$$C'(s|s_i) = \left(\frac{w'(s)}{w_i} - \frac{w_i'}{w(s)}\right)\cos\Delta\psi(s) - \left(\frac{1}{w_i w(s)} + w_i' w'(s)\right)\sin\Delta\psi(s)$$
$$S'(s|s_i) = \frac{w_i}{w(s)}\cos\Delta\psi(s) + w_i w'(s)\sin\Delta\psi(s)$$

$$\Delta\psi(s) \equiv \int_{s_i}^{s}\frac{d\tilde{s}}{w^2(\tilde{s})} \qquad \begin{aligned} w_i &\equiv w(s=s_i) \\ w_i' &\equiv w'(s=s_i) \end{aligned}$$

Using this and

$$\begin{aligned} \beta &\equiv w^2 \\ \alpha &\equiv -ww' \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \frac{w}{w_i} &= \sqrt{\frac{\beta}{\beta_i}} \\ -w_i' w &= (-w_i' w_i)\frac{w}{w_i} = \alpha_i\sqrt{\frac{\beta}{\beta_i}} \end{aligned}$$

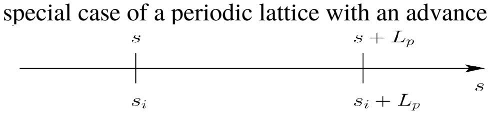
… etc.

After some algebra, we obtain the expression

$$\mathbf{M}(s|s_i) = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{\frac{\beta(s)}{\beta_i}}[\cos\Delta\psi(s) + \alpha_i\sin\Delta\psi(s)] & \sqrt{\beta_i\beta}\sin\Delta\psi(s) \\ -\frac{\alpha(s)-\alpha_i}{\sqrt{\beta_i\beta(s)}}\cos\Delta\psi(s) - \frac{1+\alpha_i\alpha(s)}{\sqrt{\beta_i\beta(s)}}\sin\Delta\psi(s) & \sqrt{\frac{\beta_i}{\beta(s)}}[\cos\Delta\psi(s) - \alpha\sin\Delta\psi(s)] \end{bmatrix}$$

- Transfer matrix is now expressed in terms of Courant-Snyder ellipse functions, their initial values, and the phase advance from the initial point.

For the special case of a periodic lattice with an advance over one period



$$\alpha(s_i) = \alpha(s) \quad \beta(s_i) = \beta(s) \quad \gamma(s_i) = \gamma(s) \quad \Delta\psi = \sigma_0$$

this expression for ***M*** reduces to

$$\mathbf{M}(s_i + L_p|s_i) = \begin{bmatrix} \cos\sigma_0 + \alpha\sin\sigma_0 & \beta\sin\sigma_0 \\ -\gamma\sin\sigma_0 & \cos\sigma_0 - \alpha\sin\sigma_0 \end{bmatrix}$$
$$= \mathbf{I}\cos\sigma_0 + \mathbf{J}(s)\sin\sigma_0$$

$$\mathbf{I} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{J} \equiv \begin{bmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{bmatrix} \qquad 1 + \alpha^2 = \gamma\beta$$

$$\sigma_0 \equiv \int_{s_i}^{s_i+L_p}\frac{d\tilde{s}}{\beta(\tilde{s})}$$

It is straightforward to verify that:

$$\det\mathbf{J} = -\alpha^2 + \gamma\beta = 1 \qquad e^{\mathbf{J}\psi(s)} = \mathbf{I}\cos\psi(s) + \mathbf{J}\sin\psi(s)$$
$$\mathbf{J}\cdot\mathbf{J} = -\mathbf{I}$$

An advance $s_i \rightarrow s + L_p$ through any interval in a periodic lattice can be resolved as:

Giving for $\mathbf{M}(s + L_p | s_i)$ advance (write in two di erent steps LHS and RHS):

$$\mathbf{M}(s + L_p | s) \cdot \mathbf{M}(s|s_i) = \mathbf{M}(s + L_p | s_i + L_p) \cdot \mathbf{M}(s_i + L_p | s_i)$$
$$= \mathbf{M}(s|s_i) \cdot \mathbf{M}(s_i + L_p | s_i) \quad \Leftarrow \quad \cdot \mathbf{M}^{-1}(s|s_i)$$

Or:

$$\mathbf{M}(\mathrm{s{+}L}_p|s) = \mathbf{M}(s|s_i) \cdot \mathbf{M}(s_i + L_p|s_i) \cdot \mathbf{M}^{-1}(s|s_i)$$

Operate with from LHS and employ

$$\mathbf{M} \cdot \mathbf{M}^{-1} = \mathbf{I}$$

Using:

$$\mathbf{M}(s + L_p|s) = \mathbf{I}\cos\sigma_0 + \mathbf{J}(s)\sin\sigma_0$$
$$\mathbf{M}(s_i + L_p|s_i) = \mathbf{I}\cos\sigma_0 + \mathbf{J}(s_i)\sin\sigma_0$$

Gives:

$$\mathbf{I}\cos\sigma_0 + \mathbf{J}(s)\sin\sigma_0 = \mathbf{M}^{-1}(s|s_i) \cdot [\mathbf{I}\cos\sigma_0 + \mathbf{J}(s_i)\sin\sigma_0] \cdot \mathbf{M}(s|s_i)$$
$$= \mathbf{I}\cos\sigma_0 + \mathbf{M}^{-1}(s|s_i) \cdot \mathbf{J}(s) \cdot \mathbf{M}(s|s_i)\sin\sigma_0$$

$\mathbf{I}\cos\sigma_0$ is on both RHS and LHS and then canceling $\sin\sigma_0$

This gives a simple expression connecting the Twiss parameters:

$$\Longrightarrow \quad \boxed{\mathbf{J}(s) = \mathbf{M}(s|s_i) \cdot \mathbf{J}(s_i) \cdot \mathbf{M}^{-1}(s|s_i)} \qquad \mathbf{J} \equiv \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}$$

- Simple formula connects the Courant-Synder functions $\gamma,\ \alpha,\ \beta$ at an initial point $s = s_i$ to any location s in the lattice in terms of the transfer matrix ***M***.
- Result does *NOT* require the lattice to be periodic. Periodic extensions can be used to generalize arguments employed to work for *any* lattice interval.

S8: Hill's Equation: The Betatron Formulation of the Particle Orbit and Maximum Orbit Excursions S8A: Formulation

The phase-amplitude form of the particle orbit analyzed in S6 of

$$\boxed{x(s) = A_i w(s)\cos\psi(s) = \sqrt{\epsilon}w(s)\cos\psi(s)} \qquad [[w]] = (\text{meters})^{1/2}$$

is not a unique choice. Here, *w* has dimensions sqrt(meters), which can render it inconvenient in applications. Due to this and the utility of the Twiss parameters used in describing orientation of the phase-space ellipse associated with the Courant-Snyder invariant (see: S7) on which the particle moves, it is convenient to de ne an alternative, Betatron representation of the orbit with:

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos\psi(s)$$

Betatron function: $\beta(s) \equiv w^2(s)$

Single-Particle Emittance: $\epsilon \equiv A_i^2 = \text{const}$

Phase: $\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{\beta(\tilde{s})} = \psi_i + \Delta\psi(s)$

- The betatron function is a Twiss "parameter" with dimension [[β]] = meters

Comments:

- Use of the symbol β for the betatron function should not result in confusion with relativistic factors such as β_b since the context of use will make clear
  - Relativistic factors often absorbed in lattice focusing function and do not directly appear in the dynamical descriptions
- The change in phase $\Delta\psi$ is the same for both formulations:

$$\Delta\psi(s) = \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})} = \int_{s_i}^{s} \frac{d\tilde{s}}{\beta(\tilde{s})}$$

From the equation for w:

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s + L_p) = w(s) \qquad w(s) > 0$$

the betatron function is described by:

$$w = \beta^{1/2}$$

$$w' = \frac{1}{2}\frac{\beta'}{\beta^{1/2}}$$

$$w'' = \frac{1}{2}\frac{\beta''}{\beta^{1/2}} - \frac{1}{4}\frac{\beta'^2}{\beta^{3/2}}$$

$$\Longrightarrow \boxed{\begin{array}{l} \frac{1}{2}\beta(s)\beta''(s) - \frac{1}{4}\beta'^2(s) + \kappa(s)\beta^2(s) = 1 \\ \beta(s+L_p) = \beta(s) \qquad \beta(s) > 0 \end{array}}$$

- The betatron function represents, analogously to the *w*-function, a special function de ned by the periodic lattice. Similar to *w*(*s*) it is a unique function of the lattice.
- The equation is still nonlinear but we can apply our previous analysis of *w*(*s*) (see S6 Appendix A) to solve analytically in terms of the principle orbits

S8B: Maximum Orbit Excursions

From the orbit equation

$$x = \sqrt{\epsilon\beta}\cos\psi$$

the maximum and minimum possible particle excursions occur where:

$$\cos\psi = +1 \quad \longrightarrow \quad \mathrm{Max}[x] = \sqrt{\epsilon\beta(s)} = \sqrt{\epsilon}w(s)$$

$$\cos\psi = -1 \quad \longrightarrow \quad \mathrm{Min}[x] = -\sqrt{\epsilon\beta(s)} = -\sqrt{\epsilon}w(s)$$

Thus, the max radial extent of *all* particle oscillations $\mathrm{Max}[x] \equiv x_m$ in the beam distribution occurs for the particle with the max single particle emittance since the particles move on nested ellipses:

$$\boxed{\begin{array}{l} \mathrm{Max}[\epsilon] \equiv \epsilon_m \\ x_m(s) = \sqrt{\epsilon_m\beta(s)} = \sqrt{\epsilon_m}w(s) \end{array}}$$

In terms of Twiss parameters:

$$\boxed{\begin{array}{l} x_m = \sqrt{\epsilon_m}w = \sqrt{\epsilon_m\beta} \\ x'_m = \sqrt{\epsilon_m}w' = -\sqrt{\frac{\epsilon_m}{\beta}}\alpha \end{array}}$$

- Assumes su cient numbers of particles to populate all possible phases
- x_m corresponds to the min possible machine aperture to prevent particle losses
  - Practical aperture choice in uenced by: resonance e ects due to nonlinear applied elds, space-charge, scattering, nite particle lifetime, ...

From:

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s + L_p) = w(s) \qquad w(s) > 0$$

We immediately obtain an equation for the maximum locus (envelope) of radial particle excursions $x_m = \sqrt{\epsilon_m}w$ as:

$$\boxed{\begin{array}{l} x''_m(s) + \kappa(s)x_m(s) - \frac{\epsilon_m^2}{x_m^3(s)} = 0 \\ x_m(s+L_p) = x_m(s) \qquad x_m(s) > 0 \end{array}}$$

Comments:

- Equation is analogous to the statistical envelope equation derived in USPAS course Beam Physics with Intense Space-Charge when a space-charge term is added and the max single particle emittance is interpreted as a statistical emittance
  - correspondence will be developed in lecture on Space Charge E ects

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/msu/phy905_2020/

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